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Sufficient condition for ε-optimality

by

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Using an extension of Young's method from [4], vol. II, ch. II, we describe characterizations of an approximate solution in problems of control. The starting point is the Pontryagin ε-maximum principle derived by Ekeland in [1]. As consequences we obtain sufficient conditions for ε-optimality in a form similar to Weierstrass conditions from the calculus of variations.

1. Introduction

Consider the following problem:

$$\text{Minimize } g\left(x(1)\right), \tag{1}$$

over an attainable set $K(a) \subset R^n$ of trajectories satisfying in $[0, 1]$

$$\dot{x}(t) = f\left(t, x(t), u(t)\right) \quad \text{a.e. (almost everywhere)}, \tag{2}$$

$$u(t) \in U(t) \quad \text{a.e.}, \tag{3}$$

$$x(0) = a, \tag{4}$$

where a is any fixed point in R^n, $x:[0, 1] \to R^n$ an absolutely continuous function, $u:[0, 1] \to R^m$ a Lebesgue measurable function (control function), $U:[0, 1] \to R^m$ a multifunction (i.e. $U(t)$ is a subset of R^m for each t in $[0, 1]$), $f:[0, 1] \times R^n \times R^m \to R^n$, and $g:R^n \to (-\infty, +\infty)$. We assume that $K(a)$ is not empty and $\inf g\left(x(1)\right) > -\infty$ on the set $K(a)$.

It is clear that there may not exist $x_0(1) \in K(a)$ such that $\min g\left(x(1)\right) = g\left(x_0(1)\right)$. But, we know that, for any $\varepsilon > 0$, there exists $x_\varepsilon(1) \in K(a)$ such that

$$g(x_\varepsilon(1)) \leqslant \inf\{g(x(1)): x(1) \in K(a)\} + \varepsilon.$$

It is Ekeland who gave in [1] an existence theorem of an ε-approximate solution $x_\varepsilon(1)$ with the property of the Pontryagin ε-maximum principle which of course, is not a necessary condition of the ε-approximate solutions.

We present an opposite approach to this problem. We start with trajectories which satisfy the Pontryagin ε-maximum principle and show that, under certain additional hypotheses imposed upon them, such $x_\varepsilon(1) \in K(a)$ satisfies

$$g(x_\varepsilon(1)) \leqslant g(x(1)) + \varepsilon(1+l),$$

for all $x(1)$ from some subsets of $K(a)$, where l is the length of $x(t)$, $t \in [0, 1]$. To this effect, on the basis of Young's work [4] vol. II, ch. II, we describe generalizations of geodesic coverings and Hilbert's integral for that ε-maximum principle and, as their consequence, we give the mentioned characterization of a concrete $x_\varepsilon(1)$. This immediately implies sufficient conditions for an ε-optimality in a form similar to Weierstrass conditions from the calculus of variations. Of course, if $\varepsilon = 0$, we obtain sufficient conditions for a strong relative minimum, extending those of [2] and [3].

Let L be the collection of Lebesgue measurable subsets of $[0, 1]$ and let B Borel subsets of R^m. $L \times B$ denotes the σ-algebra of subsets of $[0, 1] \times R^m$, generated by products of sets in L and B. We shall suppose the following basic hypothesis:

(H1) For each s in R^n, the function $(t, u) \to f(t, s, u)$ is $L \times B$-measurable. For each t in $[0, 1]$, u in $U(t)$ the function $s \to f(t, s, u)$ is continuously differentiable. The set $\{(t, u) \in [0, 1] \times R^m : u \in U(t)\}$ is $L \times B$-measurable. The function g is locally Lipschitz in R^n.

If $u(t)$ is a control function subject to (3) and $x(t)$ is an absolutely continuous function corresponding by (2) to $u(t)$, and $g(x(1))$ is finite for those functions, then the pair $x(t)$, $u(t)$ will be called admissible and $x(t)$ an admissible trajectory.

Since we shall base ourselves on [4], we use most of its original notions.

I am grateful to the refree for remarks concerning the paper [1].

2. General notions

An admissible pair $x(t)$, $u(t)$ defined in the appropriate subinterval of $[0, 1]$ with right end at 1 will be termed a line of flight, briefly l.f., if it satisfies the following ε-maximum principle (compare [1]):

there exists along $x(t)$ a conjugate vector function $y(t)$ absolutely continuous in t, with values in R^n, such that $y(t)$ is nonvanishing and

$$-\dot{y}(t) = y(t) f_s(t, x(t), u(t)) \qquad \text{a.e.,} \qquad (5)$$

$$y(t) f(t, x(t), u(t)) \geqslant \sup \{y(t) f(t, x(t), u) : u \in U(t)\} - \varepsilon \qquad \text{a.e.,} \qquad (6)$$

$$(-y(1), -1) \text{ is a normal to epi } g \text{ at the point } (x(1), g(x(1))). \qquad (7)$$

(epi g means the epigraph of g, relation (7) is a general form of a transversality condition).

We term a canonical line of flight (briefly c.l.f.), a trio $x(t)$, $y(t)$, $u(t)$ of functions such that $x(t)$, $u(t)$ define l.f. and $y(t)$ is the corresponding conjugate function satisfying (5)–(7). In a usual way we define an open arc of l.f. or c.l.f.

In what follows, we shall take into consideration not all l.f. but only those which are subject to certain conditions imposed upon them.

Denote by $T \subset R^{n+1}$ a set covered by graphs of trajectories of l.f. which may, in the sequel, by reduced to a smaller one. For each point (t_0, x_0) in T, define $G(t_0, x_0)$ equal to $g(x^0(1))$ where $x^0(t)$ is a trajectory of c.l.f. $x^0(t)$, $y^0(t)$, $u^0(t)$ such that $x^0(t_0) = x_0$. Since, through each point $(t, x) \in T$, there may pass more than one trajectory of l.f., the map $(t, x) \to G(t, x)$ may be a multifunction. To eliminate such a situation, we exlude from our further considerations (compare [4], p. 266 and [3] (H3)) those l.f. which do not admit of the following hypothesis:

H2) If, through any point $(t_0, x_0) \in T$, there pass two trajectories of l.f., then the values of $G(t_0, x_0)$ for each of them are the same, i.e. we assume the map $(t, x) \to G(t, x)$ single-valued.

