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Parameter convergence of adaptive control algorithms for plants with purely deterministic disturbances

by

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In this paper, we give the conditions of the parameter convergence of three adaptive control algorithms for plants with purely deterministic disturbances. These algorithms are the direct and the indirect model reference adaptive control algorithm and the direct pole placement adaptive control algorithm. We consider a general case in which the estimation of unknown parameters is based on a model which is non-unique with respect to parameters.

1. Introduction

In the past few years a great deal of research has been devoted to the issue of parameter convergence of adaptive control algorithms [1]–[3]. This interest is caused, first of all, by the fact that the algorithms characterized by exponential convergence of estimated parameters to the real ones are more robust with respect to time-variation in plant parameters, the existence of unmodelled dynamics and so on [4]–[6]. The aim of this paper is to investigate the parameter convergence of adaptive control algorithms for plants with purely deterministic disturbances. The algorithms that we consider are the following: the direct and the indirect model reference adaptive control (MRAC) algorithm (the MRAC algorithms for plant with purely deterministic disturbances have been considered by Goodwin and Chan [7]) and the direct pole placement adaptive control (PPAC) algorithm [8] (this algorithm is a generalization of the algorithm proposed by Elliott [9]). The method of the convergence analysis we apply utilizes recent results on peristency of excitation for plants with purely deterministic disturbances [10] as well as the ideas used in the convergence analysis of adaptive control algorithms in the disturbance free case [1]. In contrast to all previous works on this field, we consider a general case in which the estimation of unknown parameters is based on a model which is non--unique with respect to parameters.

2. Parameter convergence of MRAC algorithms

In this section, we study the parameter convergence of the direct and the indirect model reference adaptive control algorithm for plants with purely deterministic disturbances. For clarity, we divide this section into several parts. The structures of the MRAC algorithms are described in the first four parts. The conditions of the parameter convergence of the MRAC algorithms are given and established in the next part. The remarks compose the last part.

Statement of the problem of MRAC

Let us consider a single-input, single-output, discrete-time plant described by the equation

$$A(q^{-1})\, y_k = q^{-d}(q^{-1})\, u_k + d_k, \tag{1}$$

where u_k and y_k are the plant input and output, $A(q^{-1})$ and $B(q^{-1})$ are the polynomials (of the backward shift operator q^{-1}) of the form

$$A(q^{-1}) = 1 + a_1 q^{-1} + \dots + a_{n_a} q^{-n_a},\ a_{n_a} \neq 0, \tag{2}$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + \dots + b_{n_b} q^{-n_b},\ b_{n_b} \neq 0, \tag{3}$$

and d_k is a purely deterministic disturbance composed with sine waves and/or, a bias, that is

$$d_k = \sum_{i=1}^{m} g_i \sin(\omega_i k + \varphi_i). \tag{4}$$

Let us denote by $D(q^{-1})$ the polynomial of the least possible degree such that

$$D(q^{-1})\, d_k \equiv 0. \tag{5}$$

Note that the form of the disturbances d_k implies that all zeros of the polynomial $D(q^{-1})$ are single and lie strictly on the unit circle $|q| = 1$.

Assume that:

(MR1) the polynomials $A(q^{-1})$ and $B(q^{-1})$ are relatively prime, (MR2) the polynomial $B(q^{-1})$ is asymptotically stable (that is, all zeros of $B(q^{-1})$ lie strictly outside the unit disc $|q^{-1}| \leq 1$),

(MR3) upper bounds $\bar{n}_a$ and $\bar{n}_b$ of the degrees n_a and n_b are known,

(MR4) the plant delay d is known,

(MR5) the polynomial $D(q^{-1})$ can be factorized as $D(q^{-1}) = D_1(q^{-1}) \cdot D_2(q^{-1})$: $\deg D_1 \triangleq n_{d1}$, $\deg D_2 \triangleq n_{d2}$ where $D_1(q^{-1})$ is a polynomial with known coefficients and $D_2(q^{-1})$ is a polynomial with unknown ones,

(MR6) an upper bound $\bar{n}_{d2}$ of the degree of the polynomial $D_2(q^{-q})$ is known.

The assumption (MR5) denotes that the frequencies of some sine components are known a priori while those of the others ones are unknown. Thus, we consider slightly more general case than the one considered in [7] where all frequencies ω_i are assumed to be unknown. We believe that the assumption on the knowledge of ω_i is well-founded in many cases. For example, if the disturbance $\{d_k\}$ is periodic of known period K then we have $D(q^{-1}) = D_1(q^{-1}) = 1 - q^{-K}$. It should be also pointed out that the utilization of the knowledge of the frequencies ω_i enables to decrease the number of estimated parameters and, as it can be shown by simulation, to improve the transient period of the adaptive system. The assumption (MR2) is necessary since the control law described in the sequel cancels all zeros of the plant. The knowledge of the upper bounds $\bar{n}_a$, $\bar{n}_b$ and $\bar{n}_{d2}$ and the delay d will be utilized for defining parameter vectors (the vectors θ_i^* and θ_d^* in the sequel) characterizing the plant (1). The assumption (MR1) is not necessary for the design, but in view of the other assumptions it does not decrease generality.

Further. let us assume that we are given a reference model whose output y_k^r determines for us a desired trajectory of the plant output. Let this model be described by

$$A^M(q^{-1})\, y_k^r = q^{-d} B^M(q^{-1}) u_k^r, \tag{6}$$

where u_k^r is a bounded external command input and $A^M(q^{-1})$ is a monic and asymptotically stable polynomial.

The objective of the control is to determine an appropriate bounded input sequence $\{u_k\}$ in such a way that

$$\lim_{k\to\infty} (y_k - y_k^r) = 0. \tag{7}$$

Model reference control strategy for known plants

Below, we describe an appropriate control strategy for the case of known polynomial $A(q^{-1})$, $B(q^{-1})$ and $D(q^{-1})$.

After multiplication of the plant equation (1) by $D(q^{-1}) = D_1(q^{-1}) \cdot D_2(q^{-1})$ we obtain

$$A^*(q^{-1})\, y_k^f = q^{-d} B^*(q^{-1})\, u_k^f, \tag{8}$$

where

$$y_k^f = D_1(q^{-1})\, y_k,\; u_k^f = D_1(q^{-1})\, u_k, \tag{9}$$

$$A^*(q^{-1}) = A(q^{-1})\, D_2(q^{-1}),\; B^*(q^{-1}) = B(q^{-1})\, D_2(q^{-1}). \tag{10}$$

Let the polynomials $P^*(q^{-1})$ and $H^*(q^{-1})$ of degrees $d-1$ and n_a+n_d-1 ($n_d = n_{d1}+n_{d2}$) be the solution of the polynomial equation

$$A^M(q^{-1}) = A^*(q^{-1})\, D_1(q^{-1})\, P^*(q^{-1}) + q^{-d} H^*(q^{-1}). \tag{11}$$

From (8) and (11) we get

$$A^M(q^{-1})\, y_k = q^{-d}[H^*(q^{-1})\, y_k + K^*(q^{-1})\, u_k^f], \tag{12}$$

where

$$K^*(q^{-1}) = B^*(q^{-1})\, P^*(q^{-1}). \tag{13}$$

Hence, we see that the control objective will be satisfied if we apply the following control law

$$H^*(q^{-1})\, y_k + K^*(q^{-1})\, u_k^f = B^M(q^{-1})\, u_k, \tag{14a}$$

$$D_1(q^{-1})\, u_k = u_k^f. \tag{14b}$$

Indeed, (12), (14) and (6) result in $A^M(q^{-1})(y_k - y_k^r) = 0$. Therefore, in view of the asymptotic stability of $A^M(q^{-1})$ the property (7) holds. Moreover, one can easily check that the system (1), (14) is exponentially stable, so that $\{u_k\}$ is bounded provided $\{u_k^r\}$ is bounded.

