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A counterexample to transformations in multiphase and sequential linear goal programming

by

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We consider the lexicographic linear goal programming model. The two most common solution techniques to this model are the sequential process and the multiphase process. Both the methods produce, obviously, the same solution but the interior elements of the specific tableaus differ significantly. Markowski and Ignizio proposed an algorithm for transformation of the sequential tableau into the multiphase one and vice-versa. We present an example which shows failure of this algorithm with respect to saving optimality (dual feasibility) of the transformed tableau while degeneracy occurs.

1. Introduction

In this note we deal with lexicographic linear goal programming, i.e., with the specific form of linear goal programming wherein one seeks the lexicographic minimum of an ordered set of goal deviations. This approach, also described as preemptive priority based goal programming, is widely used in multiobjective optimization.

The lexicographic linear goal programming (LGP) problem is usually given as follows.

Find a vector x so as to lexicographically minimize

$$a = [g_1(n, p), g_2(n, p), ..., g_K(n, p)]^T, \tag{1}$$

subject to

$$\sum_{j \in J} c_{ij} x_j + n_i - p_i = b_i \quad \text{for } i \in I, \tag{2}$$

$$x \geqslant 0, \quad n \geqslant 0, \quad p \geqslant 0, \tag{3}$$

where

J — set of decision variable indices,

I — set of goal indices,
x_j — j-th decision variable,
c_{ij} — coefficient of variable j in the i-th goal constraint,
b_i — target for goal i,
n_i — negative deviation for goal i,
p_i — positive deviation for goal i,
$g_k(n, p)$ — linear function of the deviation variables to be minimized at priority k.

The two most common solution techniques for LGP problems are: sequential process (known as sequential linear GP or SLGP [2]) and the modified simplex procedure which is known as the multiphase process or MLGP [1]. Both the methods have the same conceptional basis.

Due to lexicographic minimization, the optimal solution to the LGP problem is defined as follows:

1) find S_1 as the optimal set to the problem
P_1: min $\{g_1(n, p)$ subject to (2) and (3)$\}$,
2) for $k = 2, 3, \ldots, K$ find S_k as the optimal set to the problem
P_k: min $\{g_k(n, p)$ subject to (2), (3) and $x \in S_{k-1}\}$,
3) any vector of the set S_K is optimal to the LGP problem.

Both the methods SLGP and MLGP are based on the above scheme. They differ only in techniques used for introducing the additional requirement $x \in S_{k-1}$ into the problem P_k.

In the SLGP approach the requirement $x \in S_{k-1}$ is represented by the equality system

$$
\begin{aligned}
g_1(n, p) &= \bar{a}_1 \\
g_2(n, p) &= \bar{a}_2 \\
&\ldots \\
g_{k-1}(n, p) &= \bar{a}_{k-1}
\end{aligned}
$$

where $\bar{a}_i$ denotes the optimal value to the problem P_i. This method is very convenient when the standard simplex codes are used. On the other hand, sensitivity and parametric analysis is extremely difficult in the SLGP approach since a dual solution to the LGP problem is then not available.

The MLGP method utilizes specifity of the simplex algorithm for introducing the requirement $x \in S_{k-1}$. Namely, only variables having all the reduced costs (optimality indices [1]) $I_{1,s}, I_{2,s}, \ldots, I_{k-1,s}$ equaled to zero are allowed to be positive in the problem P_k. Such an approach is equivalent to the lexicographic simplex method [3]. The optimal MLGP tableau contains a dual optimal solution to the LGP problem, so that sensitivity and parametric analysis can be easily carry out in the MLGP method. There are, however, difficulties with using standard simplex codes for implementation of this method.