We shall say that a rectifiable curve C lying in T is bounded if $G(t, x)$ is bounded on the graph of C, i.e. "along" C.

For each point (t_0, x_0) in T, $Y(t_0, x_0)$, $U(t_0, x_0)$ denote the sets of values of all those $y(t)$, $u(t)$ at t_0 for which $x(t)$, $y(t)$, $u(t)$ is some c.l.f. and $x(t_0) = x_0$. It is natural to expect that $Y(t, x)$ and $U(t, x)$, $(t, x) \in T$, may not be single-valued. Thus, by an admissible pair of functions

$$y(t, x) \in Y(t, x), \qquad u(t, x) \in U(t, x), \qquad (t, x) \in T, \qquad (8)$$

we shall mean single-valued functions $y(t, x)$, $u(t, x)$ in T such that, for each $(t_0, x_0) \in T$, there is some c.l.f. $x(t)$, $y(t)$, $u(t)$ for which $x(t_0) = x_0$, $y(t_0, x(t_0)) = y(t_0)$, $u(t_0, x(t_0)) = u(t_0)$.

Up to now, the basic tool for studying old and new "fields of extremals" has been the Hilbert integral in its old and new forms (see e.g. [4], [2], [3]). Hence. let C be any bounded rectifiable curve lying in T, with description $t = t(s)$, $x = x(s)$, $0 \leqslant s \leqslant l$, where s is the arc length parameter. We define on C the curvilinear integral

$$\int_C -y(t,x) f\big(t, x(t), u(t,x)\big)\, dt + y(t,x)\, dx =$$

$$= \int_0^l \Big[-y\big(t(s), x(s)\big) f\big(t(s), x(s), u\big(t(s), x(s)\big)\big) \frac{dt}{ds} + + y\big(t(s), x(s)\big) \frac{dx}{ds}\Big] ds \qquad (9)$$

for any admissible pair of functions (8) in T such that $-yf\,dt/ds + y\,dx/ds$ is a measurable function of the arc length s along C. The functional defined by (9) in the class of curves C and of functions (8) will be termed the Hilbert integral. Our task is to study its ε-independence, and this in its turn will necessitate some further definitions and concepts.

Let us fix the curve $C \subset T$ for a moment. If the expression $-yf\,dt/ds + + y\,dx/ds$ at the point (t, x) belonging to the set of values of C differs in ε from a certain value for all admissible pairs of functions (8), then the direction $e = (dt/ds, dx/ds)$ of the tangent to C at (t, x) will be called a direction of ε-univalence (compare [4], p. 270). A rectifiable curve $C \subset T$ such that, at almost all points of C, the direction of the tangent to C is a direction of ε-univalence will be called a curve of ε-univalence.

Following [4], p. 271, we introduce the notion of an ε-exact integrability of a set or simply of an ε-exact set. A subset $\dot{T}$ of T will be called an ε-exact set if all rectifiable bounded curves $C \subset \dot{T}$ are curves of ε-univalence and, in addition, for every such C with $(t_1, x_1), (t_2, x_2)$ as the initial and final points of C,

$$\Big|\int_C -y(t,x) f\big(t, x, u(t,x)\big)\, dt + y(t,x)\, dx - - G(t_1, x_1) + G(t_2, x_2)\Big| \leqslant \varepsilon l, \qquad (10)$$

for each admissible pair $y(t,x) \in Y(t,x)$, $u(t,x) \in U(t,x)$, $(t,x) \in T$, such that integral (9) is defined; l is the length of C.

3. A spray of flights

First of all, we shall describe and study some family of arcs of l.f. depending on a parameter σ. Let us define on an open set $Z \subset R^n$ a pair of continuous functions $t^-(\sigma), t^+(\sigma)$, $t^-(\sigma) < t^+(\sigma)$, $\sigma \in Z$, with values in the interval $[0, 1]$. We shall suppose that the function $t^+(\sigma)$ is C^1 in Z.

We further suppose that Z is a projection of a certain set $\tilde{Z} \subset R^{n+p}$, $p > 0$, whose elements will be denoted by (σ, ϱ). $\tilde{Z}$ does not have to be necessarily open; instead of that, we assume that the operation of projection is standard (see [4], p. 226). Let $S^- = \{(t, \sigma): t = t^-(\sigma) \geqslant 0, \sigma \in Z\}$, $S = \{(t, \sigma): t^-(\sigma) < t < t^+(\sigma), \sigma \in Z\}$, $S^+ = \{(t, \sigma): t = t^+(\sigma) \leqslant 1, \sigma \in Z\}$, $[S] = S^- \cup S \cup S^+$. Similarly, we denote by S^{*-}, S^*, S^{*+} the sets of (t, σ, ϱ) for which t satisfies the same conditions as in S^-, S, S^+, respectively, and $(\sigma, \varrho) \in \tilde{Z}$; $[S^*] = S^{*-} \cup S^* \cup S^{*+}$.

Let us consider a family Σ of arcs of l.f., given by functions

$$x(t, \sigma), \quad u(t, \sigma), \quad (t, \sigma) \in S; \tag{11}$$

σ remains constant on an arc of l.f. of Σ and this arc then corresponds to the open time interval $(t^-(\sigma), t^+(\sigma))$. By Σ^* we denote a family of arcs of c.l.f. which correspond to the arcs of Σ and which are obtained by giving with functions (11) the corresponding function

$$y(t, \sigma, \varrho), \quad (t, \sigma, \varrho) \in S^*. \tag{12}$$

The definition of the functions $x(t, \sigma)$, $y(t, \sigma, \varrho)$ will be supposed extended to the sets $[S]$, $[S^*]$. The sets of pairs (t, x), where $x = x(t, \sigma)$ with (t, σ) belonging to S^-, S, S^+, $[S]$, will be denoted by E^-, E, E^+, $[E]$, respectively, and the sets of values of triplets $(t, x(t, \sigma), y(t, \sigma, \varrho))$ with (t, σ, ϱ) in S^{*-}, S^*, S^{*+}, $[S^*]$ by E^{*-}, E^*, E^{*+}, $[E^*]$, respectively.