When the polynomials $A(q^{-1})$, $B(q^{-1})$ and $D_2(q^{-1})$ are unknown we can apply an adaptive control strategy. Roughly speaking, such strategy consists in recursive estimation of parameters characterizing the process to be controlled and in application of the time-varying control law

$$\hat{H}_k(q^{-1})\, y_k + \hat{K}_k(q^{-1})\, u_k^f = B^M(q^{-1})\, u_k^r, \tag{15a}$$

$$D_1(q^{-1})\, u_k = u_k^f, \tag{15b}$$

whose parameters $\hat{h}_{0k}, \hat{h}_{1k}, \ldots, \hat{k}_{0k}, \hat{k}_{1k}, \ldots$ (these are the coefficients of $\hat{H}_k(q^{-1})$ and $\hat{K}_k(q^{-1})$) are determined by the use of the estimated parameters as if these parameters were correct. Most of the known estimation algorithms need the following representation for the unknown parameter vector $\theta^* \in \mathbf{R}^{n_\theta}$

$$v_k = x_k^T \theta^*, \quad v_k = \mathbf{R}, \quad x_k \in \mathbf{R}^{n_\theta}, \tag{16}$$

where v_k and x_k are some variables which depend in a known way on the plant input and output and play the role of the data.

Two kinds of adaptive control algorithms based on two types of such representations are described below.

Indirect MRAC algorithm

Denoting

$$v_{ik} = y_k^f, \tag{17}$$

$$x_{ik} = [-y_{k-1}^f \dots -y_{k-\bar{n}_a-\bar{n}_{d2}}^f \, u_{k-d}^f \dots u_{k-d-\bar{n}_b-\bar{n}_{d2}}^f]^T, \tag{18}$$

and

$$\theta_i^* = [\underbrace{a_1^* \, a_2^* \dots a_{n_a+n_{d2}}^* \, 0 \dots 0}_{\bar{n}_a+\bar{n}_{d2}} \, \underbrace{b_0^* \dots b_{n_b+n_{d2}}^* \, 0 \dots 0}_{\bar{n}_b+\bar{n}_{d2}+1}]^T, \dim \theta_i^* \triangleq n_{\theta i}, \tag{19}$$

we can rewrite the equation (8) in the following equivalent form

$$v_{ik} = x_{ik}^T \theta_i^*. \tag{20}$$

The MRAC algorithm based on the estimation of the parameter vector θ_i^* from the model (20) is called 'indirect', which indicates simply that the parameters of the desired control law are not estimated explicitly.

It should be pointed here that in general the representation (20) is not unique with respect to parameters. Indeed, the set Θ_i of vectors $\hat{\theta}_i$ such that $v_{ik} = x_{ik}^T \hat{\theta}_i$ for every realization of $\{u_k\}$ has the form

$$\Theta_i = \{\hat{\theta}_i = [\hat{a}_1 \dots \hat{a}_{\bar{n}_a+\bar{n}_{d2}} \, \hat{b}_0 \dots \hat{b}_{\bar{n}_b+\bar{n}_{d2}}]^T : \hat{A}(q^{-1}) = A^*(q^{-1}) \, L(q^{-1})\},$$

and $\hat{B}(q^{-1}) = B^*(q^{-1}) \, L(q^{-1})$ for some polynomial

$$L(q^{-1}) = 1 + l_1 + \dots + l_{n_l} q^{-n_l}\} \tag{21}$$

where $n_l = \bar{n}_{d2} - n_{d2} + \min(\bar{n}_a - n_a, \bar{n}_b - n_b)$. The set Θ_i is a hyperplane of dimension n_l passing through the point θ_i^*.

Now, let $\{\hat{\theta}_{ik}\} = \{[\hat{a}_{1k} \dots \hat{a}_{\bar{n}_a+\bar{n}_{d2}k} \, \hat{b}_{0k} \dots \hat{b}_{\bar{n}_b+\bar{n}_{d2}k}]^T\}$ denote the sequence of the estimated parameter vectors generated by a recursive estimation algorithm. The scheme of the computations of the regulator parameters can be described in details as follows. First, we determine the polynomials $\hat{A}_k(q^{-1}) = 1 + \hat{a}_{1k} q^{-1} + \dots + \hat{a}_{\bar{n}_a+\bar{n}_{d2}k} q^{-\bar{n}_a-\bar{n}_{d2}}$ and $\hat{B}_k(q^{-1}) = \hat{b}_{0k} + \hat{b}_{1k} q^{-1} + \dots + \hat{b}_{\bar{n}_b+\bar{n}_{d2}k} q^{-\bar{n}_b-\bar{n}_{d2}}$. Further, we find the polynomials $\hat{P}_k(q^{-1})$ and $\hat{H}_k(q^{-1})$ of degrees $d-1$ and $\bar{n}_a + \bar{n}_d - 1$ such that $A^M(q^{-1}) = \hat{A}_k(q^{-1}) \, D_1(q^{-1}) \cdot \hat{P}_k(q^{-1}) + q^{-d} \hat{H}_k(q^{-1})$. Finally, we compute $\hat{K}_k(q^{-1}) = \hat{B}_k(q^{-1}) \, \hat{P}_k(q^{-1})$.

Direct MRAC algorithm

The direct MRAC algorithm is based on the direct estimation of the regulator parameters, i.e. of the coefficients of the polynomials $H^*(q^{-1})$ and $K^*(q^{-1})$. An appropriate model for estimating these parameters is given by the equation (12). Indeed, denoting

$$v_{dk} = A^M(q^{-1})\, y_k, \tag{22}$$

$$x_{dk} = [y_{k-d}\, y_{k-d-1} \dots y_{k-d-\bar{n}_a-\bar{n}_d+1}\, u^f_{k-d} \dots u^f_{k-2d-\bar{n}_b-\bar{n}_{d2}+1}]^T, \tag{23}$$

$$\theta^*_d = [h^*_0\, h^*_1 \dots h^*_{n_a+n_d-1}\, 0 \dots 0\, k^*_0 \dots k^*_{n_b+n_{d2}+d-1}\, 0 \dots 0]^T,\ \dim \theta^*_d \triangleq n_{\theta d}, \tag{24}$$

(here we use the notation $\bar{n}_d = n_{d1}+\bar{n}_{d2}$) we can rewrite this equation in the following equivalent form

$$v_{dk} = x^T_{dk}\, \theta^*_d. \tag{25}$$

As before, the representation (25) is non-unique with respect to parameters. One can show that the set Θ_d of the parameter vectors $\hat{\theta}_d$ such that $v_{dk} = x^T_{dk}\, \hat{\theta}_d$ for every realization of $\{u_k\}$ has the form

$$\begin{aligned}\Theta_d = \{\hat{\theta}_d = [h_0 \dots \hat{h}_{\bar{n}_a+\bar{n}_d-1}\, \hat{k}_0 \dots \hat{k}_{\bar{n}_b+\bar{n}_{d2}+d-1}]^T : A^M(q^{-1}) = \\ = A^*(q^{-1})\, D_2(q^{-1})\, L(q^{-1})+q^{-d}\hat{H}(q^{-1}) \text{ and } \hat{K}(q^{-1}) = \\ = B^*(q^{-1})\, D_2(q^{-1})\, L(q^{-1}) \text{ for some polynomial} \\ L(q^{-1}) = 1+l_1 q^{-1}+ \dots +l_{n_l+d-1}\, q^{-n_l+d-1}\}\end{aligned} \tag{26}$$

where $n_l = \bar{n}_{d2}-n_{d2}+\min(\bar{n}_a-n_a, \bar{n}_b-n_b)$. The set Θ_d is a hyperplane of dimension n_l passing through the point θ^*_d.

