Control
and Cybernetics

VOL. 17 (1988) No. 1

On the bipartite crossing number

by

MICHAEL MAY

Central Institute of Cybernetics
and Information Processes
Academy of Sciences of the G.D.R.
Kurstrasse 33, 1086 Berlin, G.D.R.

KRZYSZTOF SZKATUŁA

Systems Research Institute
Polish Academy of Sciences
ul. Newelska 6, 01-447 Warszawa, Poland

In this paper we present some results on a special crossing number of bipartite graphs. This so-called bipartite crossing number (BCN) is defined by the least number of edge-crossing appearing in the graph when a special topological representation in the plane is used. Determining the BCN of a graph is known to be NP-hard. After the statement of the problem we report several estimations of the BCN [12]. Connections to related graph and matrix problems are outlined. Then a thermodynamically motivated simulation procedure for determining the p-partite crossing number ($p \geq 2$) together with computational results is presented. Finally, we point at several applications of the BCN problem to the automatic layout of interconnected systems, such as electronic circuits, schematics, and facilities.

1. Introduction

The crossing number $\underline{\nu}(G)$ of a graph G is defined by the least possible number of crossings which appear when G is mapped into the plane ([7], [6]). The corssing number is an important topological invariant of a graph. $\underline{\nu}(G) = 0$ is valid iff G is a planar graph.

CROSSING NUMBER was shown to be NP-complete [4]. Despite intensive research there are only few graphs for which the crossing number is known and hardly any infinite families. The most studied infinite families are the complete graphs, the complete bipartite graphs, the n-dimensional

cube, and the cross products of some graphs with a simple structure. However, even for these cases only very partial results are known (cf. [16]).

In 1970 Watkins [15] introduced a special crossing number for bipartite graphs as an open research problem; the so-called bipartite crossing number (cf. [4]).

Let $G = (U, V; E)$ be a simple undirected labeled bipartite graph with the vertex set $U \cup V (U \cap V = \emptyset)$ and the edge set E ($|E| = e$). As for labeling we have $U = \{u_1, ..., u_m\}$ and $V = \{v_1, ..., v_n\}$. Let G be embedded in the plane, such that the elements of U are realized as m points on a straight line with u_i lying between u_{i-1} and u_{i+1} $(i = 2, ..., m-1)$; the elements of V become n points on another straight line parallel to the first one and ordered in the same direction. The edges of G are identified with straight line segments joining the two points corresponding to their incident vertices.

Let $N(G)$ be the number of crossings of G, that is, the number of edge pairs of G which intersect when G is represented in the manner just described, not counting the intersections generated by edges incident with a common vertex. By $\{G\}$ we denote the equivalence class of labeled bipartite graphs isomorphic to G. Then the bipartite crossing number (BCN) $\nu_2(G)$ of G is defined by

$$\nu_2(G) = \min (\mathrm{N}(H) | H \in \{G\}).$$

BIPARTITE CROSSING NUMBER is known to be NP-complete [4]. Restricted to bipartite permutation graphs the BCN can be determined in polynomial time [1].

The concept of BCN can easily be extended to p-partite graphs $G = (U, E)$ where $U = U_1 \cup ... \cup U_p$, $p \geqslant 2$, $E = E_1 \cup ... \cup E_{p-1}$ and $E_i \leqslant U_i \times U_{i+1}$, $i = 1, ..., p-1$. Any p-partite graph can be transformed to this special type of p-partite graph by replacing every edge $e = (u, v)$, $u \in U_i$, $v \in U_j$, $1 \leqslant i < j+1 \leqslant p-1$ by a path $(u, e_1, w_1, ..., e_{j-i-1}, w_{j-i-1}, e_{j-i}, v)$ where $e_1 = (u, w_1)$, $e_k = (w_k, w_{k+1})$, $k = 2, ..., j-i-1$, $e_{j-i} = (w_{j-i-1}, v)$ and $w_k \in U_{i+k}$. Analogous to the BCN ($p = 2$), let the elements of U be realized as points on p parallel straight lines, where U_1 belongs to the first line, U_2 to the second, ect. Let $N_p(G)$ be the number of crossings occuring when G is embedded in the plane as described. By $\{G\}$ we denote the equivalence class of labeled p-partite graphs isomorphic to G. Then the p-partite crossing number $\underline{\nu}_p(G)$ of G is defined by

$$\underline{\nu}_p(G) = \min (N_p(H) | H \in \{G\}).$$

— ν_p has been introduced because of its importance to practical applications.

