Control
and Cybernetics
Vol. 16 (1987) No. 3–4

Stochastic production–inventory control problem with random demand

by

TADEUSZ BANEK
Department of Technical Education
University M. Curie-Skłodowska
Narutowicza 12
20-004 Lublin, Poland
OLGIERD GEDYMIN
Department of Mechanical Engineering
Technical University
Dąbrowskiego 13
20-109 Lublin, Poland

This paper considers a finite horizon stochastic production control problem, where the demand rate is a stochastic process described by bilinear stochastic differential equation. The closed-form solution for optimal feedback policy is given.

1. Introduction

Sethi and Thompson [1], [2] considered stochastic production – inventory model with quadratic loss function defined in terms of the deviation of production and inventory levels from their rated or factory – optimal values. They assumed that the control and state variables are the production rate and the inventory level respectively. These variables are related between them in the form of the Itô stochastic differential equation. Therefore the production rate and the inventory level are stochastic processes and the demand rate was assumed constant and known. The diffusion coefficient was also constant so that the perturbation effect of the Wiener process was independent of the inventory level. Closed – form solutions for optimal feedback production policy for both finite and infinite horizon versions of the model without production nonnegativity constraint were obtained.

The model considered in this paper can be treated as an extension to the model presented in [1], [2]. We assume here that the demand rate is not constant but is a stochastic process described by means of the bilinear stochastic differential equation. Moreover, the perturbation effect of the Wiener process in the stochastic differential equation, relating the production rate and the inventory level, is supposed to depend currently on the inventory level. We give the optimal feedback solution to our finite horizon problem without production nonnegativity constraint.

2. Statement of the problem

Consider the problem of determining the minimum cost production—inventory schedule for a homogeneous commodity, over a fixed planning horizon $[0, T]$.

Define the following quantities:

x_t = inventory level at time t (state variable)
u_t = production rate at time t (control variable)
y_t = demand rate at time t (uncontrolled state variable)
z_t = factory—optimal (given) inventory rate at time t
v_t = factory—optimal (given) production rate at time t
x^0 = initial inventory level
y^0 = initial demand rate
(w_{1t}, w_{2t}) = the standard 2-dimensional Wiener process
$\sigma_1, \sigma_2, \varrho$ = the constant diffusion coefficients
α = drift coefficient
a = production cost coefficient (> 0)
b = inventory holding cost coefficient (> 0)
c = the penalty coefficient (> 0)
T = planning horizon.

The conditions of the model:

$$dx_t = (u_t - y_t)\,dt + (\sigma_1 + \varrho x_t)\,dw_{1t}, \qquad x_0 = x^0 \tag{1}$$

$$dy_t = y_t(\alpha\,dt + \sigma_2\,dw_{2t}), \qquad y_0 = y^0 \tag{2}$$

$$\min E\left\{\int_0^T [a(u_t - v_t)^2 + b(x_t - z_t)^2]\,dt + c(x_T - z_T)^2\right\}. \tag{3}$$

(1) is the balance equation in a differential form, and it expresses relation between inventory level, production rate, demand rate and random disturbances. The latter are represented by the term $(\sigma_1 + \varrho x_t)\,dw_{1t}$, which can be interpreted as "sales returns", "inventory spoilage" etc. (see [3], where $\varrho = 0$).

(2) describes the behaviour of the demand process in the form of stochastic bilinear differential equation.

The description of the random economic processes such as price or demand fluctuations in the form of linear stochastic equations (see Albouy [4], Aoki

[5]) is not adequate to the economic reality, since the trajectories of the linear stochastic processes (Ornstein–Uhlenbeck processes) can vary from $-\infty$ to ∞. For that reason the price fluctuations in Merton's portfolio selection model [6], were described by the bilinear stochastic process, with the trajectories varying from 0 to ∞.

Our motivation for modeling the demand process in the form of equation (2) can be summarized as follows:

(i) the trajectories of eq. (2) vary between 0 and ∞ (assuming that $y_0 > 0$).

(ii) Many probabilistic aspects of bilinear processes were widely discussed in various publications, e.g. see [7] and papers cited there.

(iii) The case, where an increment of the demand (in time t) is proportional to the value of the demand (in time t) with the random coefficient of proportionality ($\alpha\, dt + \sigma_2\, dw_{2t}$), is very interesting from the economic point of view.