Finally, note that denoting the sequence of estimated parameter vectors by $\{\hat{\theta}_{dk}\}$ the regulator equation (15a) can be rewritten in the following compact form

$$x^T_{dk+d}\, \hat{\theta}_{dk} = B^M(q^{-1})\, u^r_k. \tag{27}$$

Parameter convergence of MRAC algorithms

In this section, we give conditions of convergence of the estimated parameters for the direct and indirect MRAC algorithms described previously. As an example of an identifier we shall consider a recursive estimation algorithm derived recently in [11]. Assuming that the estimation the of unknown parameter vector θ^* is based on the model (16) the formulas describing this algorithm are the following

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \frac{\Gamma_{k-1}\, x_k\, (v_k - x^T_k\, \hat{\theta}_{k-1})}{1+x^T_k\, \Gamma_{k-1}\, x_k},\ \hat{\theta}_0, \tag{28a}$$

$$\Gamma_k = \frac{1}{\lambda}[P_k-(1-\lambda)\, P_k\, Q^{-1}\, P_k],\ Q^{-1} = (Q^{-1})^T > 0,\quad \frac{1}{2} \leqslant \lambda < 1, \tag{28b}$$

$$P_k = \Gamma_{k-1} - \frac{\Gamma_{k-1}\, x_k\, x^T_k\, \Gamma_{k-1}}{1+x^T_k\, \Gamma_{k-1}\, x_k},\quad \Gamma_0 = \Gamma^T_0 \geqslant Q. \tag{28c}$$

Note that if $Q^{-1}=0$ then the algorithm (28) is equivalent to the standard exponentially weighted recursive least squares algorithm with the forgetting factor λ. An application of the weighted least squares algorithm is limited, however, to the case when the parametrization (16) is unique. Indeed, it can be shown that the non-unique parametrization leads to unboundedness of gain matrix Γ_k. On the other hand, the choice $Q^{-1}>0$ ensures $\Gamma_k<0\ \forall k$ irrespectively of the fact whether the parametrization (16) is unique or not [11]. This is why we have assumed $Q^{-1}>0$ in (28). Now, let us consider the issue of the parameter convergence of the algorithm (28). It is obvious that if the parametrization (16) is non-unique it has no possibility to ensure the convergence of θ_k to θ^*. Indeed, non-uniquenes denotes that there exists a vector $\theta \neq \theta^*$ such that $v_k = x_k^T\theta\ \forall \{u_k\}$. Thus, the choice $\hat{\theta}_0=\theta$ in (28a) leads to $\hat{\theta}_k \equiv \theta$ independently of the realization of input sequence. It appears, however, that it is possible to ensure the convergence of the estimate $\hat{\theta}_k$ to one of the element of a set Θ defined as

$$\Theta = \{\theta \in \mathbf{R}^{n_\theta}: v_k = x_k^T\theta\ \ \forall\ \{u_k\}\}. \tag{29}$$

Indeed, we have the following result [11]:

Lemma 2.1. *Consider the algorithm* (28). *If there exists a positive integer N and a positive real ϱ such that*

$$\sum_{k=j}^{j+N} h^T x_k x_k^T h \geqslant \varrho h^T h \ \text{for all sufficiently large (for a.s.l.)}\ j \text{ and } \forall\, h \in \mathscr{X} \tag{30}$$

where $\mathscr{X}$ is a linear subspace of $\mathbf{R}^{n_\theta}$ defined as

$$\mathscr{X} = \{h \in \mathbf{R}^{n_\theta}: h^T(\theta^* - \theta) = 0\ \ \forall\, \theta \in \Theta\} \tag{31}$$

then $\hat{\theta}_k$ converges exponentially and $\lim_{k\to\infty} \hat{\theta}_k \in \Theta$.

Note that in the adaptive control algorithms described in the previous parts of this section the vector x_d as well as the vector x_i depend on the reference trajectory $\{u_k^r\}$ only (neglecting the initial conditions of the algorithms). Thus the expression of the convergence condition of $\hat{\theta}_{ik}$ and $\hat{\theta}_{dk}$ in terms of $\{u_k^r\}$ is of greater interest. We shall do that below. Firstly, we shall introduce the notion of persistently exciting (PE) and persistently spanning (PS) signals.

Definition 2.1. We say that the vector sequence $\{v_k\}$ is PS iff there exist a positive integer N and a positive real number such that

$$\sum_{k=j}^{j+N} v_k v_k^T \geqslant \varrho I \text{ for a.s.l. } j.$$

We say also that the scalar sequence $\{\xi_k\}$ is PE with richness m iff the vector sequence $\{[\xi_k\ \xi_{k+1} \dots \xi_{k+m-1}]^T\}$ is PS.

Now we can state the following result for the indirect MRAC algorithm.

THEOREM 2.1. *Consider the indirect MRAC system described by* (1)–(5), (15), (17), (18), *and* (28) (*with* x_k, $\hat{\theta}_k$, *and* v_k *replaced by* x_{ik}, $\hat{\theta}_{ik}$, *and* v_{ik} *in* (28)). *Assume that* $\hat{b}_{0k} \neq 0\ \forall k \geqslant 0$ ($\hat{b}_{0k}$ *denotes here the* $(\bar{n}_a + \bar{n}_{d2} + 1)$ *th component of* $\hat{\theta}_{ik}$). *If the filtered sequence* $\{B^M(q^{-1})\, D(q^{-1})\, u_k^r\}$ *is PE with richness* $\bar{n}_{d2} + n_a + n_b + 1 + max(\bar{n}_a - n_a, \bar{n}_b - n_b)$ *then the estimated parameter vector* $\hat{\theta}_{ik}$ *converges exponentially and*

$$\lim_{k \to \infty} \hat{\theta}_{ik} = \Theta_i. \tag{32}$$

The proof of Theorem 3.1 as well as the proofs of further theorems are based on four lemmas given below. The proofs of the first two lemmas are simple and are omitted. The proofs of the remaining two ones can be found in [10].

LEMMA 2.2. *Let* $v_k, w_k \in \mathbf{R}^m$. *If* $\{v_k\}$ *is PS and* $\lim_{k \to \infty} (v_k - w_k) = 0$ *then* $\{w_k\}$ *is PS.*

LEMMA 2.3. *Let* $w_k = G v_k$ *where* $w_k, v_k \in \mathbf{R}^m$ *and* G *is* $m \times m$ *nonsingular matrix. If* $\{v_k\}$ *is PS then* $\{w_k\}$ *is PS.*

LEMMA 2.4. *Let* $C(q^{-1})\, w_k = v_k$ *where* $w_k, v_k \in \mathbf{R}^m$ *and* $C(q^{-1})$ *is a polynomial of the backward shift operator. If* $\{v_k\}$ *is PS then* $\{w_k\}$ *is PS.*

LEMMA 2.5. *Assume that* d_k *and* $D(q^{-1})$ *have the same meaning as in the first part of this section. Consider the sequence of vectors* $w_k = [d_{k-1} \dots \dots d_{k-n_d} | v_k^T]^T$ *where* $v_k \in \mathbf{R}^m$ *and* $n_d = deg\, D(q^{-1})$. *If* $\{v_k\}$ *is bounded and* $\{D(q^{-1})\, v_k\}$ *is PS then* $\{w_k\}$ *is PS.*

Proof of Theorem 2.1. It can be shown by generalization of the stability results of [13] that the assumption $\hat{b}_{0k} = 0\ \forall k$ imply the boundedness of the input and the output of the plant and the fulfilment of the control objective (7). We shall utilize these properties later.

