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A free boundary problem in combustion of solid material

by

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A model for burning of a solid material in contact with gaseous oxidizer is considered. Oxidation is assumed to be concentrated on the surface of the reacting solid, driven by a vigorous Arrhenius surface reaction. The reaction contributes to a surface recession. Hence, a free boundary problem arises because of the dependence of the depletion rate of reaction surface upon the reaction itself rather than upon the temperature profile and oxidizer distribution. The model formulated covers diffusive phenomena of the coupled heat and mass transfer. One-dimensional geometry of the process is assumed.

Introduction

This paper is concerned with a model for the burning of a solid material in contact with a gaseous oxidizer. In the literature one can find a variety of models describing such a phenomenon, which give rise to one or more driving mechanisms of the burning process. In fact, depending on the type of materials one deals with, the process can be described with reasonable accuracy by one of the most popular approximations (internal gasification, surface oxidation or gas-phase reaction), see [2] and references therein.

Here we are interested in a model where the oxidation is assumed to take place on the surface of a solid material, driven by a vigorous Arrhenius surface reaction. The effect of the reaction is then a surface recession. This gives rise to a free boundary problem, since the depletion rate of this surface depends on the reaction itself and cannot be prescribed independently of the temperature profile and the oxidizer amount. Moreover we must specify some other characteristics of the process regarding the mechanisms of heat and mass transport in both the solid and the surrounding gas. We assume that both the heat and the mass transport (for the oxidizer concentration) in the surrounding gas obey to a diffusion equation. The solid is assumed to be impervious to the gas. The heat conduction inside the solid is assumed to be much faster than in the gas, so that we can assume that the temperature is everywhere the same inside the solid and equal to the surface temperature. Some mechanism of heat exchange on the solid boundary other than the reaction surface has to be assumed, for instance an adiabatic condition.

We describe the equations in an idealized situation. The geometry of the problem is the simplest one, namely a slab of solid of finite thickness, a half-space full of gas on one side of the slab, some simple heat exchange condition on the other side of the slab. The x-axis is directed normally to the surface of reaction. At time $t=0$, we assume that the origin $x=0$ and the surface of reaction coincide. The penetrating front is then located in $x=s(t)$ and moves toward the solid. $\theta(t)$ represents the temperature of the burning material, $v(x,t)$ represents the normalized temperature in the gas mixture and $u(x,t)$ a normalized concentration of oxidant.

Thus, the mathematical model can be stated as follows:

(P.) Find (T, u, v, s, θ) such that: $T>0$, $s \in C^1[0,T]$, $\theta \in C^1[0,t]$, $u \in C^{2,1}(D_T) \cap C^{1,0}(\bar{D}_T)$, $v \in C^{2,1}(D_T) \cap C^{1,0}(\bar{D}_T)$, where $D_T = \{(x,t): -\infty < x < s(t),\ 0<t<T\}$, and the following equations are satisfied:

$$v_t = v_{xx}, \qquad \text{in } D_T; \tag{1.1}$$

$$v(x,0) = \psi(x), \qquad -\infty < x < 0; \tag{1.2}$$

$$u_t = u_{xx}, \qquad \text{in } D_T; \tag{1.3}$$

$$u(x,0) = \varphi(x), \qquad -\infty < x < 0; \tag{1.4}$$

$$s(0) = 0; \tag{1.5}$$

$$v(s(t),t) = \theta(t), \qquad 0<t<T; \tag{1.6}$$

$$\alpha \cdot u_x(s(t),t) = -(\gamma + u(s(t),t)) \cdot \dot{s}(t) \tag{1.7}$$

$$\dot{\theta}(t) = h(v_x(s(t),t), s(t), \dot{s}(t)), \qquad 0<t<T; \tag{1.8}$$

$$\dot{s}(t) = f(u(s(t),t)) \cdot G(\theta(t)), \qquad 0<t<T; \tag{1.9}$$

where α and γ are positive constants and the data satisfy the assumptions:

$\psi \in C^1(-\infty, 0]$, and ψ' bounded; (H.1)

$\varphi \in C^2(-\infty, 0]$, φ bounded; (H.2)

f is a Lipschitz continuous[1] non decreasing function on $\mathbf{R}$, with $f(0)=0$; (H.3)

G is Lipschitz continuous w.r.t. θ for $\theta > 0$; (H.4)

h is Lipschitz continuous w.r.t. all its arguments. (H.5)

[1] f may not be Lipschitz continuous in 0 in some applications, e.g. $f(u) = cu^\alpha$, with $\alpha \in (0,1)$. This does not change our results, since we can bound u away from 0 on the free boundary by means of *a priori* estimate, see [4]. Here we omit this generalization for sake of simplicity.