Obviously, we have $\underline{\nu}_p(G) \geqslant \sum_{i=1}^{p-1} \underline{\nu}_2(G_i)$, $G_i = (U_i, U_{i+1}; E_i)$.

2. Some estimations for the bipartite crossing number

In the following we restrict our consideration to $\underline{\nu}_p$ where $p=2$, that is, we consider the bipartite graphs $G=(U, V; E)$ with $U=\{u_1, \ldots, u_m\}$, $V=\{v_1, \ldots, v_n\}$, and $|E|=e$. At first we give an equivalent formulation of $\underline{\nu}_2$ in terms of $m\times n$ (0, 1)-matrices. Let $\bar{A}=(\bar{a}_{ij})$ be the $(m+n)\times(m+n)$ adjacency matrix of G, where

$$\bar{a}_{ij}=\begin{cases}1 \text{ if } i\leqslant m,\ j>m,\ (u_i, v_{j-m})\in E\\ 1 \text{ if } i>m,\ j\leqslant m,\ (u_{i-m}, v_j)\in E\\ 0 \text{ otherwise}\end{cases}$$

Hence
$\bar{A}$ maybe presented in the following way

$$\bar{A}=\begin{array}{c} \\ u_1\\ \vdots\\ u_m\\ v_1\\ \vdots\\ v_n\end{array}\begin{array}{c}\begin{array}{cc}u_1 \ldots u_m & v_1 \ldots v_n\end{array}\\ \left[\begin{array}{c|c} 0 & A\\ \hline A^T & 0\end{array}\right]\end{array}$$

where A^T denotes the transpose of A. A is a $m\times n$ (0, 1)-matrix which we shall refer to as reduced adjacency matrix in what follows. There is a natural one-to-one correspondence between the set of $m\times n$ (0, 1)-matrices A and the set of labeled bipartite graphs G described above. Thus, it is sufficient to consider A instead of $\bar{A}$, where $A=(a_{ij})$ and $a_{ij}=\bar{a}_{i,j+m}$, $i=1, \ldots, m, j=1, \ldots, n$. Let $N(A)$ be the number of distinct (but not necessarily disjoint) 2×2 submatrices of A having the form $\begin{vmatrix}a & 1\\ 1 & b\end{vmatrix}$, $a, b\in\{0, 1\}$. Obviously $N(A)=N(A^T)$.

Taking into account that two edges (u_i, v_j), $(u_k, v_l)\in E$ intersect iff $i>k$, $l>j$ or $i<k$, $l<j$ it follows $N(A)=N(G)$. Hence

$$\underline{\nu}_2(G)=\min\left(N(B)|B\in\{A\}\right),$$

where $\{A\}$ is the set of all matrices obtainable from A by permutations of its rows and columns. The number of crossings of G can be obtained by

$$N(G)=\sum_{i=2}^{m}\sum_{j=1}^{n-1}\sum_{k=1}^{i-1}\sum_{l=j+1}^{n}a_{ij}\,a_{kl}=\sum_{k=1}^{m-1}\sum_{i=k+1}^{m}\sum_{i=2}^{n}\sum_{j=1}^{l-1}a_{ij}\,a_{kl}, \tag{1}$$

and hence

$$\underline{\nu}_2(G)=\min_{\substack{\pi\in S_m\\ \varphi\in S_n}}\sum_{\substack{1\leqslant k\leqslant i<m\\ 1\leqslant j\leqslant l<n}}a_{\pi(i)\varphi(j)}\,a_{\pi(k)\varphi(l)},$$

where S_n is the symmetric group of n objects.