We shall only consider the case $\bar{n}_b - n_b \geqslant \bar{n}_a - n_a$ in the sequel. The opposite case can be considered similarly. Denote

$$\bar{x}_{ik} = [-y^f_{k-\bar{n}_a-\bar{n}_{d2}+n_a+n_{d2}-1} \dots y^f_{k-\bar{n}_a-\bar{n}_{d2}}\ u^f_{k-d} \dots u^f_{k-d-\bar{n}_b-\bar{n}_{d2}}]^T, \tag{33}$$

$$\bar{\bar{x}}_{ik} = [-y^f_{k-\bar{n}_a-\bar{n}_{d2}+n_a-1} \dots y^f_{k-\bar{n}_a-\bar{n}_{d2}}\ u^f_{k-d} \dots u^f_{k-d-\bar{n}_b-\bar{n}_{d2}}]^T. \tag{34}$$

Note that utilizing the equation (8) many times the first $\bar{n}_a - n_a + \bar{n}_{d2} - n_{d2}$ components of the vector x_{ik} can be expressed in terms of the elements of the vector $\bar{x}_{ik}$. Thus

$$x_{ik} = K_{i1}\,\bar{x}_{ik}, \tag{35}$$

for a matrix K_{i1} of the form $K_{i1} = [////\,|\,I]^T$. Note that

$$\Theta_i = \{\theta \in \mathbf{R}^{n_{\theta i}} : (\theta - \theta_i^*)^T K_{i1} = 0\}. \tag{36}$$

Indeed, in view of the definition of Θ_i we have $\bar{\Theta}_i \triangleq \{\theta \in \mathbf{R}^{n_{\theta i}} : (\theta - \theta_i^*)^T K_{i1} = 0\} \subset \Theta_i$. But the set $\bar{\Theta}_i$ is a hyperplane of the same dimension as Θ_i. This implies an equivalence $\Theta_i = \bar{\Theta}_i$. Now, note that by (36) and Lemma 3.1, Theorem 3.1 will be established if we show that the vector sequence $\{\bar{x}_{ik}\}$ is PE. To this end let us observe that utilizing the equation

$$A^*(q^{-1})\,y_k^f = q^{-d} B^*(q^{-1})\,u_k^f + d_k^f \quad (d_k^f = D_1(q^{-1})\,d_k) \tag{37}$$

many times the first n_{d2} components of the vector x_{ik} can be expressed in terms of the components of the vector $\bar{\bar{x}}_{ik}$ and of the filtered disturbances $d^f_{k-\bar{n}_a-\bar{n}_{d2}+n_a+n_{d2}-1}, \ldots, d^f_{k-\bar{n}_a-\bar{n}_{d2}+n_a}$. Therefore

$$\bar{x}_{ik} = \left[\begin{array}{c|c} K_{i2} & K_{i3} \\ \hline & I \end{array}\right] \left[\begin{array}{c} d^f_{k-n_a-n_{d2}+n_a+n_{d2}-1} \\ \vdots \\ d^f_{k-n_a-n_{d2}+n_a} \\ \hline \bar{\bar{x}}_{ik} \end{array}\right] \tag{38}$$

for some matrices $K_{i2} \in \mathbf{R}^{n_{d2} \times n_{d2}}$ and $K_{i3} \in \mathbf{R}^{n_{d2} \times (n_a + n_b + n_{d2} + 1)}$. It can be easily checked that K_{i2} is uper-right triangular with unities on main diagonal. Thus the matrix $\left[\begin{array}{c|c} K_{i2} & K_{i3} \\ \hline & I \end{array}\right]$ is nonsingular. Further, from the equation $A(q^{-1})\,D_1(q^{-1})\,y_k^f = q^{-d} B(q^{-1})\,D_2(q^{-1})\,u_k^f$ we get

$$B(q^{-1})\,D_1(q^{-1})\,\bar{\bar{x}}_{ik} = \begin{bmatrix} -q^{-\bar{n}_a-\bar{n}_{d2}+n_a-1} B(q^{-1}) \\ \vdots \\ -q^{-\bar{n}_a-\bar{n}_{d2}} B(q^{-1}) \\ A(q^{-1}) \\ \vdots \\ q^{-\bar{n}_b-\bar{n}_{d2}} A(q^{-1}) \end{bmatrix} D(q^{-1})\,y_k =$$

$$= \Lambda_i \begin{bmatrix} D(q^{-1})\,y_k \\ D(q^{-1})\,y_{k-1} \\ \vdots \\ \vdots \\ \vdots \\ D(q^{-1})\,y_{k-n_a-\bar{n}_b-\bar{n}_{d2}} \end{bmatrix} \tag{39}$$

where Λ_i is a Sylvester matrix for the polynomial $A(q^{-1})$ and $q^{-\bar{n}_a-\bar{n}_{d2}+n_a-1} \times B(q^{-1})$. Recall that the assumption (MR1) implies the nonsingularity of Λ_i. Now, we have consecutively the following. By assumption on the persistency of excitation of $\{B^M(q^{-1})\, D(q^{-1})\, u_k^r\}$ and Lemma 2.4, the sequence $\{D(q^{-1}) y_k^r\}$ is PE with richness $n_a+n_b+n_{d2}+1$. Hence, by Lemma 2.2, $\{D(q^{-1})\, y_k^r\}$ is PE with the same richness $n_a+\bar{n}_b+\bar{n}_{d2}+1$. Hence, by nonsingularity of Λ_i and Lemmas 2.3 and 2.4, the vector sequence $\{\bar{\bar{x}}_k\}$ is PE. Hence, finally, by boundedness of $\{\bar{x}_{ik}\}$, identity (38) and Lemmas 2.5 and 2.3, $\{\bar{x}_{ik}\}$ is PE too. This establishes Theorem 2.1. ■

For the direct MRAC algorithm the results analogous to Theorem 2.1 is given below.

THEOREM 2.2 *Consider the direct MRAC system described by* (1)–(5), (22), (23) *and* (28) (*with x_k, $\hat{\theta}_k$ and v_k replaced by x_{dk}, $\hat{\theta}_{dk}$ and v_{dk}* in (28)). *Assume that $\hat{s}_{0k} \neq 0\, \forall k$ ($\hat{s}_{0k}$ denotes here the $(n_a+n_d+1)\cdot th$ component of $\hat{\theta}_{dk}$). If the external input $\{u_k^r\}$ is bounded and the filtered sequence $\{B^M(q^{-1})\cdot D_2(q^{-1})\, u_k^r\}$ is PE with richness $\bar{n}_d+n_a+n_b+d+max(\bar{n}_a-n_a, \bar{n}_b-n_b)$ then the estimated parameter vector $\hat{\theta}_{dk}$ converges exponentially and*

$$\lim_{k\to\infty} \hat{\theta}_{dk} \in \Theta_d, \tag{40}$$

Proof. The proof of Theorem 2.2 proceeds along the same scheme as the proof of Theorem 2.1. As before, we shall only consider the case $\bar{n}_b-n_b \geqslant \bar{n}_a-n_a$. Denote

$$\bar{x}_{dk} = [-y_{k-d-\bar{n}_a+n_a-\bar{n}_{d2}+n_{d2}} \cdots -y_{k-d-\bar{n}_a-\bar{n}_d+1}\; u_{k-d}^f \cdots \\ \cdots u_{k-2d-\bar{n}_b-\bar{n}_{d2}+1}^f]^T, \tag{41}$$

$$\bar{\bar{x}}_{dk} = [-y_{k-d-\bar{n}_a+n_a-\bar{n}_{d2}} \cdots -y_{k-d-\bar{n}_a-\bar{n}_d+1}\; u_{k-d}^f \cdots u_{k-2d-\bar{n}_b-\bar{n}_{d2}+1}^f]^T. \tag{42}$$

By arguments similar to those used in the proof of Theorem 2.1 we obtain consecutively

$$x_{dk} = K_{d1}\, \bar{x}_{dk}, \tag{43}$$

$$\bar{x}_{dk} = \left[\begin{array}{c|c} K_{d2} & K_{d3} \\ \hline & I \end{array}\right] \left[\begin{array}{c} d_{k-d-\bar{n}_a+n_a-\bar{n}_{d2}+n_{d2}}^f \\ \vdots \\ d_{k-d-\bar{n}_a+n_a-\bar{n}_{d2}+1}^f \\ \hline \bar{\bar{x}}_{dk} \end{array}\right], \tag{44}$$

$$B(q^{-1})\, D_2(q^{-1})\, \bar{\bar{x}}_{dk} = \Lambda_d\, D_2(q^{-1}) \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_{k-d-n_a-\bar{n}_b-\bar{n}_d+1} \end{bmatrix} \tag{45}$$

where K_{d1}, K_{d2}, K_{d3} and Λ_d are matrices of appropriate dimensions and K_{d1}, K_{d2} and Λ_d are nonsingular. Moreover, it can be shown by generalization of the stability results of [13] that the assumption $\hat{s}_{0k} \neq 0\ \forall k$ implies the boundedness of the input and the output and the property (7). Now, as in the proof of Theorem 2.1, we establish consecutively that $\{D_2(q^{-1})\cdot \cdot [y_k\, y_{k-1} \cdots y_{k-d-n_a-\bar{n}_b-\bar{n}_d+1}]^T\}$, $\{D_2(q^{-1})\bar{\bar{x}}_{dk}\}$ and $\{\bar{x}_{dk}\}$ are PE. The exponential convergence of $\hat{\theta}_{dk}$ is a direct consequence of the peristency of excitation of $\{\bar{x}_{dk}\}$.