Condition (1.9) is a generalization of the Arrhenius reaction law, the typical form of the function G being $G(\theta) = c_1 \exp(-c_2/\theta)$. Condition (1.8) is a generic form for the energy balance in the solid which can account for a variety of heat exchange mechanisms on the non reacting surface of the solid. (1.6) represents the continuity of the temperature on the reacting surface and (1.7) is the mass conservation on it. Note that the r.h.s. of (1.7) is the sum of the transported mass $u(s(t), t)\dot{s}$, and the burned oxidizer $\gamma\dot{s}$.

With this condition, and under our assumptions we can state an existence and uniqueness result:

THEOREM 1.1. *Problem (P.) has a unique local solution for any set of data satisfying assumption H.*

Let us sketch the proof of the Theorem. The first step is to transform the problem by introducing a new dependent variable

$$z(x, t) = -\int_x^{s(t)} \big(u(\xi, t) + \gamma\big)\, d\xi. \tag{1.10}$$

Then we construct a fixed point machinery for the free boundary $x = s(t)$ in the following way: start by fixing $s(t)$ in some Banach space, solve the problem for $z(x, t)$ in $-\infty < x < s(t)$, $t > 0$, and define the function $F(t) = f\big(z_x(s(t), t) - \gamma\big)$. Then solve an auxiliary free boundary problem (see next section), which is given by the "v's equations", and finally compare this free boundary with the previously fixed $s(t)$.

By means of (1.10), the problem transforms in the equivalent problem (P'), where equations (1.3), (1.4), (1.7), (1.9) are replaced by

$$z_t = z_{xx}, \qquad \text{in } D_T; \tag{1.3'}$$

$$z(x, 0) = \bar{\varphi}(x), \qquad -\infty < x < 0; \tag{1.4'}$$

$$z\big(s(t), t\big) = 0, \qquad 0 < t < T; \tag{1.7'}$$

$$\dot{s}(t) = f\big(z_x(s(t), t) - \gamma\big) \cdot G\big(\theta(t)\big), \qquad 0 < t < T; \tag{1.9'}$$

where

$$\bar{\varphi}(x) = -\int_x^0 \big(\varphi(\xi) + \gamma\big)\, d\xi. \tag{1.11}$$

We start our proof in the next section by investigating the "v problem".

2. An auxiliary problem

In this section we investigate the following free boundary problem:
(P.A.) Find (T, v, s, θ) such that: $T > 0$, $s \in C^1[0, T]$, $\theta \in C^1[0, t]$, $v \in C^{2,1}(D_T) \cap C^{1,0}(\bar{D}_T)$, where $D_T = \{(x, t)\colon -\infty < x < s(t), 0 < t < T\}$ and the

following equations are satisfied:

$$v_t = v_{xx}, \qquad \text{in } D_T; \tag{2.1}$$

$$v(x, 0) = \psi(x), \qquad -\infty < x < 0; \tag{2.2}$$

$$s(0) = 0; \tag{2.3}$$

$$v\big(s(t), t\big) = \theta(t), \qquad 0 < t < T; \tag{2.4}$$

$$\theta(t) = h\big(v_x(s(t), t), s(t), \dot{s}(t)\big), \qquad 0 < t < T; \tag{2.5}$$

$$\dot{s}(t) = F(t) \cdot G\big(\theta(t)\big), \qquad 0 < t < T, \tag{2.6}$$

where

$\psi \in C^1(-\infty, 0]$, and ψ' bounded; (A.1)

$F \in C^0\,[0, T']$ for a sufficiently large T'; (A.2)