Whether one employs the SLGP or MLGP method the final solutions to the LGP problem are, obviously, the same. However, the interior elements of the specific tableaus will differ considerably. Markowski and Ignizio [4] proposed some algorithm which allows to transform the SLGP tableau into the MLGP tableau. Such a transformation is very useful since it makes possible to utilize extremely efficient standard simplex codes for solving the problem via SLGP approach and next to perform sensitivity analysis using the MLGP tableau. Unfortunately, we have found out that the algorithm fails while degeneracy occurs. Namely, for some problems it transforms the optimal (final) SLGP tableau into a MLGP tableau which does not satisfy optimality conditions (i.e., generated dual solution is infeasible). In this note we present such an example.

2. The counterexample

Consider the following LGP problem
lexmin $[(n_1+n_2), (n_2+p_2+n_3)]'$ subject to

$$\begin{aligned} x_1 + \quad + n_1 - p_1 &= 1 \\ x_1 + x_2 + n_2 - p_2 &= 2 \\ x_2 + n_3 - p_3 &= 1 \\ x \geqslant 0, \quad n \geqslant 0, \quad p &\geqslant 0 \end{aligned}$$

We solve this problem via the SLGP approach. First, the problem P_1 is solved, i.e., the achievement function $a_1 = n_1 + n_2$ is minimized. The initial simplex tableau appears in Table 1 (in the same form as in [4]).

Table 1. Priority level one: initial tableau

	x_1	x_2	p_1	p_2	p_3	
n_1	1	0	−1	0	0	1
n_2	1	1	0	−1	0	2
n_3	0	1	0	0	−1	1
	2	1	−1	−1	0	3

x_1 enters the basis, replacing n_1, and thereby yielding the second tableau given in Table 2.

Table 2. Priority level one: second tableau

	x_2	n_1	p_1	p_2	p_3	
x_1	0	1	−1	0	0	1
n_2	1	−1	1	−1	0	1
n_3	1	0	0	0	−1	1
	1	−2	1	−1	0	1

Now x_2 is chosen to enter the basis and n_2 leaves the basis. The iteration is performed to yield the tableau given in Table 3.

Table 3. Priority level one: third tableau

	n_1	n_2	p_1	p_2	p_3	
x_1	1	0	−1	0	0	1
x_2	−1	1	1	−1	0	1
n_3	1	−1	−1	1	−1	0
	−1	−1	0	0	0	0

The last tableau is optimal to the problem P_1 (priority level one). The solution is: $x = (1, 1)^T$, $n = (0, 0)^T$, $p = (0, 0)^T$ and $a_1 = 0$.

Next we solve the problem P_2. Similarly as in [4], we introduce two additional equalities:

$$-n_1 - n_2 + r_- = 0$$

$$n_1 + n_2 + r_+ = 0$$

and compute reduced costs for the achievement function

$$a_2 = n_2 + p_2 + n_3$$

The initial tableau to the problem P_2 appears in Table 4.

Table 4. Priority level two: initial tableau.

	n_1	n_2	p_1	p_2	p_3	
x_1	1	0	−1	0	0	1
x_2	−1	1	1	−1	0	1
n_3	1	−1	−1	1	−1	0
r_-	−1	−1	0	0	0	0
r_+	1	1	0	0	0	0
	1	−2	−1	0	−1	0

This tableau is nonoptimal. The variable n_1 enters basis, replacing n_3, and thereby yielding the second tableau given in Table 5.

Table 5. Priority level two: second (final) tableau.

	n_2	n_3	p_1	p_2	p_3	
x_1	1	−1	0	−1	1	1
x_2	0	1	0	0	−1	1
n_1	−1	1	−1	1	−1	0
r_-	−2	1	−1	1	−1	0
r_+	2	−1	1	−1	1	0
	−1	−1	0	−1	0	0

The last tableau proves to be optimal to the problem P_2 and thereby it is an SLGP final tableau. The optimal solution to the original LGP problem is:

$$\bar{x} = (1, 1)^T,\ \bar{n} = (0, 0)^T,\ \bar{p} = (0, 0)^T \text{ and } \bar{a} = (0, 0)^T.$$

We now transform Table 5 into the corresponding MLGP tableau using the algorithm proposed in [4]. The main operation performed in the algorithm depends on forcing r_- and r_+ into the basis. In our case, however, r_- and r_+ have already stayed in the basis. So, the algorithm transforms only form of the tableau without any change of the basis structure. In effect we get the MLGP tableau given in Table 6. The tableau is evidently nonoptimal since there are some positive elements in the P_1 index row.