Remarks

REMARK 2.1. Theorems 2.1 and 2.2 are a generalization of the results of [1] where convergence of the MRAC algorithm has been studied for the case of $d=1$, $n_a=\bar{n}_a$, $n_b=\bar{n}_b$, $n_d=1$ (disturbance free and unique parametrization case). To our mind, an investigation of the case with disturbances has required most of all an application of new ideas.

REMARK 2.2. Since the parameter convergence of the direct and the indirect MRAC algorithm depends on the persistency of excitation of the appropriately filtered external input, the following result is of interest. The filtered variable $\{T(q^{-1})\xi_k\}$ ($\xi_k \in \mathbf{R}^1$ and $T(q^{-1})$ is a polynomial) is PE with richness m if $\{\xi_k\}$ is PE with richness $m+n_z$ where n_z is a number of zeros of the polynomial $T(q^{-1})$ lying on the unit circle. It should be pointed out, however, that the richness m of the persistency of excitation of $\{\xi_k\}$ is sufficient for $\{T(q^{-1})\xi_k\}$ to be PE with richness m in most cases. For example, if ξ_k consists of entire $[(m+1)/2]$ distinct sine waves with randomly chosen frequencies then $\{T(q^{-1})\xi_k\}$ is PE with richness m almost surely.

REMARK 2.3. Note that in the unique parametrization case both in the direct and the indirect MRAC algorithm a richness of the persistency of excitation of external input $\{u_k^r\}$ (strictly speaking, of the filtered external input) necessary for ensuring the parameter convergence is n_{d2} lower than the number of estimated parameters. This difference is a consequence of the fact that the disturbance components of unknown frequencies caused an additional, useful from the convergence point of view, plant excitation.

REMARK 2.4. If external input $\{u_k^r\}$ is not PE with sufficient richness then we can add an auxiliary signal $\{\eta_k\}$ additively to the regulator equation to ensure an additional plant excitation. It should be pointed out, however, that instead of the property (7) we have then only $\lim_{k\to\infty}(e_k^0-\bar{\eta}_k)=0$ where $e_k^0=y_k-y_k^r$ and $A^M(q^{-1})\bar{\eta}_k=\eta_k$.

REMARK 2.5. As we have been pointed out, in the proofs of Theorems 2.1 and 2.2, the boundedness of the plant input and output and the property (7) hold if $\hat{b}_{0k}\neq 0\ \forall k$ in the case of the indirect MRAC algorithm and if $\hat{s}_{0k}\neq 0\ \forall k$ in the case of the direct one. This condition can be guaranteed, for example, by introducing an appropriate projection to the estimation algorithm (28) (see [14]).

3. Parameter convergence of the direct PPAC algorithm for plants with purely deterministic disturbances

In this section, we study parameter convergence of the direct adaptive pole placement control algorithm for plants with purely deterministic disturbances. This algorithm has been derived recently in [8] and is recalled shortly below.

Structure of the algorithm

As before, we assume that the plant to be controlled is described by the equations (1)–(5). All assumptions required for the design are listed below:

(PP1) the polynomials $A(q^{-1})$ and $B(q^{-1})$ are relatively prime,

(PP2) upper bounds $\bar{n}_a$ and $\bar{n}_b$ of n_a and n_b such that $\min(\bar{n}_a - n_a, \bar{n}_b - n_b) = 0$ are known,

(PP3) a lower bound of the plant time delay is known,

(PP4) the polynomial $D(q^{-1})$ occuring in the disturbance model (5) can be factorized as $D(q^{-1}) = D_1(q^{-1}) D_2(q^{-1})$: $\deg D_1 \triangleq n_{d1}$, $\deg D_2 \triangleq n_{d2}$ where $D_1(q^{-1})$ is a polynomial with known coefficients and $D_2(q^{-1})$ is a polynomial with unknown ones,

(PP5) an upper bound $\bar{n}_{d2}$ of the degree of the polynomial $D_2(q^{-1})$ is known,

(PP6) the polynomial $B(q^{-1})$ and $D_2(q^{-1})$ are relatively prime.

Note that the assumptions on the asymptotic stability of the polynomial $B(q^{-1})$ occurring in the MRAC designs are not necessary now. Thus, the PPAC algorithm can be applied for a considerably broader class of plants than the MRAC algorithms. On the other hand, however, an application of the PPAC algorithm requires more precise prior knowledge of the degrees of the polynomials $A(q^{-1})$ and $B(q^{-1})$ than the application of the MRAC ones (compare the assumptions (MR3) and (PP2)). Note also that since we do not assume that $b_0 \neq 0$, hence without loss of generality we can assume that the integer d in the equation (1) equals exactly the lower bound of the delay of the plant.

Firstly, we shall describe an appropriate pole placement control strategy for the case of known $A(q^{-1})$, $B(q^{-1})$ and $D_1(q^{-1})$. Let $C(q^{-1})$ be a monic and asymptotically stable polynomial of degree n_c which represents the desired denumerator of the transfer function from the external input $\{u_k^r\}$ to the output $\{y_k\}$. Consider the following control law for the system (1)–(5)

$$H^*(q^{-1})\, y_k + K^*(q^{-1})\, u_k^f = u_k^r, \tag{46a}$$

$$D_1(q^{-1})\, u_k = u_k^f, \tag{46b}$$

where $H^*(q^{-1})$ and $K^*(q^{-1})$ are polynomials of degrees $\bar{n}_a+n_{d1}-1$ and $\bar{n}_b+d-1$ respectively. Combining (1) and (46) we get

$$[A(q^{-1})D_1(q^{-1})K^*(q^{-1})+q^{-d}B(q^{-1})H^*(q^{-1})]\,y_k = \\ = q^{-d}B(q^{-1})u_k^r+K^*(q^{-1})D_1(q^{-1})d_k. \tag{47}$$

Thus, the control objective will be satisfied if the polynomials $H^*(q^{-1})$ and $K^*(q^{-1})$ fulfil

$$A(q^{-1})D_1(q^{-1})K^*(q^{-1})+q^{-d}B(q^{-1})H^*(q^{-1})=C(q^{-1}). \tag{48}$$

Finally, note that the control law (46) removes the known frequency components of the disturbance from the output.

When $A(q^{-1})$ and $B(q^{-1})$ are unknown we can apply the adaptive control strategy which consists in recursive estimation of the coefficients of $H^*(q^{-1})$ and $K^*(q^{-1})$ and in application of the control law

$$u_k^f = u_k^r - \sum_{i=0}^{\bar{n}_a+n_{d1}-1} \hat{h}_{ik}\, y_{k-i} - \sum_{j=1}^{\bar{n}_b+d-1} \hat{k}_{jk}\, u_{k-j}^f, \tag{49a}$$

$$D_1(q^{-1})\,u_k = u_k^f, \tag{49b}$$

where $\hat{h}_{ik}$ and $\hat{k}_{jk}$ are the estimates of h_i^* and k_j^* (we recall that $k_0^*=1$). An appropriate model for estimating the coefficients of $H^*(q^{-1})$ and $K^*(q^{-1})$ is given in the following theorem [8].