We shall write $\tilde{f}(t, \sigma)$ for $f(t, x(t, \sigma), u(t, \sigma))$ when $(t, \sigma) \in [S]$.

We now suppose the following additional hypotheses satisfied:

(H3) The function $\tilde{f}(t, \sigma)$ is continuous in $[S]$ and there exist in S continuous derivatives $\tilde{f}_\sigma(t, \sigma)$ and $(\partial/\partial\sigma) f(t, x, u(t, \sigma))$ for each fixed $(t, x) \in E$, satisfying at $x = x(t, \sigma)$ the relation $\partial\tilde{f}/\partial\sigma = (\partial/\partial\sigma) f(t, x, u(t, \sigma)) + f_x(t, x, u(t, \sigma)) x_\sigma(t, \sigma)$.

(H4) The function $y(t, \sigma, \varrho)$ is continuous in $[S^*]$, the function $x(t, \sigma)$ is C^1 in $[S]$ and $u(t, \sigma)$ is Borel measurable in $[S]$.

(H5) The maps $S^- \to E^-$, $S \to E$ defined by $(t, \sigma) \to (t, x(t, \sigma))$ are descriptive (see [4], p. 266).

(H6) For each fixed $(\sigma, \varrho) \in \tilde{Z}$ and for $x = x(t, \sigma)$ we have: for each $t' \in (t^-(\sigma), t^+(\sigma))$ and each vector $(\alpha, \beta) \in R^{n+1}$, $\beta = (\beta_1, \ldots, \beta_n)$, $\alpha^2 + \beta_1^2 + \ldots + \beta_n^2 = 1$, there exists a function $\alpha(t)$ of bounded variation, defined in $[t', t^+(\sigma)]$, with values $\alpha(t') = \alpha$, $\alpha(t) \in R$ for t in $(t', t^+(\sigma))$, $\alpha(t^+(\sigma)) = t_\sigma^+(\sigma)\beta$, such that

$$|y(t, \sigma, \varrho)(\partial/\partial\sigma) f(t, x, u(t, \sigma))\beta| \leqslant$$
$$\leqslant -\varepsilon(\partial/\partial t)\sqrt{(1 + |x_t(t, \sigma)|^2)(\alpha(t))^2 + |x_\sigma(t, \sigma)\beta|^2},$$

for almost all t in $[t', t^+(\sigma)]$. (We assume that the derivative on the right-hand side of the last inequality exists).

Notice that, in view of (H3), (6) and Theorem 2.2 from [1], hypothesis (H6) is not essentially strong.

If hypotheses (H3)–(H6) together with those on $t^-(\sigma)$, $t^+(\sigma)$, Z, $\tilde{Z}$ are satisfied, the family Σ is called a spray of flights and the family Σ a canonical spray of flights.

For $(t,x)\in[E]\subset T$, let $Y_\Sigma(t,x)\subset Y(t,x)$ and $U_\Sigma(t,x)\subset U(t,x)$ stand for the sets of values of $y(t,\sigma,\varrho)$ and $u(t,\sigma)$ at those $(t,\sigma,\varrho)\in[S^*]$, $(t,\sigma)\in[S]$ for which $x(t,\sigma)=x$. By

$$y_\Sigma(t,x)\in Y_\Sigma(t,x), \quad u_\Sigma(t,x)\in U_\Sigma(t,x) \tag{13}$$

we denote an admissible pair of functions $y(t,x)$, $u(t,x)$ defined in $[E]$ and call them functions relative to Σ. At a point $(t,x)\in[E]$, we call direction of relative ε-univalence a direction $e=(dt/ds, dx/ds)$ such that, for all admissible pairs of functions (13), the expression $-y\,f\,dt/ds+y\,dx/ds$ differs in ε from a certain value. We term curve of relative ε-univalence a rectifiable curve $C\subset[E]$ such that, at almost all points of C, the direction of the tangent to C is a direction of relative ε-univalence. An ε-exact subset $\dot{T}\subset[E]$ for admissible pairs $y(t,x)\in Y_\Sigma(t,x)$, $u(t,x)\in U_\Sigma(t,x)$, $(t,x)\in[E]$, will be called a relative ε-exact set.

We notice, by the definition of the function $G(t,x)$ and hypothesis (H2), that $G\big(t,x(t,\sigma)\big)$ for each fixed $\sigma\in Z$ takes the same value for all $t^-(\sigma)\leqslant t\leqslant t^+(\sigma)$. Thus we can consider it only for $t=t^+(\sigma)$. So, let $G^+(\sigma)=G\big(t^+(\sigma),x\big(t^+(\sigma),\sigma\big)\big)$.

Now, we shall study a spray of flights with the help of Hilbert's integral (9). Thus, assume we are given a spray of flights for which the set E^+ is relative ε-exact.

LEMMA 3.1. *Let $\sigma(\lambda)$ be any Lipschitz function in $[0,h]$ with values in Z. Then $\lambda\to G^+(\sigma(\lambda))$ is locally Lipschitz in $[0,h]$.*