G is Lipschitz continuous w.r.t. θ for $\theta > 0$; (A.3)

h is Lipschitz continuous w.r.t. all its arguments. (A.4)

THEOREM 2.1. *Problem (P.A.) has a unique local solution, continuously dependent on the data.*

Let us first give a sketch of the proof. We will use a fixed point argument for the function θ in a closed subset of $C^1\,[0, T]$. This goes as follows:

First define a set:

$$\mathscr{X}(T, B) = \{\theta \in C^1\,[0, T] : \|\theta - \theta_0\|_{C^1} \leqslant B, \text{ with } \theta_0 \equiv \psi(0), \theta(0) = \theta_0\}, \tag{2.7}$$

where T and B are constants to be fixed in the sequel; of course $\mathscr{X}$ is a closed subset of $C^1\,[0, T]$.

For any $\theta \in \mathscr{X}$ we define $s(t)$ by

$$s(t) = \psi(0) + \int_0^t F(\tau) \cdot G\big(\theta(\tau)\big)\, d\tau. \tag{2.8}$$

Next step is to solve the parabolic equation (2.1) with b.d. (2.2), (2.4) (with the fixed θ) in the domain D_T $\big(s$ is given by (2.8)$\big)$, namely

$$v_t = v_{xx}, \qquad -\infty < x < s(t), \qquad 0 < t < T; \tag{2.9}$$

$$v(x, 0) = \psi(x), \qquad -\infty < x < 0; \tag{2.10}$$

$$v\big(s(t), t\big) = \theta(t), \qquad 0 < t < T. \tag{2.11}$$

This is a classical moving (not "free") boundary problem, and can be transformed into an integral equation problem by means of the fundamental solution

$$\Gamma(x,t;\xi,\tau)=\frac{1}{2\sqrt{\pi(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right).$$

Finally we look for a fixed point in $\mathscr{X}$ of the transformation

$$\left[\begin{array}{l}\mathscr{M}:\theta\mapsto\tilde{\theta}\\ \tilde{\theta}(t)=\psi(0)+\int_0^t h\left(v_x(s(\tau),\tau),s(\tau),\dot{s}(\tau)\right)d\tau.\end{array}\right. \tag{2.12}$$

The first part of the proof consists in showing that $\mathscr{M}$ maps $\mathscr{X}$ into itself. We start with the integral representation of the solution of (2.9)–(2.11):

$$v(x,t)=\int_{-\infty}^{0}\Gamma(x,t;\xi,0)\psi(\xi)\,d\xi+\int_0^t\Gamma\left(x,t;s(\tau),\tau\right)\theta(\tau)\dot{s}(\tau)\,d\tau+$$

$$+\int_0^t\Gamma\left(x,t;s(\tau),\tau\right)v_\xi\left(s(\tau),\tau\right)d\tau-\int_0^t\Gamma_\xi\left(x,t;s(\tau),\tau\right)\theta(\tau)\,d\tau. \tag{2.13}$$

Equation (2.13) can be differentiated term by term w.r.t. x for $x<s(t)$. Moreover we can perform integration by parts to obtain

$$v_x(x,t)=\int_{-\infty}^{0}\Gamma(x,t;\xi,0)\psi'(\xi)\,d\xi+\int_0^t\Gamma_x\left(x,t;s(\tau),\tau\right)v_x\left(s(\tau),\tau\right)d\tau+$$

$$+\int_0^t\Gamma\left(x,t;s(\tau),\tau\right)\dot{\theta}(\tau)\,d\tau. \tag{2.14}$$

Now we can pass to the limit in (2.14) using the jump relation for Γ_x [F]:

$$\frac{1}{2}v_x\left(s(t),t\right)=\int_{-\infty}^{0}\Gamma\left(s(t),t;\xi,0\right)\psi'(\xi)\,d\xi+$$

$$+\int_0^t\Gamma_x\left(s(t),t;s(\tau),\tau\right)v_x\left(s(\tau),\tau\right)d\tau+\int_0^t\Gamma\left(s(t),t;s(\tau),\tau\right)\dot{\theta}(\tau)\,d\tau. \tag{2.15}$$

The first term of (2.15) is obviously bounded by a constant C_1 times $\|\psi'\|_{C^0}$, while the third term is bounded by a constant C_3 times B times $\sqrt{t}$.