For bipartite multigraphs the a_{ij} in formula (1) must be replaced by the weights (multiplicities) b_{ij} of the edges (u_i, v_j). For the complete bipartite graph $K_{m,n}$ we have $a_{ij} = 1$ for all i, j. Then (1) immediately yields

$$\underline{\nu}_2(K_{m,n}) = \binom{m}{2}\binom{n}{2}.$$

Clearly, for every bipartite graph G it holds

$$\underline{\nu}(G) \leqslant \underline{\nu}_2(G) \leqslant \underline{\nu}_2(K_{m,n}).$$

In the following we summarize some of the results obtained in [12]. The BCN can easily be determined for circuits. By C_l we denote a circuit of lenght $l \geqslant 3$. C_l is bipartite iff $l = 2L$, $L \geqslant 2$. We have

LEMMA 1. $\underline{\nu}_2(C_{2L}) = L-1$, $L \geqslant 2$. [12].

Since the existence of circuits in G is sufficient for the appearance of crossings in G the investigation of the circuits of G yields some nontrivial bounds on $\underline{\nu}_2$ for general bipartite graphs.

A chord of a circuit C is a path with its end-vertices belonging to C but its edges and inner vertices do not.

Let $k_C(G)$ be the number of circuits of G containing no chord of lenght 1 (sometimes called diagonal).

LEMMA 2. $\underline{\nu}_2(G) \geqslant k_C(G)$. [12].

THEOREM 1. *Let $k_4(G)$ be the number of all different circuits C_4 of G. Then*

$$\underline{\nu}_2(G) \geqslant k_4(G) = \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\binom{k_{ij}}{2} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\binom{k'_{ij}}{2}, \tag{3}$$

where

$$k_{ij} = \sum_{l=1}^{n} a_{il}\, a_{lj}, \quad i, j = 1, \ldots, m,$$

$$k'_{ij} = \sum_{l=1}^{m} a_{li}\, a_{lj}, \quad i, j = 1, \ldots, n. \text{ [12].}$$

It is desirable to have estimations of $\underline{\nu}_2$ in terms of some other characteristics of the graph than k_{ij}, e.g. the degree of its vertives or the number of edges. This can most easily be done by further estimating k_4, i.e. the right side of (3). For this reason the following identity proves helpful

$$\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} k_{ij} = \sum_{l=1}^{n}\binom{d(v_l)}{2}, \quad v_l \in V, \quad l = 1, \ldots, n,$$

where (v) denotes the degree of the vertex v. In this way we get from (3)

$$\underline{\nu}_2(G) \geqslant \binom{m}{2}\binom{[a]}{2}, \quad a = \binom{m}{2}^{-1} \sum_{l=1}^{n} \binom{d(v_l)}{2}, \quad [12], \tag{5}$$

where $[a]$ denotes the integer part of a. Note that (5) holds also when m, n, and v_l are replaced by n, m, and u_l, respectively.

Let $k(G)$ be the cyclomatic number of G. For a connected graph $G=(V,E)$ it holds $k(G)=|E|-|V|+1$ [7].

THEOREM 2. *Let G be a connected bipartite graph. Then*

$$\underline{\nu}_2(G) \geqslant k(G) = e-(m+n)+1. \quad [12] \tag{6}$$

COROLLARY. *Let t be the number of the components of the bipartite graph G.*

Then $\underline{\nu}_2(G) \geqslant e-(m+n)+t$. For detailed proofs and some further results see [12].

We note that including the chordless circuits of G with length $l \geqslant 6$ into the consideration would result in stronger bounds on the BCN.