Proof. Let c be a small curve in S^+ with description $t=t^+(\sigma(\lambda))$, $\lambda\in J=[h_0,h_1]\subset[0,h]$, and let C be the image of c in E^+ under the map $(t,\sigma)\to\big(t,x(t,\sigma)\big)$ with ends (t_0,x_0), (t_1,x_1). Since E^+ is relative ε-exact, we have

$$\Big|\int_C -y_\Sigma(t,x)\,f\big(t,x,u_\Sigma(t,x)\big)\,dt+y_\Sigma(t,x)\,dx-G(t_0,x_0)+G(t_1,x_1)\Big|\leqslant\varepsilon l,$$

for each admissible pair (13), such that integral (9) is defined, where l is the length of C; by relative ε-univalence and the fact that $x_t(t,\sigma)=\tilde{f}(t,\sigma)$ along c, we get

$$\begin{aligned}\Big|\int_c y\big(t,\sigma,\varrho(\sigma)\big)\,x_\sigma(t,\sigma)\,d\sigma-G^+\big(\sigma(h_0)\big)+G^+\big(\sigma(h_1)\big)\Big|&\leqslant\\ \leqslant\varepsilon\int_J\sqrt{\big(1+\big|x_t\big(t^+(\sigma(\lambda)),\sigma(\lambda)\big)\big|^2\big)\big(t_\sigma^+(\sigma(\lambda))\,\sigma_\lambda(\lambda)\big)^2+}&\\ \overline{+\big|x_\sigma\big(t^+(\sigma(\lambda)),\sigma(\lambda)\big)\,\sigma_\lambda(\lambda)\big|^2}\,d\lambda&\end{aligned} \tag{14}$$

where $\varrho(\sigma)$ is a continuous function of σ suitably chosen along $\sigma(\lambda)$, according to the standard projection. From (14) we infer that the ratio $|G^+(\sigma(h_1)) - G^+(\sigma(h_0))|/|h_1 - h_0|$ is uniformly bounded for all sufficiently small $|h_1 - h_0|$, as asserted. ■

COROLLARY 3.1. *We are given any point* $(\sigma_0, \varrho_0) \in S^{*+}$ *and any sufficiently small curve* $\gamma \subset Z$ *which issues from* σ_0 *with description* $\sigma(\lambda), \lambda \in [0, h]$; $\sigma(\lambda)$ *is a Lipschitz function,* $\sigma(0) = \sigma_0$, 0 *is the point of approximate continuity of* $\sigma(\lambda)$ *and* $(d/d\lambda)\, G^+(\sigma(\lambda))$. *Then*

$$|y(t^+(\sigma_0), \sigma_0, \varrho_0)\, x_\sigma(t^+(\sigma_0), \sigma_0)\, \sigma_\lambda(0) + (d/d\lambda)\, G^+(\sigma(0))| \leqslant$$
$$\leqslant \varepsilon \sqrt{\left(1 + |x_t(t^+(\sigma_0), \sigma_0)|^2\right)\left(t_\sigma^+(\sigma_0)\, \sigma_\lambda(0)\right)^2 + |x_\sigma(t^+(\sigma_0), \sigma_0)\, \sigma_\lambda(0)|^2}. \tag{15}$$

Proof. According to the standard projection, there exists on γ a continuous function $\varrho(\sigma)$ such that $\varrho(\sigma_0) = \varrho_0$. Thus, analogously as in the proof of Lemma 3.1 we have inequality (14) which evidently implies (15). ■

LEMMA 3.2. *Let* C *be a rectifiable curve lying, together with its terminal points, in* E^- *or* E. *Then* C *is bounded and there exist Borel measurable functions* $y_\Sigma(t, x), u_\Sigma(t, x)$ *along* C *relative to* Σ, *and* $y_\Sigma(t, x)$ *and* $f(t, x, u_\Sigma(t, x))$ *are bounded along it.*

Proof. For each point (t, x) of C in E^- or E, there exists a neighbourhood on C that is the image of some curve c from S^- or S (see (H5)). We represent c in terms of its arc length λ by functions $t(\lambda)$, $\sigma(\lambda)$, $\lambda \in [0, h]$. By the note before Lemma 3.1, $G(t, x(t, \sigma)) = G^+(\sigma)$ on c and, by Lemma 3.1, $G^+(\sigma(\lambda))$ is bounded on $[0, h]$. From Borel's covering theorem it follows that $G(t, x)$ is bounded on C. The proof of the second assertion is analogous of that of Lemma 25.1 of [4], vol. II. ■

LEMMA 3.3. *On each arc of the canonical spray of flights* Σ^* *we have: for each* $t' \in (t^-(\sigma), t^+(\sigma))$ *and each vector* $(\alpha, \beta) \in R^{n+1}$, $\beta = (\beta_1, \ldots, \beta_n)$, $\alpha^2 + \beta_1^2 + \ldots + \beta_n^2 = 1$, *there exists a function* $\alpha(t)$ *of bounded variation, defined in* $[t', t^+(\sigma)]$, *with values* $\alpha(t') = \alpha$, $\alpha(t) \in R$ *for* t *in* $(t', t^+(\sigma))$, $\alpha(t^+(\sigma)) = t_\sigma^+(\sigma)\, \beta$, *such that*

$$|(\partial/\partial t)\left(y(t, \sigma, \varrho)\, x_\sigma(t, \sigma)\, \beta\right)| \leqslant$$
$$\leqslant -\varepsilon\, (\partial/\partial t) \sqrt{\left(1 + |x_t(t, \sigma)|^2\right)(\alpha(t))^2 + |x_\sigma(t, \sigma)\, \beta|^2} \tag{16}$$

for almost all t *in* $[t', t^+(\sigma)]$.

Proof. Let (t', σ', ϱ') be any point of S^* and $x'(t)$, $y'(t)$, $u'(t)$ the corresponding values of the functions $x(t, \sigma')$, $y(t, \sigma', \varrho')$, $u(t, \sigma')$, $t \in [t', t^+(\sigma'))$. Let further r stand for any coordinate of the vector $\sigma \in Z$. By performing in different orders the operations of integration in t and differentiation in r on relation (2), and then differentiating in t, we get the following relation

$$\frac{\partial}{\partial t} x_r(t, \sigma) = \frac{\partial}{\partial r} \tilde{f}(t, \sigma), \tag{17}$$

calculated at the point (t, σ', ϱ'), $t \in [t', t^+(\sigma'))$. From (5) we obtain at (t, σ', ϱ'), for almost all t in $[t', t^+(\sigma'))$,

$$x_r(t, \sigma) \frac{\partial}{\partial t} y'(t) = -y'(t) f_s(t, x'(t), u'(t)) x_r(t, \sigma). \tag{18}$$