3. Solution to the problem

The solution of the model (1), (2), (3) will be carried out via the development of the Hamilton–Jacobi–Bellman equation. Let $W = W(t, x, y)$ denote the expected value of the control problem (1), (2), (3), so that $W(t, x_0, y_0)$ represents the value of the objective function (3) subject to the state equation (1), (2). Then it can be shown that $W(t, x, y)$ satisfies the following Hamilton–Jacobi–Bellman equation (see [8]):

$$W_t + \frac{\sigma_1^2 + 2\varrho\sigma_1 x + \varrho^2 x^2}{2} W_{xx} + \frac{\sigma_2^2 y^2}{2} W_{yy} + \min_u \{(u-y) W_x + \alpha y W_y + a(u-v)^2 + b(x-z)^2\} = 0 \tag{4}$$

with the boundary condition

$$W_T = c(x_T - z_T)^2. \tag{5}$$

From (4) we have

$$u_{\text{opt}} = -\frac{W_x}{2a} + v.$$

Substituting u_{opt} into (4) yields the following Hamilton–Jacobi–Bellman equation

$$W_t + \frac{\sigma_1^2 + 2\varrho\sigma_1 x + \varrho^2 x^2}{2} W_{xx} + \frac{\sigma_2^2 y^2}{2} W_{yy} - \frac{W_x^2}{4a} - yW_x + \alpha y W_y + b(x-z)^2 + vW_x = 0. \tag{6}$$

To solve (6) we let

$$W(t, x, y) = \alpha_{11} x^2 + \alpha_{12} xy + \alpha_{22} y^2 + \beta_1 x + \beta_2 y + \gamma.$$

After computing all the derivatives of W, we substitute them into (6) and after collecting terms we get

$$\dot{\alpha}_{11} x^2 + \dot{\alpha}_{12} xy + \dot{\alpha}_{22} y^2 + \dot{\beta}_1 x + \dot{\beta}_2 y + \dot{\gamma} + \frac{\sigma_1^2}{2} 2\alpha_{11} +$$

$$+ 2\varrho\sigma_1 \alpha_{11} x + \sigma_2^2 \alpha_{22} y^2 - \frac{1}{4a}(2\alpha_{11} x + \alpha_{12} y + \beta_1)^2 + \varrho\alpha_{11} x^2$$

$$- y(2\alpha_{11} x + \alpha_{12} y + \beta_1) + \alpha y(\alpha_{12} x + 2\alpha_{22} y + \beta_2) +$$

$$+ b(x-z)^2 + v(2\alpha_{11} x + \alpha_{12} y + \beta_1) = 0. \tag{7}$$

Since (7) must hold for any value of x and y, then by comparing the coefficients in (7) we have

$$\dot{\alpha}_{11} = \frac{\alpha_{11}^2}{a} - b - \varrho^2 \alpha_{11}, \quad \alpha_{11}(T) = C \tag{8}$$

$$\dot{\alpha}_{12} = \left(\frac{\alpha_{11}}{a} - \alpha\right) \alpha_{12} + 2\alpha_{11}, \quad \alpha_{12}(T) = 0 \tag{9}$$

$$\dot{\alpha}_{22} = -(2\alpha + \sigma_2^2)\alpha_{22} + \frac{\alpha_{12}^2}{4a} + \alpha_{12}, \quad \alpha_{22}(T) = 0 \tag{10}$$

$$\dot{\beta}_1 = \frac{\alpha_{11}}{a} \beta_1 + 2bz - 2\alpha_{11} v - 2\varrho\sigma_1 \alpha_{11}, \quad \beta_1(T) = -2cz_T \tag{11}$$

$$\dot{\beta}_2 = -\alpha\beta_2 + \left(\frac{\alpha_{12}}{2a} + 1\right)\beta_1 - \alpha_{12} v, \quad \beta_2(T) = 0 \tag{12}$$

$$\dot{\gamma} = \frac{\beta_1^2}{4a} - \sigma_1^2 \alpha_{11} - bz^2 - \beta_1 v, \quad \gamma(T) = cz_T^2. \tag{13}$$

Integrating (8)–(13) we get the following formulas:

$$\alpha_{11}(t) = \frac{\theta_2(c-\theta_1) - \theta_1(c-\theta_2)\exp\left[(\theta_2-\theta_1)\frac{t-T}{a}\right]}{(c-\theta_1) - (c-\theta_2)\exp\left[(\theta_2-\theta_1)\frac{t-T}{a}\right]} \tag{14}$$

where

$$\theta_1 = \tfrac{1}{2}\left[a\varrho^2 - \sqrt{(a\varrho^2)^2 + 4ab}\right]$$

$$\theta_2 = \tfrac{1}{2}\left[a\varrho^2 + \sqrt{(a\varrho^2)^2 + 4ab}\right]$$

$$\alpha_{12}(t) = -2\int_t^T \exp\left[-\int_t^s\left(\frac{\alpha_{11}(\tau)}{a}-\alpha\right)d\tau\right]\alpha_{11}(s)\,ds \tag{15}$$