The kernel in the second term of the sum can be bounded as well by some constant C_2 times $(t-\tau)^{-1/2}$, because of the Lipschitz continuity of the function $s(t)$ as defined by (2.8): in fact we have

$$\|\dot{s}\|_{C^0}\leqslant k(B,T)\sup_{t\in[0,T]}|F(t)|=K, \tag{2.16}$$

where $k(B,T)$ is the supremum of $|G|$ in the set of all possible values of θ, which depends on the parameter B and T[2].

Then from the integral representation (2.15) we get the inequality

$$|v_x(s(t), t)| \leqslant C_1 \|\psi'\|_{C^0} + C_3 B\sqrt{t} + C_2 \int_0^t \frac{|v_x(s(\tau), \tau)|}{\sqrt{(t-\tau)}}\, d\tau. \tag{2.17}$$

We can now apply Abel's integral techniques of [1] to obtain

$$|v_x(s(t), t)| \leqslant (C_1 \|\psi'\| + C_3 B\sqrt{t}) \cdot (1 + 2C_2\sqrt{t}) \cdot \exp(\pi C_2 t) \leqslant \\ \leqslant C_4 (\|\psi'\| + B\sqrt{T}), \tag{2.18}$$

where C_4 depends on B an T, but remains bounded for B and T bounded.

Now we are in a position to prove that $\mathcal{M}(\mathcal{X}) \subset \mathcal{X}$: in fact,

$$|\hat{\theta}(t)| \leqslant |h(v_x(s(t), s(t), \dot{s}(t))) - h(v_x(0,0), 0, \dot{s}(0))| + \\ + |h(v_x(0,0), 0, \dot{s}(0))| \leqslant \mathrm{Lip}\,\{|v_x(s(t), t)| + |\psi'(0)| + |\dot{s}(t) - \dot{s}(0)|\} + \\ + |s(t)| + |h(\psi'(0), 0, F(0)\cdot G(\psi(0))| \leqslant \\ \leqslant \mathrm{Lip}\cdot\{C_5 + C_4(\|\psi'\| + B\sqrt{T})\} = \bar{C}(1 + B\sqrt{T}), \tag{2.19}$$

where Lip is the Lipschitz constant of h.

From (2.19) we get the first result by choosing B greater than $2\cdot\bar{C}$ and T less than $1/B^2$.

Let us now prove the contractive character of $\mathcal{M}$: to do this, let θ_1, θ_2 be functions in $\mathcal{X}$, and indicate with s_1, s_2 the corresponding boundaries defined via (2.8), v_1, v_2 the solutions of (2.9)-(2.11); then the difference of the corresponding transformed functions is given by:

$$(\mathcal{M}\theta_1)(t) - (\mathcal{M}\theta_2)(t) = \int_0^t \{h_1(\tau) - h_2(\tau)\}\, d\tau,$$

$$\text{where } h_i(t) = h(v_{ix}(s_i(t), t), s_i(t), \dot{s}_i(t)), \quad i = 1, 2.$$

Then

$$\left|\frac{d(\mathcal{M}\theta_1)}{dt}(t) - \frac{d(\mathcal{M}\theta_2)}{dt}(t)\right| \leqslant \mathrm{Lip}\,\{|v_{1x}(s_1(t), t) - v_{2x}(s_2(t), t)| + \\ + |s_1(t) - s_2(t)| + |\dot{s}_1(t) - \dot{s}_2(t)|\}. \tag{2.20}$$

The second term in the sum is dominated by t times the third term so we can neglect it for the moment. The third term can be estimated using (2.8), which gives

[2] Of course, $k(B, T)$ is increasing in B, but we can make it less than some fixed quantity, say k' restricting the maximal time T.

$$|\dot{s}_1(t)-\dot{s}_2(t)| \leqslant \text{Const.} \cdot \|\theta_1-\theta_2\|_{C^0} \leqslant \text{Const.} \cdot t \cdot \|\dot{\theta}_1-\dot{\theta}_2\|_{C^0}. \tag{2.21}$$