Multiplying (17) by $y'(t)$ and adding the results to both sides of (18), we obtain at the same (t, σ', ϱ')

$$\frac{\partial}{\partial t}(y'(t) x_r(t, \sigma)) = y'(t) \frac{\partial}{\partial r}(\tilde{f}(t, \sigma) - f(t, x(t, \sigma), u'(t))).$$

By (H3), $(\partial/\partial t)(y' x_r) = y' (\partial/\partial r) f(t, x'(t), u(t, \sigma))$ at (t, σ', ϱ') for almost all t in $[t', t^+(\sigma'))$. Hence and by (H6), we obtain the assertion of the lemma. ■

COROLLARY 3.2. *Let any point $(\sigma_0, \varrho_0) \in S^*$ and any $\alpha \in [-1, 1]$ be given. Let $\gamma \subset Z$ be any sufficiently small curve which issues from σ_0, with description $\sigma(\lambda)$, $\lambda \in [0, h]$, and such that $\sigma(\lambda)$ is a Lipschitz function, $\sigma(0) = \sigma_0$, 0 is the point of approximate continuity of $\sigma_\lambda(\lambda)$ and $(d/d\lambda) G^+(\sigma(\lambda))$, and $\alpha^2 + |\sigma_\lambda(0)|^2 = 1$. Then*

$$\left|y(t, \sigma_0, \varrho_0) x_\sigma(t, \sigma_0) \sigma_\lambda(0) + (d/d\lambda) G^+(\sigma(0))\right| \leqslant \\ \leqslant \varepsilon \sqrt{(1 + |x_t(t, \sigma_0)|^2) \alpha^2 + |x_\sigma(t, \sigma_0) \sigma_\lambda(0)|^2} \tag{19}$$

for all t in $(t^-(\sigma_0), t^+(\sigma_0))$.

Proof. Let $t' \in (t^-(\sigma_0), t^+(\sigma_0))$ be arbitrarily fixed and let $\beta = \sigma_\lambda(0)$. Integrating (16) in the interval $[t', t^+(\sigma_0)]$ and using (15), we find (19). ■

Let C be any rectifiable curve contained in E^- or E with parametric description $t = t(s)$, $x = x(s)$, $0 \leqslant s \leqslant l$, where s is the arc length parameter. Then the function $G(t, x)$ restricted to the curve C becomes the function $\tilde{G}(s) = G(t(s), x(s))$ of the variable s in $[0, l]$, i.e. "along" C. We shall need the following important

THEOREM 3.1. *The function $\tilde{G}(s)$ is absolutely continuous along C and, for almost all s in $[0, l]$,*

$$\Big| y_\Sigma(t(s), x(s)) \frac{dx}{ds} - y_\Sigma(t(s), x(s)) f\big(t(s), x(s), \\ u_\Sigma(t(s), x(s))\big) \frac{dt}{ds} + \frac{d}{ds} \tilde{G}(s) \Big| \leqslant \varepsilon \tag{20}$$

for any admissible $y_\Sigma(t, x) \in Y_\Sigma(t, x)$, $u_\Sigma(t, x) \in U_\Sigma(t, x)$, $(t, x) \in [E]$.

Proof. Let $e(s)=(dt/ds, dx/ds)$ stand for the direction of the tangent to C defined for $s\in[0,l]$ a.e. Let s_0 be any point in $[0,l]$ such that $e(s)$ is approximately continuous at it. We set $t_0=t(s_0)$, $x_0=x(s_0)$, $e_0=e(s_0)$, $\dot{t}_0=dt(s_0)/ds$, $\dot{x}_0=dx(s_0)/ds$. Let y_0, u_0 be any admissible vectors from the sets $Y_\Sigma(t_0,x_0)$, $U_\Sigma(t_0,x_0)$ and let (t_0,σ_0,ϱ_0) be any point in S^* for which $y(t_0,\sigma_0,\varrho_0)=y_0$, $u(t_0,\sigma_0)=u_0$. We also put $f_0=f(t_0,x_0,u_0)$.

Denote by c a rectifiable curve in S such that small arcs of the curve C, issuing from (t_0,x_0), are, in accordance with (H5), the images under the map $(t,\sigma)\to(t,x(t,\sigma))$ of small arcs of c issuing from (t_0,σ_0). Let now γ be a sufficiently small arc of c issuing from (t_0,σ_0), with the arc length parametric description

$$t=\bar{t}(\lambda),\qquad \sigma=\bar{\sigma}(\lambda),\qquad \lambda\in J=[0,h],$$

such that the point (t_0,σ_0) should correspond to the value 0. Next, define an increasing continuous function $s=s(\lambda)$, $\lambda\in J$, such that $s(0)=s_0$, satisfying in J the relations

$$t(s(\lambda))=\bar{t}(\lambda),\qquad x(s(\lambda))=x(\bar{t}(\lambda),\bar{\sigma}(\lambda)). \tag{21}$$

Denote by Δs and $\Delta\tilde{G}$ the corresponding differences in s and in $\tilde{G}(s)$ at the ends of a small arc of C issuing from (t_0,x_0), being the image of γ. In view of the note preceding Lemma 3.1, we have $\int_J (d/d\lambda)\,G^+(\bar{\sigma}(\lambda))\,d\lambda=\Delta\tilde{G}$. Thus, by Corollary 3.2, we conclude that

$$\Big|\int_J yx_\sigma\,d\bar{\sigma}+\Delta\tilde{G}\Big|\leqslant$$
$$\leqslant\varepsilon\int_J\sqrt{\big(1+|x_t(\bar{t}(\lambda),\bar{\sigma}(\lambda))|^2\big)(\bar{t}_\lambda(\lambda))^2+|x_\sigma(\bar{t}(\lambda),\bar{\sigma}(\lambda))\,\bar{\sigma}_\lambda(\lambda)|^2}\,d\lambda$$

and further, since (21) holds and $x_t(t,\sigma)=\tilde{f}(t,\sigma)$ along γ, that

$$\Big|\int_J[-y\tilde{f}\,dt/ds+y\,dx/ds]\,ds(\lambda)+\Delta\tilde{G}\Big|\leqslant\varepsilon\Delta s. \tag{22}$$

Since $y(t,\sigma,\varrho(\sigma))$ ($\varrho(\sigma)$ is suitably chosen), $\tilde{f}(t,\sigma)$ are continuous on γ, we deduce that they are bounded on J. This, along with (22), implies the uniform boundedness of the ratio $\Delta\tilde{G}/\Delta s$ for all sufficiently small Δs. Thus $\tilde{G}(s)$ is locally Lipschitz in $[0,l]$ and, hence, absolutely continuous there. This proves the first assertion of the theorem.