THEOREM 3.1. *Consider the plant* (1)–(5). *Let the polynomials* $H^*(q^{-1})$ *and* $K^*(q^{-1})$ *satisfy the equation* (48). *Then there exist polynomials* $V^*(q^{-1})$ *and* $W^*(q^{-1})$ *of degrees* $\bar{n}_a+\bar{n}_d-1$ *and* $\bar{n}_b+\bar{n}_{d2}-1$ *respectively such that*

$$V^*(q^{-1})C(q^{-1})\,y_{k+d}+W^*(q^{-1})C(q^{-1})\,u_k^f = \\ = H^*(q^{-1})\,y_{k-\bar{n}_a-\bar{n}_b-\bar{n}_d+1}+K^*(q^{-1})\,u_{k-\bar{n}_a-\bar{n}_b-\bar{n}_d+1}^f \tag{50}$$

for every plant input sequence $\{u_k\}$.

The equation (50) can be rewritten in the following form

$$v_{pk} = x_{pk}^T\,\theta_p^*, \tag{51}$$

if we assume

$$v_{pk} = u_{k-\bar{n}_a-\bar{n}_b-\bar{n}_d-d+1}^f, \tag{52}$$

$$x_{pk} = [-y_{k-\bar{n}_a-\bar{n}_b-\bar{n}_d-d+1} \dots -y_{k-2\bar{n}_a-\bar{n}_b-2n_{d1}-\bar{n}_{d2}-d+2} \\ -u_{k-\bar{n}_a-\bar{n}_b-\bar{n}_d-d}^f \dots -u_{k-\bar{n}_a-2\bar{n}_b-\bar{n}_d-2d+2}^f\; C(q^{-1})\,y_k \dots \\ \dots C(q^{-1})\,y_{k-\bar{n}_a-\bar{n}_d+1}\; C(q^{-1})\,u_{k-d}^f \dots C(q^{-1})\,u_{k-\bar{n}_b-\bar{n}_{d2}-d+1}^f]^T, \tag{53}$$

$$\theta_p^* = [h_0^* \dots h_{\bar{n}_a+n_{d1}-1}^*\; k_1^* \dots k_{\bar{n}_b+d-1}^*\; v_0^* \dots \\ \dots v_{\bar{n}_a+\bar{n}_d-1}^*\; w_0^* \dots w_{\bar{n}_b+\bar{n}_{d2}-1}^*]^T. \tag{54}$$

The full adaptive control algorithm is described by the set of the equations (52), (53), (49), (28) with v_k, x_k, $\hat{\theta}_k$ replaced by v_{pk}, x_{pk}, $\hat{\theta}_{pk}$ in (28).

REMARK 3.1: The PPAC algorithm described above removes the known frequency components of the disturbance $\{d_k\}$ from the plant output.

Parameter convergence

THEOREM 3.2. *Consider the adaptive system* (1)–(5), (28), (49), (52), (53) (*with* v_k, x_k, $\hat{\theta}_k$ *replaced by* v_{pk}, x_{pk}, $\hat{\theta}_{pk}$ *in* (28)). *If* $\{u_k\}$, $\{y_k\}$ *and* $\{u_k^r\}$ *are bounded and the sequence* $\{D_2(q^{-1})\, u_k^r\}$ *is PE with richness* $2\bar{n}_a+2\bar{n}_b+2n_{d1}+\bar{n}_{d2}+d-1$ *then the estimated parameters* $\hat{h}_{0k}, \ldots, \hat{h}_{n_a+n_{d1}-1k}, \hat{k}_{1k}, \ldots, \hat{k}_{n_b+d-1k}$ *converge exponentially to* $h_0^*, \ldots, h_{n_a+n_{d1}-1}^*, k_1^*, \ldots, k_{n_b+d-1}^*$.

Proof. The proof of Theorem 3.2 differs slightly froom the proofs of Theorems 2.1 and 2.2. Namely, instead of the property (7) which has no equivalent now we shall utilize the following properties of the estimation algorithm

1. $\lim_{k\to\infty} (\hat{\theta}_{k+1}-\hat{\theta}_k) = 0,$ (55)
2. the sequence $\{\hat{\theta}_k\}$ is bounded. (56)

(In fact, the properties (55) and (56) have been also used implicitly in the proofs of Theorem 2.1 and 2.2, since they just imply the boundedness of the plant input and output and the convergence property (7)). Note that (55) and (56) denote that for large k the control law (49) is approximately time-invariant. To not lenghten the proof too much we shall act in the sequel as if the control law (49) were time-invariant exactly, that is as if

$$\hat{H}(q^{-1})\, y_k + \hat{K}(q^{-1})\, u_k^f = u_k^r, \tag{57}$$

for every k. It should be pointed that this simplification is possible owing to the assumption on the boundedness of $\{y_k\}$, $\{u_k\}$ and $\{u_k^r\}$ (for details see [15], the proof of Lemma 3.3).

Let $\bar{x}_{pk}$ and $\bar{\bar{x}}_{pk}$ denote the column vectors which can be obtained from x_{pk} by removing the components $C(q^{-1})\, y_k, \ldots, C(q^{-1})\, y_{k-\bar{n}_{d2}+n_{d2}+1}$ and the components $C(q^{-1})\, y_k, \ldots, C(q^{-1})\, y_{k-\bar{n}_{d2}+1}$ respectively. Note that utilizing the equations $A(q^{-1})\, D(q^{-1})\, [C(q^{-1})\, y_k] = q^{-d} B(q^{-1})\, D_2(q^{-1})\, [C(q^{-1})\, u_k^f]$ many times each of the components $C(q^{-1})\, y_k, \ldots, C(q^{-1})\, y_{k-\bar{n}_{d2}+n_{d2}}$ of the vector x_{pk} can be expressed in terms of its remaining components, that is of the components of $\bar{x}_{pk}$. Therefore

$$x_{pk} = K_{p1}\, \bar{x}_{pk}, \tag{58}$$

for a full rank matrix K_{p1}. We shall show that $\{\bar{x}_{pk}\}$ is PS. Utilizing the equation $A(q^{-1})\, D_1(q^{-1})\, [C(q^{-1})\, y_k] = q^{-d} B(q^{-1})\, [C(q^{-1})\, u_k^f] + C(q^{-1})\, d_k^f$

each of the components $C(q^{-1})\, y_{k-\bar{n}_{d2}+n_{d2}}, \ldots, C(q^{-1})\, y_{k-\bar{n}_{d2}}$ can be expressed in terms of the components of the vector $\bar{x}_{pk}$ and of $C(q^{-1})\, d^f_{k-\bar{n}_{d2}+n_{d2}}, \ldots$ $\ldots, C(q^{-1})\, d^f_{k-\bar{n}_{d2}}$. Therefore

$$\bar{x}_{pk} = K_{p2} \begin{bmatrix} C(q^{-1})\, d^f_{k-\bar{n}_{d2}+n_{d2}} \\ \vdots \\ C(q^{-1})\, d^f_{k-\bar{n}_{d2}} \\ \hline \bar{\bar{x}}_{pk} \end{bmatrix} \tag{59}$$

for a square matrix K_{p2}. It can be shown by inspection that K_{p2} is nonsingular. Further, from (8), (10) and (57) we get

$$\hat{C}(q^{-1})\, D_2(q^{-1})\, y_k = q^{-d} B(q^{-1})\, D_2(q^{-1})\, u^r_k, \tag{60a}$$

$$\hat{C}(q^{-1})\, D_2(q^{-1})\, u^f_k = A(q^{-1})\, D_2(q^{-1})\, u^r_k \tag{60b}$$

where $\hat{C}(q^{-1}) = A(q^{-1})\, D_1(q^{-1})\, \hat{K}(q^{-1}) + q^{-d} B(q^{-1})\, \hat{H}(q^{-1})$. Hence

