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Computation of the time-optimal control for some linear systems

by

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The paper deals with the method of time-optimal control computation for a linear, time-invariant undisturbed. system, whose state-matrix is simple and has only real negative eigenvalues. The fundamental matrix was found according to [3] and the proposed procedure is based on the minimization of the norm representing the distance between the desired final state and the state at the end of the last switching instant.

Introduction

Let us consider the linear, time-invariant system described by the state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where x, u are respectively n-dimensional state and r-dimensional control vector, A, B are constant matrices with coresponding dimensions, t is the time.

We assume that:

— the state is unconstrained

$$x(t) \in R^n \tag{2}$$

— on the control vector components the following inequality constraints are imposed

$$|u_k(t)| \leqslant U_{k\max} \quad k = 1, 2, \ldots, r, \tag{3}$$

— the state matrix A is simple and all its eigenvalues are real negative

$$\begin{array}{l} \operatorname{Re} s_l < 0 \\ \operatorname{Im} s_l = 0 \end{array} \quad l = 1, 2, \ldots, n.$$

— the system (1) satisfies the condition of the normal time-optimal control

$$\det [b_k : Ab_k : A^2 b_k : \ldots\ldots : A^{n-l} b_k] \neq 0, \quad \forall k = 1, 2, \ldots, r \tag{5}$$

We have to find the time-optimal control $u^*(t)$ satisfying the constraints (3), which transfers the system from the initial state x_0 at $t = t_0$ to the given final state x_f at $t = t_f$, while minimizing the performance index

$$I = \int_{t_0}^{t_f} dt = \text{minimum}. \tag{6}$$

The state-and the costate vectors $x^*(t)$, $\lambda^*(t)$ corresponding to $u^*(t)$ must satisfy the canonical equations

$$\left.\begin{array}{l} \dot{x}^*(t) = Ax^*(t) + Bu^*(t) \\ \dot{\lambda}^*(t) = -A^T \lambda^*(t) \end{array}\right\}, \tag{7}$$

and it is known, that the time-optimal control for the considered system is of the bang-bang type

$$u_k^*(t) = -U_k \operatorname{sign}(\lambda^*(t)\, b_k), \tag{8}$$

and that the number of switching intervals is at most n.

With the exception of some low-order systems, where it is possible to find the analytic expressions for $u^*(t)$, the numerical methods must be applied in order to solve the above time-optimal problem. Several computational procedures, based on the numerical solution of the set of $2n$ ordinary differential equations obtained from (7), have been developed. The difficulties in applications of these procedures are due to the fact, that the adjoint system is unstable (if the primary system is stable) and that the initial and final conditions x_0, x_f are given for the state variables. Hence the problem arises, whether it would be possible to apply the procedure of determining the switching instants based on the known solution of the equation (1)

$$x(t) = \Phi(t - t_0)\, x(t_0) + \int_{t_0}^{t} \Phi(t - \tau)\, Bu(\tau)\, d\tau, \tag{9}$$

where $\Phi(t)$ denotes the fundamental matrix

$$\Phi(t) = e^{At}. \tag{10}$$

Unfortunately the evaluation of the fundamental matrix by known methods becomes tedious in the case where the state matrix A is of higher dimension. On the other hand the computation of the fundamental matrix poses some essential problems shown by Moler and Van Loon [2] and by Laub [1]. But for the considered system with a simple state matrix A (i.e. having only distinct eigenvalues) the fundamental matrix can be found by the straightforward computation-according to the results presented in [3]. That enables to formulate the procedure of determining the switching instants, where the state vector is computed according to the formula (9).