It remains to estimate $|v_{1x}(s_1(t), t)-v_{2x}(s_2(t), t)|$. To this aim we introduce the coordinate transformation $y = x-s(t)$ and define $V(y, t) = v(x, t)$, then we have

$$|v_{1x}(s_1(t), t)-v_{2x}(s_2(t), t)| = |V_{1x}(0, t)-V_{2x}(0, t)| = |w_x(0, t)|, \tag{2.22}$$

where $w(y, t) = V_1(y, t)-V_2(y, t)-(\theta_1(t)-\theta_2(t))$, and w solves the problem:

$$w_t = w_{yy}+\dot{s}_1 w_y+(\dot{s}_1-\dot{s}_2) V_{2y}-(\dot{\theta}_1(t)-\dot{\theta}_2(t)), \quad -\infty < y < 0, 0 < t < T; \tag{2.23}$$

$$w(y, 0) = 0, \quad -\infty < y < 0; \quad w(0, t) = 0, \quad 0 < t < T. \tag{2.24}$$

Solution of problem (2.23), (2.24) can be represented using the Green function for second quarter of the plane, which we indicate by $G(x, t; \xi, \tau)$. Then

$$w(x, t) = \int_0^t \int_{-\infty}^0 G(x, t; \xi, \tau)[\mathscr{F}(\xi, \tau)+\dot{s}_1(\tau) w_x(\xi, \tau)] \, d\xi \, d\tau, \tag{2.25}$$

where $\mathscr{F}$ indicates the source term $(\dot{s}_1-\dot{s}_2) V_{2y}-(\dot{\theta}_1(t)-\dot{\theta}_2(t))$ in eq. (2.23), and

$$w_x(x, t) = \int_0^t \int_{-\infty}^{+\infty} \Gamma_x(x, t; \xi, \tau) \overline{[\mathscr{F}(\xi, \tau)+\dot{s}_1(\tau) w_x(\xi, \tau)]} \, d\xi \, d\tau \tag{2.26}$$

where $\bar{f}(\xi)$ indicates the odd prolongation of $f(\xi)$ over $0 < \varepsilon$. Then

$$|w_x(x, t)| \leqslant \sqrt{\frac{2}{\pi}} \left\{ 2 \sup_{\xi, \tau} |\mathscr{F}(\xi, \tau)| \sqrt{t} + \|\dot{s}_1\|_{C^0} \int_0^t \frac{\sup_\xi |w_x(\xi, \tau)|}{\sqrt{t-\tau}} \, d\tau \right\}. \tag{2.27}$$

From inequality (2.27) we have, by the same techniques we used in inequality (2.17),

$$\sup_x |w_x(x, t)| \leqslant \text{Const.} \sqrt{t} \sup_{x, \tau} |\mathscr{F}(x, \tau)|. \tag{2.28}$$

It remains to estimate the term $\sup|\mathscr{F}|$ in (2.28). But, from the definition of $\mathscr{F}$ we have

$$|\mathscr{F}| \leqslant |\dot{s}_1-\dot{s}_2| \, |v_{2x}|+|\dot{\theta}_1-\dot{\theta}_2|, \tag{2.29}$$

$|v_{2x}|$ is bounded because of (2.18) and the maximum principle, moreover the difference $|\dot{s}_1-\dot{s}_2|$ is bounded from above by a constant times $|\dot{\theta}_1-\dot{\theta}_2|$ because of the definition (2.8) of $\dot{s}_i$.

All this computation can be resumed in the following inequality

$$|h_1(t)-h_2(t)| \leqslant \text{Const.} \cdot \sqrt{t} \, \|\theta_1-\theta_2\|_{C^1[0, T]}. \tag{2.30}$$

From (2.30) we get

$$\|\mathscr{M}\theta_1-\mathscr{M}\theta_2\|_{C^1[0, T]} \leqslant \lambda \, \|\theta_1-\theta_2\|_{C^1[0, T]}.$$

Finally we can choose T small enough in such a way that the constant λ is less than 1. This proves the local existence and uniqueness result.