Table 6. Final SLGP tableau transformed into the MLGP form

		$P2$	1	1	0	1	0	
		$P1$	1	0	0	0	0	
$P2$	$P1$	V	n_2	n_3	p_1	p_2	p_3	$\bar{b}$
0	0	x_1	1	-1	0	-1	1	1
0	0	x_2	0	1	0	0	-1	1
0	1	n_1	-1	1	-1	1	-1	0
		$P1$	-2	1	-1	1	-1	0
		$P2$	-1	-1	0	-1	0	0

In other words, we get a tableau which generates an optimal solution to the LGP problem but the dual solution generated by the tableau is infeasible and cannot be used in sensitivity analysis. This phenomenon can be easily explained by careful analysis of the transformation proposed in [4]. As we have already mentioned the main operation performed in the algorithm depends on forcing r_- and r_+ variables into the basis. Such an operation guarantees that the transformation yields some basis to the optimal solution of the MLGP problem. However, if degeneracy occurs then some bases to the optimal solution can be nonoptimal. So, Markowski and Ignizio use degeneracy of the SLGP problem for forcing the r_- and r_+ variables into basis and, simultaneously, they ignore consequences of degeneracy in the original LGP problem. Thus advantages of the transformation proposed in [4] seem to be limited to rather theoretical class of nondegenerated LGP problems whereas real-life LGP problems are usually strongly degenerated.

Moreover, note that the basis consisting of the variables x_1, x_2 and n_1 is optimal in the SLGP approach and it is nonoptimal in the MLGP approach. One can easily verify that the tableau given in Table 4, which is nonoptimal to the SLGP approach, would be transformed into an optimal MLGP tableau. So, the basis optimal in one approach can be nonoptimal in the second approach and vice versa. The above proves that these two approaches to the LGP problems should be regarded as inconvertible with respect to optimal basis structure.

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Kontrprzykład na przekształcenia tablic sympleksowych w hierarchicznym i wielofazowym podejściu do leksykograficznego programowania celowego

Leksykograficzne programowanie celowe (LPC) jest szeroko stosowanym narzędziem analizy problemów wielokryterialnych. Istnieją dwie podstawowe metody rozwiązywania zadań LPC: optymalizacja hierarchiczna i wielofazowy algorytm sympleks. Oba podejścia wyznaczają te same wektory optymalne, ale odpowiadają im różne tablice sympleksowe. Tablica sympleksowa dla wielofazowego algorytmu sympleks zawiera jednocześnie rozwiązanie dualne i inne elementy potrzebne do analizy wrażliwości. Własności tej nie posiada tablica otrzymana w wyniku realizacji łatwiej impelementowanej optymalizacji hierarchicznej. Okazuje się, że nie ma możliwości wzajemnego przekształcania tych tablic (po rozwiązaniu zadania), gdyż każda z nich może być generowana przez inną bazę optymalną.

Контрпример преобразования симплексных таблиц в иерархическом и многофазном подходе к лексикографическому целевому программированию

Лексикографическое целевое программирование (ЛЦП) является широкоприменяемым инструментом анализа многокритериальных задач. Существуют два основные метода решения задач ЛЦП: иерархическая оптимизация и многофазный симплекс-алгоритм. Оба подхода определяют те же оптимальные векторы, однако им соответствуют разные симплексные таблицы. Симплексная таблица для многофазного симплекс-алгоритма одновременно содержит дуальное решение и другие элементы, необходимые для анализа чувствительности. Этим свойством не обладает таблица получаемая в результате реализации, более удобной в применении, иерархической оптимизации. Оказывается, что отсутствует возможность взаимного преобразования этих таблиц (после решения задачи), поскольку каждая из них может быть генерирована другой оптимальной базой.