Computation of the state-equation solution

It was proved in [3] that in the case where the state matrix A is simple, the fundamental matrix of the system (1) can be expressed by the formula

$$\Phi(t) = \left[[f_{i1l}] \cdot \begin{bmatrix} e^{s_1 t} \\ e^{s_2 t} \\ \vdots \\ e^{s_n t} \end{bmatrix} \;\vdots\; [f_{i2l}] \cdot \begin{bmatrix} e^{s_1 t} \\ e^{s_2 t} \\ \vdots \\ e^{s_n t} \end{bmatrix} \;\vdots\; \cdots \;\vdots\; [f_{inl}] \cdot \begin{bmatrix} e^{s_1 t} \\ e^{s_2 t} \\ \vdots \\ e^{s_n t} \end{bmatrix} \right] \tag{11}$$

$$i = 1, 2, \ldots, n, \quad l = 1, 2, \ldots, n,$$

with

$$f_{ijl} = \frac{p_{il} \operatorname{cof} p_{jl}}{\det P}, \tag{12}$$

$$i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n, \quad l = 1, 2, \ldots, n$$

where P is the nonsingular modal matrix whose columns $p_1, p_2, \ldots, p_n$ are eigenvectors of A

$$P = [p_1\; p_2 \cdots p_n]. \tag{13}$$

Hence with

$$p_j = [p_{1j}\; p_{2j} \cdots p_{nj}]^T \tag{14}$$

we have

$$P = [p_{ij}], \tag{15}$$

$$i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n$$

We find the coefficients f_{ijl} from (12) and that enables us to compute — with desired accuracy — the trajectory $x(t)$ in the general case, where all the control vector's components are functions of bounded variation on any bounded interval of time. We choose the computational interval Δt and at the sampling times

$$t_1 = \Delta t, t_2 = t_1 + \Delta t, \ldots, t_\mu = t_{\mu-1} + \Delta t,$$

replace the control components $u_q(t)$, $q = 1, 2, ..., r$ by constant functions on the particular intervals and find $x(t)$ at $t_1, t_2, ...$ from the formula in [5]

$$x(t_\mu) = \Phi(\Delta t)\, x(t_{\mu-1}) + D(\Delta t)\, u(t_{\mu-1}), \tag{16}$$

where

$$D(\Delta t) = \int_0^{\Delta t} \Phi(\vartheta)\, B\, d\vartheta, \tag{17}$$

and the elements of the matrix $D(\Delta t)$ are

$$d_{iq}(\Delta t) = \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{f_{ijl}}{s_l} (e^{s_l \Delta t} - 1)\, b_{jq}$$
$$i = 1, 2, ..., n, \qquad q = 1, 2, ..., r$$

In the case of the bang-bang control the particular control components $u_q(t)$, $q = 1, 2, ..., r$ are constant on the intervals between the switching instants. That simplifies the computation of $x(t)$ because the sampling intervals Δt can be chosen equal to the corresponding switching intervals.

Time-optimal control of the single-input system

In the single-input system the control is a scalar $u(t)$ and with $U_{k\max} = U_{\max}$ the constraint (3) reduces to

$$|u(t)| \leqslant U_{\max}. \tag{19}$$

We assume that the number of switching intervals is equal to n and denote by $t_{\mathrm{I}}^*, t_{\mathrm{II}}^*, ..., t_n^*$ the switching instants of the time-optimal bang-bang control, given by

$$u^*(t) = \begin{cases} 0, & \forall t < t_0 \\ \sigma U_{\max}, & \forall t \in [t_0^+, t_{\mathrm{I}}^*) \\ -\sigma U_{\max}, & \forall t \in [t_{\mathrm{I}}^*, t_{\mathrm{II}}^*) \\ \vdots & \vdots \\ (-1)^{n-1} \sigma U_{\max}, & \forall t \in [t_{n-1}^*, t_n^*) \\ 0, & \forall t \geqslant t_n^* \end{cases} \tag{20}$$