$$\hat{C}(q^{-1})\, D_2(q^{-1})\, \bar{\bar{x}}_{pk} = \begin{bmatrix} q^{-\bar{n}_a-\bar{n}_b-\bar{n}_d-d+1} B(q^{-1}) \\ \vdots \\ q^{-2\bar{n}_a-\bar{n}_b-2n_{d1}-\bar{n}_{d2}-d+2} B(q^{-1}) \\ -q^{-\bar{n}_a-\bar{n}_b-\bar{n}_d} A(q^{-1})\, D_1(q^{-1}) \\ \vdots \\ -q^{-\bar{n}_a-2\bar{n}_b-\bar{n}_d-d+2} A(q^{-1})\, D_1(q^{-1}) \\ q^{-\bar{n}_{d2}} C(q^{-1})\, B(q^{-1}) \\ \vdots \\ q^{-\bar{n}_a-\bar{n}_d+1} C(q^{-1})\, B(q^{-1}) \\ C(q^{-1})\, A(q^{-1})\, D_1(q^{-1}) \\ \vdots \\ q^{-\bar{n}_b-\bar{n}_{d2}+1} C(q^{-1})\, A(q^{-1})\, D_1(q^{-1}) \end{bmatrix} u^r_{k-d} = $$

$$= \Lambda_p \begin{bmatrix} u^r_{k-d} \\ u^r_{k-d-1} \\ \vdots \\ \vdots \\ \vdots \\ u^r_{k-d-m+1} \end{bmatrix} \tag{61}$$

where Λ_p is a square matrix defined by the second identity in (61) and

$m=2\bar{n}_a+2\bar{n}_b+2n_{d1}+\bar{n}_{d2}+d-1$. We shall show that Λ_p is nonsingular. To this end, it is sufficient to show that the components of the polynomial vector in the middle term of (61) are linearly independent over reals, or equivalently, that the polynomial equation

$$
\begin{aligned}
&[A(q^{-1})D_1(q^{-1})\tilde{W}(q^{-1})+q^{-n_{d2}}B(q^{-1})\tilde{V}(q^{-1})]\,C(q^{-1})=\\
&\quad=q^{-\bar{n}_a-\bar{n}_b-\bar{n}_d}[A(q^{-1})D_1(q^{-1})\tilde{K}(q^{-1})+q^{-d+1}B(q^{-1})\tilde{H}(q^{-1})] \qquad (62)
\end{aligned}
$$

has no solution with respect to the polynomials $\tilde{V}(q^{-1})$, $\tilde{W}(q^{-1})$, $\tilde{H}(q^{-1})$ and $\tilde{K}(q^{-1})$ of degrees $\bar{n}_a+n_{d1}-1$, $\bar{n}_b+\bar{n}_{d2}-1$, $\bar{n}_a+n_{d1}-1$ and $\bar{n}_b+d-2$ respectively. We have the following: $f\tilde{V}(q^{-1})$ and $\tilde{W}(q^{-1})$ do not both equal zero then the first term of the polynomial on the left side of (62) is of degree non lower than $\bar{n}_a+\bar{n}_b+\bar{n}_d-1$, but, on the other hand, each term of the polynomial on the right side of (62) is of degree non greater than $\bar{n}_a+\bar{n}_b+\bar{n}_d$. Therefore $\tilde{V}(q^{-1})=\tilde{W}(q^{-1})=0$. Hence, in view of the restrictions $\deg\tilde{K}=\bar{n}_b+d-2$ and $\deg\tilde{H}=\bar{n}_a+n_{d1}-1$, $\tilde{H}(q^{-1})=\tilde{K}(q^{-1})=0$. Consequently, the matrix Λ_{pk} is nonsingular. Now, applying Lemmas 2.2 and 2.4 and the assumptions of Theorem 3.2 one establishes consecutively that $\{\hat{C}(q^{-1})D_2(q^{-1})\bar{x}_{pk}\}$, $\{\bar{x}_{pk}\}$ and, finally, $\{x_{pk}\}$ are PE. Thus, by Lemma 2.1 the sequence $\{\hat{\theta}_{pk}\}$ converges exponentially and $\lim_{k\to\infty}\hat{\theta}_{pk}\in\Theta_p$ where

$$\Theta_p=\{\hat{\theta}\in\mathbf{R}^{n_{\theta p}}:(\theta_p^*-\hat{\theta})^T K_{p1}=0\}. \qquad (63)$$

Let us consider the set

$$
\begin{aligned}
\bar{\Theta}_p=\{[\hat{h}_0\,\ldots\,\hat{h}_{\bar{n}_a+n_{d1}-1}\;\hat{k}_1\,\ldots\,\hat{k}_{\bar{n}_b+d-1}\;\hat{v}_0\,\ldots\,\hat{v}_{\bar{n}_a+\bar{n}_d-1}\;\hat{w}_0\,\ldots\\
\ldots\,\hat{w}_{\bar{n}_b+\bar{n}_{d2}-1}]^T:\hat{H}(q^{-1})=H^*(q^{-1}),\,\hat{K}(q^{-1})=\\
=K^*(q^{-1}),\,A(q^{-1})D_1(q^{-1})\hat{V}(q^{-1})L(q^{-1})+\\
+B(q^{-1})\hat{W}(q^{-1})L(q^{-1})=q^{-\bar{n}_a-\bar{n}_b-\bar{n}_d+1}\text{ for some polynomial}\\
L(q^{-1})=1+l_1q^{-1}+\ldots+l_{\bar{n}_{d2}-n_{d2}}q^{-\bar{n}_{d2}+n_{d2}}\}. \qquad (64)
\end{aligned}
$$

Using the arguments of the proof of Theorem 3.1 (see [8]) it can be shown that $\forall\hat{\theta}_p\in\bar{\Theta}_p$ and $\forall\{u_k\}$ the identity (50) holds. Therefore, $\bar{\Theta}_p\subset\Theta_p$. On the other hand, $\bar{\Theta}_p$ and $\bar{\Theta}_p$ are some hyperplanes of the same dimension $\bar{n}_{d2}-n_{d2}$. Consequently, $\Theta_p=\bar{\Theta}_p$. Hence, in light of (64) and $\lim_{k\to\infty}\hat{\theta}_{pk}\in\Theta_p$, Theorem 3.2 holds. ∎

In the proof given above, similarly as in the proofs of Theorems 2.1 and 2.2, for establishing that the external excitation causes the proper plant excitation one utilizes the assumption on the plant signal boundedness. Contrary to the MRAC algorithms, however, in the PPAC algorithm the sequences $\{y_k\}$ and $\{u_k\}$ can be unbounded when the plant excitation condition is not satisfied and the initial parameter error $\tilde{\theta}_0 = \theta - \hat{\theta}_0$ is sufficiently large. Because of this we have had to insert the assumption on the boundedness of $\{y_k\}$ and $\{u_k\}$ directly into the text of Theorem 3.2. This assumption can be, however, omitted when instead of (49a) we apply a piece-wise constant parameter control law similar to that proposed in [16]. Namely, let us assume that (49a) is replaced by

$$\hat{H}_{im}(q^{-1})\, y_k + \hat{K}_{im}(q^{-1})\, u_k^f = u_k^r,\ k = im, \ldots, (i+1)m-1,\ i = 0, 1, \ldots \tag{65}$$

where $m = 2n_a + 2n_b + 2n_{d1} + n_{d2} + d - 1$. Now, combining (8), (10) and (65) we get (without assuming that $\hat{H}_k$ and $\hat{K}_k$ are time-invariant as in (57))

$$\hat{C}_{im}(q^{-1})\, D_2(q^{-1})\, \bar{x}_{p(i+1)m+d-1} = \Lambda_p \begin{bmatrix} u_{(i+1)m-1} \\ \vdots \\ u_{im} \end{bmatrix} \qquad i = 0, 1, \ldots$$

where $\hat{C}_{im} = AD_1 \hat{K}_{im} + q^{-d} B\hat{H}_{im}$. Hence, the sequence of vectors $x_{p(i+1)m+d-1}$, $i = 0, 1, \ldots$ (and consequently the sequence $\{x_{pk}\}$) is PS. This implies $\hat{h}_{ik} \to h_i^*$ and $\hat{k}_{jk} \to k_j^*$.

Discussion of stability

Below, we shall discuss briefly the issue of the stability of the PPAC algorithm considered in this section in the case of absence of the external excitation.