To prove the second one, it is enough to show that

$$\lim\frac{1}{\Delta s}\int_J\left[-y\tilde{f}\frac{dt}{ds}+y\,\frac{dx}{ds}\right]ds(\lambda)=[-y_0f_0\,\dot{t}_0+y_0\,\dot{x}_0]\quad\text{as}\quad\Delta s\to0.$$

But this is quite analogous to the corresponding part of the proof of Lemma 25.3 in [4], vol II, p. 274 if we take there $\varphi=\varphi(\lambda)=$ $=[-y\tilde{f}\,dt/ds+y\,dx/ds]-[-y_0f_0\,\dot{t}_0+y_0\,\dot{x}_0]$. ■

COROLLARY 3.3. *The sets E^- and E are relative ε-exact.*

Proof. By (20), each rectifiable curve C lying in E^- or E is a curve of relative ε-univalence. By Theorem 3.1, $\tilde{G}(s)$ is absolutely continuous. In view of Lemma 3.2, we may integrate (20) and then we find the inequality which defines the relative ε-exactness. ■

4. A chain of flights

In the preceding section we described and discussed a fixed spray of flights Σ. However, the family of l.f. defined in Section 2 may consist of a greater number of sprays of flights satisfying conditions (H3)–(H6), whose graphs of trajectories are contained in T.

We recall (see [4], vol II, § 27) that a finite or countable sequence of sprays of flights in T

$$\Sigma_1, \Sigma_2, \ldots, \Sigma_N, \ldots$$

will be termed a chain of flights and the corresponding sequence of canonical sprays a canonical chain if, for $i = 1, 2, \ldots, N, \ldots$, they fit together in inverse order so that the set E_i^{*-} corresponding to Σ_i^* contains E_{i+1}^{*+} corresponding to Σ_{i+1}^*. This implies that the set E_{i+1}^+ of Σ_{i+1} is relative ε-exact for Σ_i as well as for Σ_{i+1}.

Let Z_1 be an open set of parameters σ^1, associated with a spray Σ_1, and let $\tilde{Z}_1$ be a set of parameters (σ^1, ϱ^1), associated with a canonical spray Σ_1^*. We suppose one more hypothesis satisfied:

(H7) The set E_1^+ of the spray Σ_1 of a chain is contained in $K(a)$ and the function $g^+(\sigma^1) = g(x(1, \sigma^1)) = G(1, x(1, \sigma^1))$ has a continuous derivative $g_{\sigma 1}^+$ in Z_1.

A chain of flights which satisfies (H7) will be called a distinguished chain of flights.

We notice that if, in the set $S_1^{*+} = \{(t, \sigma^1, \varrho^1) : t = 1, (\sigma^1, \varrho^1) \in \tilde{Z}_1\}$ of Σ_1^* of a distinguished chain of flights, the quantity $yx_{\sigma 1} + g_{\sigma 1}^+$ is identically zero, then the assertions of Lemma 3.1 and Corollary 3.1 hold, and so, by Corollary 3.3, the sets E_1^-, E_1 of Σ_1 are relative ε-exact. In consequence, all the sets E_i^- and E_i of Σ_i, $i = 2, \ldots, N, \ldots$, of that chain are also relative ε-exact; such a distinguished chain of flights will be termed an ε-exact chain of flights.

The sets E_i^- and E_i of Σ_i will be termed constituent sets of a chain and E_i^{*-}, E_i^* of Σ_i^* canonical constituent sets of a canonical chain.

LEMMA 4.1. *Given any distinguished chain of flights, the quantity $yx_{\sigma 1} + g_{\sigma 1}^+$ is identically zero in the set S_1^{*+} of Σ_1^*.*

Proof. Notice that, by (H4) and (H7), the smooth function $(x(1,\sigma^1), g^+(\sigma^1))$, $\sigma^1 \in Z_1$, is a parametric description of the graph of the function g restricted to the projection of E_1^+ of Σ_1 onto the x-space, and that the tangent space at each point of this graph is spanned by the columns of the matrix $(x_{\sigma 1}, g_{\sigma 1}^+)$; $(x_{\sigma 1}, g_{\sigma 1}^+)$ is the $(n+1)\times n$ — matrix with the $(n+1)$-th row equal to $g_{\sigma 1}^+$. In view of (7), the vector $(-y, -1)$ is orthogonal to the columns of $(x_{\sigma 1}, g_{\sigma 1}^+)$. Hence we conclude the assertion of the lemma. ■

From Lemma 4.1 and the above note we obtain.

THEOREM 4.1. *Every distinguished chain of flights is an ε-exact chain of flights.*

5. A concourse of flights

The concept of a concourse of flights originates from L. C. Young [4], vol. II, § 28 where there are many details on it. Here we only give a sketch of this theory to formulate further results.

Denote by K the family of all bounded rectifiable curves lying in T, and by T_n, $n = 1, 2, \ldots$, a finite or countable system of disjoint subsets of T whose union is T. Of course, any T_n should be a subset of some constituent set of a chain or a subset of a few constituent sets of different chains.

A curve $C \subset K$ will be called a fragment if its interior lies in some T_n. The class of such fragments will be denoted by K_0. We need a situation in which K can be derived from K_0. To this effect, we shall need two forms of the addition of curves: fusion and embellishment and two subtraction operations: cutting and trimming (see [4], p. 277).