In the formula (20) is σ the sign of $u^*(t)$ in the first interval $[t_0^+, t_{\mathrm{I}}^*)$. According to (16) the state vector at switching instants will be

$$\left.\begin{aligned} x^*(t_{\mathrm{I}}^*) &= \Phi(\Delta t_{\mathrm{I}}^*)\, x(t_0) + D(\Delta t_{\mathrm{I}}^*)\, \sigma U_{\max} \\ x^*(t_{\mathrm{II}}^*) &= \Phi(\Delta t_{\mathrm{II}}^*)\, x^*(t_{\mathrm{I}}^*) + D(\Delta t_{\mathrm{II}}^*)\, (-1)\, \sigma U_{\max} \\ &\vdots \\ x^*(t_n^*) &= \Phi(\Delta t_n^*)\, x^*(t_{n-1}^*) + D(\Delta t_n^*)\, (-1)^{n-1} \sigma U_{\max} \end{aligned}\right\} \tag{21}$$

where

$$\left.\begin{aligned} \Delta t_{\mathrm{I}}^* &= t_{\mathrm{I}}^* - t_0 \\ \Delta t_{\mathrm{II}}^* &= t_{\mathrm{II}}^* - t_{\mathrm{I}}^* \\ \vdots\quad & \quad\vdots \quad \vdots \\ \Delta t_n^* &= t_n^* - t_{n-1}^* \end{aligned}\right\} \tag{22}$$

At $t = t_n^*$ the time-optimally controlled system arrives to the given final state $x^*(t_n^*) = x_f$. It means, that the euclidean norm in R^n representing the distance of the state $x^*(t_n^*)$ from x_f

$$N(t_n^*) = \|x_f - x^*(t_n^*)\| = \sqrt{\sum_{i=1}^{n} \left(x_{fi} - x_i^*(t_n^*)\right)^2} \tag{23}$$

is at $t = t_n^*$ equal to zero.

But if the switching instants (or at least one of them) differ from the optimal ones, then generally $t_n \neq t_n^*$ and at $t = t_n$ will be $x(t_n) \neq x_f$ implying $\|x_f - x(t_n)\| \neq 0$. Hence the numerical procedure can be based on the minimization of the norm (23) for $t = t_n$ and must enable us to find the switching instants according to the imposed accuracy of the final result, defined by

$$N(t_n) = \|x_f - x(t_n)\| \leqslant \varepsilon, \tag{24}$$

where ε is a given sufficiently small positive number. The switching instants corresponding to the bang-bang control satisfying the condition (24) will be accepted as optimal:

$$t_g \cong t_g^*, \qquad g = \mathrm{I}, \mathrm{II}, .., n.$$

In order to obtain the above solution we proceed in the following way: First we have to find the unknown value of σ. In some cases it can be possible to evaluate σ directly for the given initial and final states — after considerations based on the system's properties. But generally we will fix σ definitely, comparing the results obtained for its possible values. For the first computation we can choose σ according to system's response on the constant input signal. With $u(t) = \sigma U_{\max} = \text{const}$ we find for both $\sigma = +l$ and $\sigma = -l$ the trajectories $x(t)$ starting from x_0. Then we accept the value of σ corresponding to that of the above two trajectories whose minimal distance $d_{\min}$ (at $t = t_d$) from the final state

$$d_{\min} = \min \|x_f - x(t_d)\| \tag{25}$$

is smaller.

We choose approximatively the initial switching instants. It can be done arbitrarily or from the formulae

$$\left.\begin{array}{l} t'_n = \alpha t_d \\ t'_{\mathrm{I}} = \beta t_n \\ t'_{\mathrm{II}} = t'_{\mathrm{I}} + \beta\,(t'_n - t'_{\mathrm{I}}) \\ t'_{\mathrm{III}} = t'_{\mathrm{II}} + \beta\,(t'_n - t'_{\mathrm{II}}) \\ \vdots \quad \vdots \quad \vdots \\ t'_j = t'_{j-1} + \beta\,(t'_n - t'_{j-1}) \\ \quad 2 \leqslant j \leqslant n-1 \end{array}\right\} \tag{26}$$

where α and β-constant coefficients, which could be chosen $\alpha \in [0, 8, 1, 8]$, $\beta \in [0, 4, 0, 8]$.
For the bang-bang control with above switching instants we find the state vector and the norm (23) at $t = t'_n$. If $N(t'_n) > \varepsilon$ we apply the computational procedure minimizing this norm as function of switching instants--arriving finally to the result satisfying the condition (24) for the imposed value of ε. Next we compute (applying the analogous procedure) the switching instants satisfying the condition (24), in the case of the opposite value of σ. Comparing the obtained results we fix definitely the right σ.