First of all, note that the local stability result (local with respect to the initial values of the estimation algorithm (28)) follows directly from the following property of the estimation algorithm (28).

$$\|\theta^* - \hat{\theta}_k\|_{Q^{-1}} \leqslant \|\theta^* - \hat{\theta}_0\|_{Q^{-1}}. \tag{66}$$

Indeed, note that the system (1), (49) with $\hat{h}_{ik} = h_i^*$ and $\hat{k}_{ik} = k_i^*$ is stable exponentially. Thus, by linearity of the equations (1) and (49) the system (1), (49) remains stable if $|\hat{h}_{ik} - h_i^*| \leqslant \varepsilon$ and $|\hat{k}_{ik} - k_i^*| \leqslant \varepsilon$ for a sufficiently small ε or, in view of (66), if the initial parameters estimation error $\|\theta_p^* - \hat{\theta}_{p0}\|$ is sufficiently small. Utilizing additionally the property (55) (that is the asymptotic time-invariantness of (49)) one can show easily that the PPAC system is stable if $\hat{\theta}_{p0}$ belongs to the region

$$\{\hat{\theta}_p \in \mathbf{R}^{n_{0p}}: \text{the polynomial } A(q^{-1})\, D_1(q^{-1})\, \tilde{K}(q^{-1}) + q^{-d} B(q^{-1})\, \tilde{H}(q^{-1})$$
$$\text{is asymptotically stable for every } \tilde{\theta}_p = [\tilde{h}_0 \ldots \tilde{w}_{\bar{n}_b + n_{d2} - 1}]^T$$
$$\text{such that } \|\theta_p^* - \tilde{\theta}_p\|_{Q^{-1}} \leqslant \|\theta_p^* - \hat{\theta}_p\|_{Q^{-1}}\}.$$

The stronger result can be received if together with (55) and (66) one utilizes also the following property of the algorithm (28)

$$\lim_{k \to \infty} (v_{pk} - x_k^T \hat{\theta}_k) = 0. \tag{67}$$

Namely, using (55), (66) and (67) one can establish the following stabilizing property of the PPAC algorithm

THEOREM 3.3. *Consider the adaptive system described by* (1)–(5), (28) (49), (52), (53) (*with* v_k, x_k, $\hat{\theta}_k$ *replaced by* v_{pk}, x_{pk}, $\hat{\theta}_{pk}$ *in* (28)). *If there exists a closed subset* $\hat{\varphi}$ *of the set*

$$\varphi = \{\hat{\theta}_p = [\hat{h}_0 \dots \hat{w}_{\bar{n}_b+\bar{n}_{d2}-1}]^T: \text{the polynomials } A(q^{-1})\, D_1(q^{-1})\, \hat{K}(q^{-1}) + $$
$$+q^{-d} B(q^{-1})\, \hat{H}(q^{-1}) \text{ and } A(q^{-1})\, D_1(q^{-1})\, \hat{W}(q^{-1}) + B(q^{-1})\, \hat{V}(q^{-1})$$
$$\text{have no common zeros in the region } |q^{-1}| \leqslant 1\}, \tag{68}$$

such that $\hat{\theta}_{pk} \in \hat{\varphi}$ *for all sufficiently large* k *then* $\{u_k\}$ *and* $\{y_k\}$ *are bounded.*
Theorem 3.3 implies that the PPAC system is stable if $\hat{\theta}_{p0}$ belongs to the region

$$\{\hat{\theta}_p \in \mathbf{R}^{n_{\theta p}}: \tilde{\theta} \in \varphi \quad \forall \tilde{\theta} \text{ such that } \|\theta_p^* - \tilde{\theta}\|_{Q^{-1}} \leqslant \|\theta_p^* - \hat{\theta}_p\|_{Q^{-1}}\}. \tag{69}$$

For more detailed discussion on stability of the PPAC algorithm the reader is referred to [17].

References

[1] ANDERSON B. D. O., JOHNSON C. R. Jr. Exponential convergence of adaptive identification and control algorithms. *Automatica*, **18**, (1982) 1–18.

[2] BOYD S., SASTRY S. On parameter convergence in adaptive control. *Systems Contr. Letters*, **3** (1983), 311–319.

[3] JANECKI D. Globally stable and exponentially convergent adaptive control. *Int. J. Contr.*, **43** (1986), 611–613.

[4] ANDERSON B. D. O. Exponential convergence and persistent excitation. Proc. 21st IEEE CDC, pp. 12–17, 1982.

[5] ANDERSON B. D. O. Adaptive systems, lack of persistency of excitation and bursting phenomena. *Automatica*, **21** (1985), 247–258.

[6] ANDERSON B. D. O., JOHNSTON R. M. Adaptive systems and time-varying plants. *Int. J. Contr.*, **37** (1983) 367–377.

[7] GOODWIN G. C. CHAN S. W. Model reference adaptive control of system having purely deterministic disturbances. *IEEE Trans. Automat. Contr.*, **AC-28** (1983), 855–858.

[8] JANECKI D. Global adaptive pole placement algorithm for plants with purely deterministic disturbances. *IEEE Trans. Automat. Contr.*, **AC-32** (1987), 187–189.

[9] ELLIOTT H. Direct adaptive pole placement with application to nonminimumphase systems. *IEEE Trans. Automat. Contr.*, **AC-27** (1982), 720–722.

[10] JANECKI D. Remarks on persistency of excitation for nonminimal models of systems having purely deterministic disturbances. *IEEE Trans. Automat. Contr.*, to be published.

[11] JANECKI D. New recursive parameter estimation algorithm with bounded gain matrix. *Int. J. Contr.*, to be published.

[12] GOODWIN G. C., PALANISWAMI D. J. A perspective on convergence of adaptive control algorithms. *Automatica*, **20** (1984), 519–511.

[13] GOODWIN G. C., RAMADGE P. J., CAINES P. E. Discrete-time multivariable adaptive control. *IEEE Trans. Automat. Contr.*, **AC-25** (1980), 449–592.

[14] ELLIOTT H. Hybrid adaptive control of continuous time systems. *IEEE Trans. Automat. Contr.*, **AC-27** (1982), 2.

[15] ANDERSON B. D. O., JOHNSTONE R. M. Global adaptive pole positioning. *IEEE Trans. Automat. Contr.*, **AC-30** (1985), 11–21.

[16] ELLIOTT H., CRISTI R., DAS M. Global stability of adaptive pole placement algorithms. *IEEE Trans. Automat. Contr.*, **AC-30** (1985), 348–356.

[17] JANECKI D. Stability analysis of Elliott's direct adaptive pole placement algorithm. *System Contr. Letters*, submitted for publication.

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Zbieżność adaptacyjnych algorytmów sterowania dla obiektów z czysto deterministycznymi zakłóceniami

W pracy podano warunki parametrycznej zbieżności trzech adaptacyjnych algorytmów sterowania dla obiektów z zakłóceniami, które mogą być modelowane w postaci sumy sygnałów sinusoidalnych. Rozpatrzono następujące algorytmy: algorytm sterowania typu bezpośredniego z modelem odniesienia, algorytm sterowania typu pośredniego z modelem odniesienia oraz algorytm sterowania typu bezpośredniego zapewniający przesunięcie biegunów w zadane położenie. W odróżnieniu od wcześniejszych prac dotyczących zbieżności algorytmów adaptacyjnych, w pracy rozważany jest ogólny przypadek, gdy estamacja parametrów obiektu (regulatora) przeprowadzana jest na podstawie modelu, który jest niejednoznaczny względem parametrów.

Сходимость адаптивных алгоритмов управления для объектов со строго детерминированными помехами

В работе приведены условия параметрической сходимости трех адаптивных алгоритмов управления для объектов с помехами, которые могут моделироваться в виде суммы синусоидальных сигналов. Рассмотрены следующие алгоритмы: алгоритм управления непосредственного типа с моделью отнесения, алгоритм управления посредственного типа с моделью отнесения, а также алгоритм управления непосредственного типа, обеспечивающий сдвиг полюсов в заданное положение. В отличие от более ранних работ, касающихся сходимости адаптивных алгоритмов в работе рассматривается общий случай, когда оценка параметров объекта (регулятора) проводится на основе модели, которая является неоднозначной по отношению к параметрам.