The remaining part of this section is devoted to prove that the solution of (P.A.) depends continuously on $F(t)$. This fact will be crucial in proving the existence and uniqueness of a local solution to problem (P).

Suppose that F_1, F_2 are two continuous bounded functions, $|F_i| \leqslant H$, on $[0, T]$. Then we define by (v_i, θ_i, s_i), $i = 1, 2$, the corresponding solution to problem $(P.A.)$, with the same data. The computation done to prove (2.28) can be repeated, with no substantial change, to obtain (in the following $\|\cdot\|$ means C^0-norm)

$$\left|v_{1x}(s_1(t), t) - v_{2x}(s_2(t), t)\right| \leqslant \text{Const.} \sqrt{t}\, \{\|\dot{s}_1 - \dot{s}_2\| + \|\theta_1 - \theta_2\|\}. \tag{2.31}$$

Now the term $\|\theta_1 - \theta\|$ can be dominated because of the Lipschitz continuity of the function h in (2.5), this gives an inequality of the form

$$\|\theta_1 - \theta_2\| \leqslant \text{Const.} \left\{\left\|v_{1x}(s_1(\cdot), \cdot) - v_{2x}(s_2(\cdot), \cdot)\right\| + \|\dot{s}_1 - \dot{s}_2\| + \|s_1 - s_2\|\right\}.$$

The term $\left\|v_{1x}(s_1(t), t) - v_{2x}(s_2(t), t)\right\|$ is now replaced by means of (2.31), and the time T is chosen small enough in order to have

$$\|\theta_1 - \theta_2\| \leqslant \text{Const.} \|\dot{s}_1 - \dot{s}_2\|. \tag{2.32}$$

We repeat the same computation starting from equation (2.6), and, possibly after restricting the maximal time T, we obtain

$$\|\dot{s}_1 - \dot{s}_2\|_{C^0} \leqslant \text{Const.} \|F_1 - F_2\|_{C^0}, \tag{2.33}$$

and

$$\|\theta_1 - \theta_2\|_{C^0} \leqslant \text{Const.} \|F_1 - F_2\|_{C^0}. \tag{2.34}$$

From the last inequalities an uniform estimate for the difference $v_1 - v_2$ is easily obtained via the maximum principle. ■

3. Proof of Theorem 1.1

Let $\mathscr{D}(A, T) = \{s \in C^1[0, T] : \|s\|_{C^1} \leqslant A,\ s(0) = 0,\ \dot{s}(0) = v_0\}$, where A and T are constants to be determined. $\mathscr{D}$ is a closed subset of $C^1[0, T]$. We define a map $\mathscr{T} : \mathscr{D} \mapsto \mathscr{D}$ in the following way:

First solve the moving-boundary Cauchy-Dirichlet problem

$$z_t = z_{xx}, \qquad \text{in } D_T; \tag{3.1}$$

$$z(x, 0) = \bar{\varphi}(x), \qquad -\infty < x < 0; \tag{3.2}$$

$$z(s(t), t) = 0, \qquad 0 < t < T; \tag{3.3}$$

where s is any element of $\mathscr{D}$ and the datum $\bar{\varphi}$ is the one defined in

(1.11). Then use the $z_x(s(t), t)$ to define $F(t) = f(z_x(s(t), t) - \gamma)$. Finally solve the auxiliary problem $(P.A.)$ with this $F(t)$. Let $\mathscr{F}(s)$ be the free boundary in the solution of $(P.A.)$.

We are going to prove the following properties of $\mathscr{F}$:

(i) $\mathscr{F}$ maps $\mathscr{D}$ into itself, i.e. $\bar{s} = \mathscr{F}(s) \in \mathscr{D}$; (3.4)

(ii) $\mathscr{F}$ is a contractive map of $\mathscr{D}$. (3.5)

In order to prove property (i), we start with an estimate on z_x:

$$|z_x(s(t), t)| \leqslant \text{Const. } \|\varphi + \gamma\| \left(1 + \text{const. } A\sqrt{t}\right) \exp(\pi \text{ const. } At), \tag{3.6}$$

which follows from the integral representation of the solution of (3.1)–(3.3), using the same computation leading to inequality (2.18).