REMARK. The above procedure was presented under assumption that the number of switching intervals is equal to n. If in some particular cases this number is $\nu < n$, we will find it in the final result of the computation.

Time-optimal control of the multi-input system

According to the assumption (5) the considered multi-input system is controllable with respect to each of the control vector's components. Hence for all particular r single-input systems we can apply the above described procedure and obtain the switching instants corresponding to imposed values ε_k, $k = 1, 2, \ldots, r$ in the conditions-like (24) — of desired accuracy. Because these particular results are needed for the approximative choice of initial data for further computation we can fix the values of ε_k appropriately bigger than ε — given for the multiple input system.
Next we find the optimal bang-bang control in the case where two input signals are active (e.g. u_a, u_b — corresponding to shortest final times t_{na}, t_{nb} with $t_{na} < t_{nb}$). For this computation the final switching time — the same for the both control signals — can be chosen equal to γt_{na}, where γ is a constant coefficient, which could be put $\gamma \in [0{,}6, 0{,}9]$. The other switching instants can be approximately evaluated, according to the relations obtained for the considered single-input systems. The value ε_{ab} representing the desired accuracy for this two-input system can be fixed bigger than ε. The results obtained by minimization of the norm $\|x_f - x(t_n)\|$ as function of two switching instants sets enable us to choose the initial data for the

computation in the case where three input signals are active. According to the consecutive results we proceed analogously in the cases where the number of active control signals will be increased-up to r, finding finally the switching instants for our multiple-input system with the accuracy corresponding to ε in the condition (24).

Example

The system described by the state-equation (1), where

$$A = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & -4 & 3 & 3 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 2 & 4 & 1 \\ 5 & 1 & 3 \end{bmatrix}$$

has to be transferred in the shortest time from the initial state at $t_0 = 0$

$$x_0 = [20 \ \ -10 \quad 40 \ \ -30]^T,$$

to the final state $x_f = 0$. On the control components the constraints of the form (3) are imposed, with $U_{1\max} = 1{,}5$, $U_{2\max} = 7$, $U_{3\max} = 8$. The final accuracy is given by the value $\varepsilon = 0.1 \cdot 10^{-2}$ in (24). With eigenvalues $s_1 = -1$, $s_2 = -2$, $s_3 = -3$, $s_4 = -4$ of the state matrix A we find from (11)

$$\Phi(t) = \left[\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-2t} \\ e^{-3t} \\ e^{-4t} \end{bmatrix} \middle| \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-2t} \\ e^{-3t} \\ e^{-4t} \end{bmatrix} \middle| \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-2t} \\ e^{-3t} \\ e^{-4t} \end{bmatrix} \middle| \begin{bmatrix} 2 & -2 & 0 & 0 \\ 0 & 1{,}5 & 0 & -1{,}5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-2t} \\ e^{-3t} \\ e^{-4t} \end{bmatrix} \right]$$

or

$$\Phi(t) = \left[\begin{array}{c|c|c|c} e^{-t} & 0 & 0 & 2e^{-t} - 2e^{-2t} \\ 0 & e^{-4t} & 3e^{-3t} - 3e^{-4t} & 1{,}5e^{-2t} - 1{,}5e^{-4t} \\ 0 & 0 & e^{-3t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{array} \right]$$