In the sequel, we shall assume about K and K_0 that if a curve belongs to K or K_0, then each arc of the curve, and also its inverse arc, is an element of K or K_0, respectively. Moreover, we shall assume that the operations of embellishment and trimming can be carried out countably often under the restriction that from elements of K we shall again obtain elements of K.

By means of the finite fusion and the countable embellishment, from the elements of K_0 let us compose a class K_1. From K_1 we then define a subclass K_2 of K whose members are obtained by at most countable trimming. The method described by Young [4] can be applied only when $K_2 = K$.

In such a situation, K_0 is called a repairable class of fragments, and the decomposition of the set T into disjoint subsets T_n - *a* repairable decomposition. Then the set T will be termed the unimpaired union of the sets T_n.

A concourse of flights is a finite or countable infinite system of distinguished chains of flights, such that T is the unimpaired union of the constituent sets of these chains, and the set covered by graphs of c.l.f., i.e. by graphs of pairs of functions $x(t)$, $y(t)$, is the union of their canonical constituent sets.

Let $t(s)$, $x(s)$, $0 \leqslant s \leqslant l$, be the arc length description of any bounded rectifiable curve C in T. We introduce the last hypothesis whose object is to ensure that a certain integral exists along each bounded rectifiable curve in T.

(H8) There exists in T an admissible pair of functions $y(t, x) \in Y(t, x)$, $u(t, x) \in U(t, x)$, $(t, x) \in T$, such that the expression

$$\left|-y\big(t(s), x(s)\big) f\big(t(s), x(s), u\big(t(s), x(s)\big)\big)\, dt/ds + y\big(t(s), x(s)\big)\, dx/ds\right|$$

does not exceed along C some integrable function of the arc length s of C.

In optimal control we are concerned with the actual problem of a minimum in the entire set T. Considering a spray of flights Σ, we have discussed the model of the classical calculus of variations only as to what happens in a certain subset T_n whose union turns out to be T. This means that we have information about the class of fragments K_0, and that we seek information about the class of our original curves K.

Proceeding similarly as in the proof of Theorem 29.1 [4], p. 280–281, we prove

THEOREM 5.1. *Assume that a concourse of flights exists. Then $\tilde{G}(s)$ is continuous in $[0, l]$ and it has the derivative $d\tilde{G}/ds$ a.e. which satisfies*

$$\left|\frac{d\tilde{G}(s)}{ds} + y\big(t(s), x(s)\big)\frac{dx}{ds} - \right.$$

$$\left. - y\big(t(s), x(s)\big) f\big(t(s), x(s), u\big(t(s), x(s)\big)\big)\frac{dt}{ds}\right| \leqslant \varepsilon \quad \text{a.e.} \tag{23}$$

for each admissible pair $y(t, x) \in Y(t, x)$, $u(t, x) \in U(t, x)$, $(t, x) \in T$.

Proof. By the definition of a concourse of flights, there exists a repairable decomposition of T into disjoint T_n, each of which is a subset of each constituent set of the chain of the concourse which it meets. Analogously as above define, for the class K, the class of curves K_0.

Let C_0 be a subarc of C which is an element of K_0. Let Σ be any spray of flights of one of our chains, such that the curve C_0 intersects either the set E^- or the set E corresponding to Σ. It follows from our hypothesis that C_0 lies in some T_n wholly contained in E^- or E. By Theorem 4.1, the sets E^- and E are relative ε-exact, so we have, in view of Theorem 3.1, that $\tilde{G}(s)$ is absolutely continuous along C_0 and

$$\left|\frac{d\tilde{G}(s)}{ds} + y_\Sigma \frac{dx}{ds} - y_\Sigma f \frac{dt}{ds}\right| \leqslant \varepsilon \quad \text{a.e..} \tag{24}$$

This is also the case for each spray Σ whose constituent set contains the T_n considered. There is an at most countable number of sprays, so we exclude from our considerations only a countable number of sets of s of measure zero. This means that, for all sprays Σ under consideration, (24) is valid for almost all s of C_0 since each admissible pair $y(t, x)$, $u(t, x)$ has the form $y_\Sigma(t, x)$, $u_\Sigma(t, x)$ at $(t, x) \in T$ for some spray Σ whose corresponding set E^- or E contains (t, x). Hence we find (23). It is easily observed that if $\tilde{G}(s)$ is absolutely continuous along curves of K_0, then it is continuous along curves of K. ■

COROLLARY 5.1. *Suppose that a concourse exists and hypothesis* (H8) *is satisfied. Then the set T is ε-exact.*

Proof. We define, for the class K, the classes of curves K_0, K_1, K_2. Let $C \in K_0$. In the proof of Theorem 5.1 we noted that $\tilde{G}(s)$ is absolutely continuous along C. Integrating (23) along C with ends (t_1, x_1), (t_2, x_2) and the length l, we have

$$\left|\int_C -y(t, x) f(t, x, u(t, x))\, dt + y(t, x)\, dx + G(t_2, x_2) - G(t_1, x_1)\right| \leqslant \varepsilon l \tag{25}$$

for each admissible pair $y(t, x)$, $u(t, x)$, $(t, x) \in T$, such that integral (9) is defined. By additions, (25) will also hold if C is a finite fusion of members of K_0. In particular, if C is closed, we have

$$\left|\int_C -y f\, dt + y\, dx\right| \leqslant \varepsilon l.$$

Thus relation (25) is unaffected by an at most countable embellishment of C (of course l is then the length of a new curve C), i.e. (25) holds, by (H8), for all curves C of K_1. Since trimming is, in fact, removing of an embellishment (see [4], pp. 277, 278), therefore (25) does not change, either, by an at most countable trimming, i.e. for all $C \in K_2$. By assumption, $K_2 = K$, i.e. we have obtained what was asserted. ■

COROLLARY 5.2. *Let us adopt the same assumptions. If C is any arc of an admissible trajectory $x(t)$ under control $u(t)$, $t \in [t_1, t_2]$, whose graph is contained in T, then*

$$\Big|\int_{t_1}^{t_2} \Big(-y(t, x(t)) f\big(t, x(t), u(t, x(t))\big) +$$
$$+ y(t, x(t)) f(t, x(t), u(t))\Big)\, dt + G(t_2, x(t_2)) - G(t_1, x(t_1))\Big| \leqslant$$
$$\leqslant \varepsilon \int_{t_1}^{t_2} \sqrt{1 + |f(t, x(t), u(t))|^2}\, dt,$$

for some admissible pair $y(t, x), u(t, x), (t, x) \in T$.