$$\alpha_{22}(t) = -\int_t^T \exp\left[\int_t^s(2\alpha+\sigma_2^2)\,d\tau\right]\left[\alpha_{12}(s)+\frac{\alpha_{12}^2(s)}{4a}\right]ds \tag{16}$$

$$\beta_1(t) = -2cz_T\exp\left[-\int_t^T\frac{\alpha_{11}(s)}{a}ds\right]-2\int_t^T\exp\left[\int_t^s\frac{\alpha_{11}(s)}{a}ds\right]\cdot$$

$$\cdot[bz_s-\alpha_{11}(s)\,v_s-\varrho\sigma_1\alpha_{11}(s)]\,ds \tag{17}$$

$$\beta_2(t) = -\int_t^T \exp\left(\int_t^s\alpha\,d\tau\right)\left\{\beta_1(s)\left(1+\frac{\alpha_{12}(s)}{2a}\right)-\beta_1(s)\,v(s)\right\}ds \tag{18}$$

$$\gamma(t) = cz_T^2-\int_t^T\left[\frac{\beta_1^2(s)}{4a}-\sigma_1^2\alpha_{11}(s)-bz_s^2-\beta_1(s)\,v(s)\right]ds. \tag{19}$$

Thus the optimal control may be expressed using (14)–(19) in the form

$$u_{\text{opt}} = -\frac{W_x}{2a}+v_t = -\frac{1}{2a}(2\alpha_{11}x+\alpha_{12}y+\beta_1)+v_t =$$

$$= -\frac{\alpha_{11}(t)}{a}x-\frac{\alpha_{12}(t)}{2a}y-\frac{\beta_1(t)}{2a}+v_t.$$

4. Remarks

The production rate u in the model presented in this paper was not restricted to be nonnegative. The inventory level was allowed to be negative, i.e. backlogging of demand was permitted. The stochastic production planning problem with constant demand, considered by Sethi and Thompson [1], [2], was extended by Bensoussan, Sethi, Vickson and Derzko [3] to include the constraint that production rate must be nonnegative. Unfortunately, a closed-form solution for optimal feedback policy was not obtained explicitly. The authors showed that an optimal feedback solution exists for the problem, and this solution was characterized. A policy iteration procedure was used to obtain computational solutions to the related problems with upper bounds on the production rate. As the stochastic production – inventory control problem with stochastic demand, presented in this paper is much more complicated than that considered in [3], it goes without saying that obtaining a closed-form solution for optimal feedback production policy for our problem with added production rate nonnegativity constraint, is a very difficult task. It seems that for the time being, the solution to such a problem has to be sought by means of numerical methods.

References

[1] SETHI S. P., THOMPSON G. L. Simple models in stochastic production planning. In: Applied Stochastic Control in Econometrics and Management Science, A. Bensoussan, P. Kleindorfer and C. Tapiero, eds. New York, North-Holland, 1981, pp. 295–304.

[2] SETHI S. P., THOMPSON G. L. Optimal Control theory. Applications to management science. Boston, Martinus Nijhoff Pub. 1981.

[3] BENSOUSSAN A., SETHI S. P., VICKSON R., DERZKO N. Stochastic production planning with production constrains. *SIAM J. Control and Optimization*, **22** (1984) 6.

[4] ALBOUY M. La régulation économique dans l'entreprise. Paris, Dunod, 1972.

[5] AOKI M. Optimal control and system theory in dynamic economic analysis. North-Holland 1976.

[6] MERTON R. C. Optimal consumption and portfolio rules in a continuous–time model. *J. Economic Theory* **3** (1971), 373–413.

[7] LE BRETON A., MUSIELA M. A study of one-dimensional bilinear differential model for stochastic processes. *Probability and Mathematical Statistics*, **4** (1984) Fasc. 1, 91–107.

[8] FLEMING W. H., RISHEL R. W. Deterministic and stochastic optimal control. N. York, Springer–Verlag, 1975.

Received, August 1985.

Stochastyczne zadanie sterowania systemem produkcja–magazyn z losowym popytem

W pracy rozpatrzono stochastyczne zadanie sterowania produkcją ze skończonym horyzontem sterowania i popytem opisanym przez liniowe stochastyczne równanie różniczkowe. Podano analityczne rozwiązanie tego zadania w postaci optymalnego sterowania ze sprzężeniem zwrotnym.

Стохастическая задача управления системой производство–склад со случайным спросом

В работе рассмотрена стохастическая задача управления производством с конечным горизонтом управления и спросом, описываемым линейным стохастическим дифференциальным уравнением. Дается аналитическое решение этой задачи в виде оптимального управления с обратной связью.