For the three single-input systems with $u_k = \sigma_k U_{k\max} = \text{const}$, $k = 1, 2, 3$ and both $\sigma_k = +1$, $\sigma_k = -1$ we find the trajectories $x(t)$ and the values of $d_{k\min}$ and $t_{dk\min}$

Table I

u_k	σ_k	$d_{k\min}$	$t_{dk\min}$
$u_1 = \sigma_1 U_{1\max}$	$\sigma_1 = +1$	3.58	1.45
$u_2 = 0,\ u_3 = 0$	$\sigma_1 = -1$	9.165	
$u_2 = \sigma_2 U_{2\max}$	$\sigma_2 = +1$	15.54	0.9
$u_1 = 0,\ u_3 = 0$	$\sigma_2 = -1$	19.96	0.35
$u_3 = \sigma_3 U_{3\max}$	$\sigma_3 = +1$	11.47	0.7
$u_1 = 0,\ u_2 = 0$	$\sigma_3 = -1$	22.72	0.50

For the above single-input systems the initial sets of switching instants were chosen according to (26) with $\alpha = 1.5,\ \beta = 0.6$

Table II

$u_k(t)$	t'_{I}	t'_{II}	t'_{III}	t'_{IV}
$u_1 = +1{,}5,\ \sigma_1 = 1$ $u_2 = u_3 = 0$	1.305	1.827	2.036	2.175
$u_2 = \pm 7,\ \sigma_2 = 1$ $u_1 = u_3 = 0$	0.810	1.134	1.264	1.350
$u_3 = \pm 8,\ \sigma_3 = 1$ $u_1 = u_2 = 0$	0.630	0.882	0.983	1.050

Minimizing the norm N from (24) with $\varepsilon_1 = 0.2$ as function of switching instants we find for the three single-input systems the following results:
For $u_1(t) = \pm 1.5,\ \sigma_1 = 1,\ u_2 = u_3 = 0$:
$t_{\mathrm{I}1} = 1.128440,\ t_{\mathrm{II}1} = 2.244316,\ t_{\mathrm{III}1} = 2.547527,\ t_{\mathrm{IV}1} = 2.653167,\ x_1(t_{\mathrm{IV}1}) = -0.7252744 \cdot 10^{-1},\ x_2(t_{\mathrm{IV}1}) = 0.1518576,\ x_3(t_{\mathrm{IV}1}) = 0.2894551 \cdot 10^{-2},\ x_4(t_{\mathrm{IV}1}) = -0.1294738 \cdot 10^{-1},\ N(t_{\mathrm{IV}1}) = 0.1688105.$
For $u_2(t) = \pm 7,\ \sigma_2 = 1,\ u_1 = u_3 = 0$:
$t_{\mathrm{I}2} = 1.328270,\ t_{\mathrm{II}2} = 2.013882,\ t_{\mathrm{III}2} = 2.371399,\ t_{\mathrm{IV}2} = 2.482855,\ x_1(t_{\mathrm{IV}2}) = -0.2380848 \cdot 10^{-1},\ x_2(t_{\mathrm{IV}2}) = 0.5296028 \cdot 10^{-1},\ x_3(t_{\mathrm{IV}2}) = 0.5902600 \cdot 10^{-1},\ x_4(t_{\mathrm{IV}2}) = -0.17698108,\ N(t_{\mathrm{IV}2}) = 0.1953283.$
For $u_3(t) = \pm 8,\ \sigma_3 = 1,\ u_1 = u_2 = 0$:
$t_{\mathrm{I}3} = 1.036907,\ t_{\mathrm{II}3} = 1.618649,\ t_{\mathrm{III}3} = 1.939305,\ t_{\mathrm{IV}3} = 2.036321,\ x_1(t_{\mathrm{IV}3}) = -0.3559242 \cdot 10^{-1},\ x_2(t_{\mathrm{IV}3}) = 0.6684756 \cdot 10^{-1},\ x_3(t_{\mathrm{IV}3}) = 0.1454541,\ x_4(t_{\mathrm{IV}3}) = -0.1057484,\ N(t_{\mathrm{IV}3}) = 0.1951283.$
In order to check the choice of particular σ we compute the switching instants for the above three single-input systems with opposite values of σ_1, σ_2, σ_3.
For $u_1 = \pm 1.5,\ \sigma_1 = -1,\ u_2 = u_3 = 0$:
$t_{\mathrm{I}1} = 1.080000,\ t_{\mathrm{II}1} = 2.481783,\ t_{\mathrm{III}1} = 2.945350,\ t_{\mathrm{IV}1} = 3.085350,\ x_1(t_{\mathrm{IV}1}) =$