Proof. By the definition of a concourse of flights and Lemma 3.2, any arc of $x(t)$ is a bounded rectifiable curve as it is only a finite fusion of its subarcs which are contained in constituent sets. For the same reason, there exists an admissible pair $y(t, x), u(t, x)$ in T which is Borel measurable along $x(t)$. ■

As a consequence of Corollary 5.2 we obtain the following sufficient conditions for a strong relative ε-optimality of g.

THEOREM 5.2. *Suppose that a concourse of flights exists and hypothesis (H8) is satisfied. Let some* c.l.f. $x_\varepsilon(t)$, $y_\varepsilon(t)$, $u_\varepsilon(t)$, $t \in [0, 1]$, $x_\varepsilon(0) = a$, *be a member of our concourse and let* $K_T(a)$ *be the subset of* $K(a)$ *of those points* $x(1)$ *for which the graphs of* $x(t)$, $t \in [0, 1]$ *are contained in* T. *Then*

$$g(x_\varepsilon(1)) \leqslant g(x(1)) + \varepsilon\left(1 + \int_0^1 \sqrt{1 + |f(t, x(t), u(t))|^2}\, dt\right),$$

for all $x(1)$ *in* $K_T(a)$.

Proof. Let $x(t)$, $t \in [0, 1]$, be any admissible trajectory such that $x(1) \in K_T(a)$. By the definition of $G(t, x)$, $G(0, a) = g(x_\varepsilon(1))$ and $G(1, x(1)) = g(x(1))$. So, taking into account the Corollary 5.2 with $t_1 = 0$, $t_2 = 1$ and (6), we have

$$g(x_\varepsilon(1)) - g(x(1)) = G(0, a) - G(1, x(1)) \leqslant \varepsilon\left(1 + \int_0^1 \sqrt{1 + |f(t, x(t), u(t))|^2}\, dt\right),$$

as asserted. ■

Now, we give a simple example to illustrate the above theory.

Example: Let $U = (0, 1)$; admissible controls are measurable functions $u: [0, 1] \to (0, 1)$; admissible trajectories are absolutely continuous functions $x: [0, 1] \to R$ satisfying $\dot{x} = u$. We find an approximate minimum of $g(x) = -x^k (k \geqslant 1)$ over an attainable set $K(0)$ of admissible trajectories such that $x(0) = 0$.

First, we calculate canonical lines of flight. Now, (5) has the form $-\dot{y}(t) = 0$ a.e., i.e. $y(t) = \text{const}$ and (6), (7) are as follows:

$$y(t)\, u(t) \geqslant \sup\{y(t)\, u : u \in (0, 1)\} - \varepsilon, \tag{26}$$

$$y(1) = k(x(1))^{k-1}, \tag{27}$$

respectively. We notice that each $1 - \frac{\varepsilon}{k} \leqslant \bar{u} < 1$, $\bar{u} \in [0, 1]$ and $\bar{x}(1) = \bar{u}$, satisfies (26), (27). Let u_ε denote any such $\bar{u}$. Take $u(t, \sigma) = u_\varepsilon$, $x(t, \sigma) = tu_\varepsilon + \sigma$, $t \in [0, 1]$, $\sigma \in Z = (-u_\varepsilon, 1 - u_\varepsilon)$ and $y(t, \sigma, \varrho) = k(u_\varepsilon + \sigma)^{k-1}$, $t \in [0, 1]$, $(\sigma, \varrho) \in \tilde{Z} = Z \times Z$. Of course, $x(t, \sigma)$, $u(t, \sigma)$, $y(t, \sigma, \varrho)$ satisfy (26), (27).

We assume $t^-(\sigma)=0$, $t^+(\sigma)=1$, $\sigma\in Z$. We easily check that hypotheses (H1)–(H5) and (H7) are satisfied. Since in our case $f(t,x,u)=u$, therefore $(\partial/\partial\sigma)f(t,x,u(t,\sigma))=0$. So, if we take $\alpha(t')=\alpha$, $\alpha(t)=-t+2$, $t\in(t',1)$, $\alpha(1)=0$, then hypothesis (H6) is satisfied, too.

Hence the family Σ of $x(t,\sigma)$, $u(t,\sigma)$, $y(t,\sigma,\varrho)$ is a canonical spray of flights. In this case, concourse of flights consists only of Σ, thus hypothesis (H8) is automatically fulfilled (by Lemma 3.2).

Applying Theorem 5.2, we obtain that $x_\varepsilon(1)=u_\varepsilon$ satisfies

$$g(x_\varepsilon(1))\leqslant \inf\{g(x(1)):x(1)\in K(0)\}+3\varepsilon.$$

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Dostateczne warunki ε-optymalności

Posługując się rozszerzeniem metody Younga opisanej pierwotnie w [4], tom II, rozdz. II, opisano własności zadań sterowania rozwiązań przybliżonych. Punktem wyjścia jest zasada ε-maksimum wyprowadzona przez Ekelanda w [1]. W konsekwencji otrzymuje się wystarczające warunki ε-optymalności w postaci podobnej do warunków Weierstrassa z rachunku wariacyjnego.

Достаточные условия ε-оптимальности

Используя расширение метода Юнга, ранее описанного в [4], том II, глава II, представлены свойства задач управления приближенных решений. Исходной точкой является принцип ε-максимум, введенный Экелендом в [1]. В результате получаем достаточные условия ε-оптимальности, аналогичного вида, как для условий Вееритрасса в вариационном исчислении.