As a consequence of (3.6), we have that $F(t) = f(z_x(s(t), t) - \gamma)$ is uniformly bounded by a constant which does not depend on A, if the maximal time T is small enough. Then, because of (2.16), the solution $\bar{s}$ of the auxiliary problem is uniformly bounded in the C^1-norm, and the constant A can be chosen larger than this upper bound, thus proving (i).

To prove (ii), we have to compare $\|\bar{s}_1 - \bar{s}_2\|$ for any two given s_1, s_2 in $\mathscr{D}$. Let us denote by z_1 and z_2 the corresponding solutions to (3.1)–(3.3) and by F_1 and F_2 the corresponding functions $F_i(t) = f(z_{ix}(s(t), t) - \gamma)$. Then, because of inequality (2.33), we have

$$\|\dot{\bar{s}}_1 - \dot{\bar{s}}_2\|_{C^0} \leqslant \text{Const. } \|F_1 - F_2\|_{C^0}, \tag{3.7}$$

We now have to dominate the r.h.s. of (3.7) in terms of $\|s_1 - s_2\|_{C^1}$. To do this, we define $Z_i(y, t) = z_i(x - s_i(t), t)$ and $Z(y, t) = z_1(y, t) - z_2(y, t)$, then Z solves the problem

$$Z_t = Z_{xx} + \dot{s}_1 Z_x + Z_{2x}(\dot{s}_1 - \dot{s}_2), \tag{3.8}$$

with homogeneous initial and boundary conditions.

We can again use the integral representation technique, to obtain

$$|Z_x(x, t)| \leqslant \sqrt{\frac{2}{\pi}} \left\{ 2 \sup_{\xi, \tau} |Z_x(\xi, \tau)| \sqrt{t}\ \|\dot{s}_1 - \dot{s}_2\|_{C^0} + \|\dot{s}_1\|_{C^0} \int_0^t \frac{\sup_\xi |Z_x(\xi, \tau)|}{\sqrt{t - \tau}} d\tau \right\}, \tag{3.9}$$

which gives

$$\|z_{x1}(s_1(t), t) - z_{x2}(s_2(t), t)\| \leqslant \text{Const. } \sqrt{T}\ \|\dot{s}_1 - \dot{s}_2\|_{C^0}, \tag{3.10}$$

where the constant is bounded when T tends to zero.

Finally we use the Lipschitz continuity of the function $f(u)$ to conclude that

$$\|\dot{\bar{s}}_1 - \dot{\bar{s}}_2\|_{C^0} \leqslant \text{Const. } \sqrt{T}\ \|\dot{s}_1 - \dot{s}_2\|_{C^0}, \tag{3.11}$$

which proves our theorem. ∎

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Zagadnienie ze swobodną granicą w teorii spalania ciała stałego

W artykule rozważa się model spalania materiału stałego w kontakcie z gazowym utleniaczem. Zakłada się, że utlenianie następuje na powierzchni materiału i ma charakter burzliwej reakcji powierzchniowej Arrheniusa. W wyniku postępującej reakcji ma miejsce cofanie się jej frontu. Prowadzi to do modelu ze swobodną granicą, ponieważ prędkość przemieszczania się powierzchni reakcji zależy od całego jej przebiegu a nie tylko od rozkładu temperatury i ilości utleniacza. Przedstawiony model ma charakter dyfuzyjny i opisuje sprzężony transport ciepła i masy. Zakładany jest jednowymiarowy charakter geometryczny procesu.

Задача со свободной границей в теории сжигания твердого тела

В статье рассматривается модель сжигания твердого материала в контакте с газовым окислителем. Предполагается, что окисление происходит на поверхности материала и носит характер бурной поверхностной реакции Аррениуса. По ходу реакции имеет место отсупление её фронта. Это позволяет применить модель со свободной границей, поскольку скорость смещения поверхности реакции зависит от полного её прохождения, а не только от распределения температуры и количества окислителя. Представленная модель носит диффузионный характер и описывает сопряженный перенос тепла и массы. Предполагается одномерный геометрический вид процесса.