$= -0.1237813$, $x_2(t_{IV1}) = 0.5512518 \cdot 10^{-1}$, $x_3(t_{IV1}) = 0.1202387 \cdot 10^{-1}$, $x_4(t_{IV1}) = 0.1337598$, $N(t_{IV1}) = 0.1907796$.

For $u_2 = \pm 7$, $\sigma_2 = -1$, $u_1 = u_3 = 0$:

$t_{I2} = 1.302201$, $t_{II2} = 2.423386$, $t_{III2} = 2.837964$, $t_{IV2} = 2.955946$, $x_1(t_{IV2}) = 0.1222067$, $x_2(t_{IV2}) = 0.3676271 \cdot 10^{-1}$, $x_3(t_{IV2}) = -0.1146388$, $x_4(t_{IV2}) = 0.5634868 \cdot 10^{-1}$, $N(t_{IV2}) = 0.1805636$.

For $u_3 = \pm 8$, $\sigma_3 = -1$, $u_1 = u_2 = 0$:

$t_{I3} = 0.7527493$, $t_{II3} = 1.851073$, $t_{III3} = 2.317695$, $t_{IV3} = 2.459695$, $x_1(t_{IV3}) = -0.2518293 \cdot 10^{-1}$, $x_2(t_{IV3}) = -0.1552670$, $x_3(t_{IV3}) = 0.3724819 \cdot 10^{-1}$, $x_4(t_{IV3}) = 0.1173010$, $N(t_{IV3}) = 0.1997222$.

We compare the obtained results and conclude, that the right values are $\sigma_1 = 1$, $\sigma_2 = 1$, $\sigma_3 = 1$.

The two shortest final times are t_{IV3} and t_{IV2}. We choose with $\gamma = 0{,}8$ the initial sets of switching instants in the case where both control signals u_2, u_3 are active and $u_1 = 0$:

$$t'_{IV23} = 0.8 \cdot 2.036321 \cong 1.629, \quad t'_{I3} = 1.629 \, \frac{1.036907}{2.036321} \cong 0.83,$$

$$t'_{I2} = 1.629 \, \frac{1.328270}{2.482855} \cong 0.871 \text{ and analogously we find}$$

$t'_{II3} \cong 1.295$, $t_{III3} \cong 1.551$, $t_{II2} \cong 1.321$, $t_{III2} \cong 1.556$.

The results obtained for $\varepsilon_{23} = 0.1$ and $u_2(t) = \pm 7$, $\sigma_2 = 1$, $u_3(t) = \pm 8$, $\sigma_3 = 1$, $u_1 = 0$, are:

$t_{I2} = 0.6975282$, $t_{II2} = 1.222194$, $t_{III2} = 1.435299$, $t_{I3} = 0.7643130$, $t_{II3} = 1.192512$, $t_{III3} = 1.457720$, $t_{IV2} = t_{IV3} = t_{IV23} = 1.511214$,

$x_1(t_{IV23}) = 0.1194000 \cdot 10^{-2}$, $x_2(t_{IV23}) = -0.5240560 \cdot 10^{-2}$, $x_3(t_{IV23}) = 0.7260203 \cdot 10^{-2}$, $x_4(t_{IV23}) = 0.1238382 \cdot 10^{-1}$, $N(t_{IV23}) = 0.1532836 \cdot 10^{-1}$.

For the last computation in the case where all control signals are active we choose — with $\gamma = 0.8$ — the set of initial switching instants:

$$t'_{IV1} = t'_{IV2} = t'_{IV3} = t'_{IV123} = 0.8 \cdot 1.511214 \cong 1.209, \quad t'_{I1} = 1.209 \, \frac{1.128440}{2.653167} \cong 0.514,$$

$$t'_{I2} = 1.209 \, \frac{0.6975282}{1.511214} \cong 0.558, \quad t'_{I3} = 1.209 \, \frac{0.7643130}{1.511214} \cong 0.611,$$

and analogously we find

$t'_{II1} \cong 1.023$, $t'_{III1} \cong 1.161$, $t'_{II2} \cong 0.978$, $t'_{III2} \cong 1.148$, $t'_{II} \cong 0.954$, $t'_{III3} \cong 1.166$.

The results obtained for $\varepsilon = 0.01$ and $u_1(t) = \pm 1.5$, $\sigma_1 = 1$, $u_2(t) = \pm 7$, $\sigma_2 = 1$, $u_3(t) = \pm 8$, $\sigma_3 = 1$ are:

$t^*_{I1} = 0.5590975$, $t^*_{II1} = 1.10712$, $t^*_{III1} = 1.347534$, $t^*_{I2} = 0.6151865$. $t^*_{II2} = 1.126013$, $t^*_{III} = 1.316871$, $t^*_{I3} = 0.7521544$, $t^*_{II3} = 1.110476$, $t^*_{III} = 1.351378$, $t^*_{IV1} = t^*_{IV2} = t^*_{IV3} = t^*_{IV} = 1.389023$,

$x^*_1(t^*_{IV}) = -0.3997028 \cdot 10^{-2}$, $x^*_2(t^*_{IV}) = -0.6423354 \cdot 10^{-2}$, $x^*_3(t^*_{IV}) = -0.2423406 \cdot 10^{-2}$, $x^*_4(t^*_{IV}) = -0.1702905 \cdot 10^{-2}$; $N(t^*_{IV}) = 0.8124561 \cdot 10^{-2}$.

Conclusive remarks

The presented numerical procedure is easily implementable and enables to find the time-optimal control for linear time-invariant systems in the case where all eigenvalues of the state matrix A are real negative and the fundamental matrix $\Phi(t)$ is known. By appropriate choices of initial data and of imposed accuracy for consecutive steps the cost and duration of the computation can be reduced.
For systems, whose dimension of the state vector is high, the evaluation of $\Phi(t)$ poses essential problems. But if we confine ourselves to the typical case where the matrix A is simple, we can apply for a given real system the method from (3) and compute the elements of the fundamental matrix effectively.

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Numeryczna metoda wyznaczania sterowania czasowo-optymalnego dla pewnych układów liniowych

W pracy podano metodę numeryczną wyznaczania sterowania czasowo-optymalnego dla inwariantnego w czasie i nie poddanego działaniu zakłóceń zewnętrznych układu liniowego, którego macierz stanu ma wszystkie wartości własne jednokrotne, rzeczywiste ujemne. Macierz podstawową znaleziono w sposób podany w [3]. Proponowana metoda opiera się na minimizacji normy reprezentującej odległość między zadanym stanem końcowym a stanem, jaki układ osiąga po zakończeniu procesu przełączania sterowania.

Численный метод решения задачи оптимального по быстродействию управления для некоторых линейных систем

В статье представлен метод вычисления оптимального по быстродействию управления инвариантной по времени линейной системы, на которую не воздействуют внешние возмущения. Все собственные значения матрицы состояния этой системы являются однократными и отрицательными. Базовая матрица определяется согласно методу представленному в [3]. Предлагаемая вычислительная процедура основана на минимизации нормы, характеризирующей расстояние между заданной целью и состоянием системы, после завершения процесса переключения управления.