3. Related matrix and graph problems

Since the goal function (1) for the BCN problem is very costly from the computational point of view $(0(m^2 \cdot n^2))$, it would be desirable to approximate an optimal solution for BCN by some appropriate function of less computational complexity. For this purpose we make use of the following observation. In general, an optimal or near-optimal solution to the BCN problem means a clustering of the non-zero entries of the reduced adjacency matrix A of G along its "main diagonal". This diagonal is well-defined only for quadratic matrices. So we introduce a generalization of this concept for arbitrary $m \times n$ matrices A. The main diagonal of A is defined by the linear function $f_{m,n}$ where

$$f_{m,n}(i) = \frac{n-1}{m-1} i + \frac{m-n}{m-1},$$

and i corresponds to the i-th row of A. It holds $f_{m,n}(1)=1$ and $f_{m,n}(m)=n$. This diagonal is also determined by $f_{n,m}(j)$, where j corresponds to the j-th column of A. For quadratic matrices we have $f_{m,m}(i)=i$.

Now we state several problems of minimizing some characteristic of $A=(a_{ij})_{m,n}$ by permuting its rows and columns. Some special cases of these problems are well-known problems in matrix and graph theory.

Bandwidth of A

$$\beta_{m,n}(A) = \min_{\pi,\varphi} \max_{i,j} \{a_{ij} \cdot |f_{m,n}(\pi(i)) - \varphi(j)|\}.$$

Here and in the following problems we have $\pi \in S_m$ and $\varphi \in S_n$. For $m = n$ this problem is known as the bandwidth minimization problem for quadratic matrices. In this case A can be considered as the adjacency matrix of a directed graph G'. Then $\beta(G') = \beta_{m,m}(A)$ is called directed bandwidth of G'. If A is symmetric and $\pi = \varphi$ then $\beta(G')$ is said to be the bandwidth of G'.

Total bandwidth of A

$$\beta^*_{m,n}(A) = \min_{\pi,\varphi} \{\sum_i \max_j (a_{ij} \cdot |f_{m,n}(\pi(i)) - \varphi(j)|) + \sum_j \max_i (a_{ij} \cdot |\pi(i) - f_{n,m}(\varphi(j))|)\},$$

$\beta^*_{m,n}$ is a measure of how far the most distant non-zero elements in every row and column of A are from the diagonal. For symmetric matrices this concept is similar but not equivalent to the profile of A, which is defined only for positive defined A.

Linear r-arrangement of A

$$l^r_{m,n}(A) = \min_{\pi,\varphi} \sum_{i,j} a_{ij} \cdot |f_{m,n}(\pi(i)) - \varphi(j)|^r.$$

If A is a symmetric matrix, $\pi = \varphi$, and $r = 1$ this problem is known as linear optimal arrangement of the corresponding graph G'. Obviously, the greater r the more is contributed to the goal function by those elements of A having longer distance from the main diagonal.

All the problems stated in terms of A have an analogous formulation for the corresponding bipartite graph G.

Note that the extension of all these problems to p-partite graphs $G = (U_1, \ldots, U_p; E)$ (chapter 1), that is, a series of coupled reduced adjacency matrices A_s, $s = 1, \ldots, p-1$, with the size $n_s \times n_{s+1}$, $n_s = |U_s|$ is straightforward.

How these problems can be used to find an approximate solution to the BCN problem and computational experience is reported in chapter 5.

4. A thermodynamically motivated heuristic algorithm

In the previous chapters we have introduced several NP-hard combinatorial optimization problems. They have different goal functions, but a feasible solution to one of them is also a feasible solution to the other ones and

can always be presented in form of bipartite graphs or reduced adajcency matrices (in case of p-partite crossing number $p > 2$), as a p-partite graph $G = (U_1, ..., U_p; E)$ or a set of coupled reduced adjacency matrices A_s, $s = 1, ..., p-1$. Every feasible solution corresponds to a certain permutation of vertices in the sets U_i, $i = 1, ..., p$, $p \geqslant 2$, and of the corresponding rows and columns of the adjacency matrices A_s, $s = 1, ..., p-1$, respectively.

For such a combinatorial optimization problem a Monte-Carlo method can be easily applied, that is, a random local search algorithm. In the initialization step a feasible solution is to be selected as the current one. Then in every step of the algorithm a new feasible solution of the problem from the neighbourhood of the current one is randomly choosen. It is accepted as a new current solution only if it improves the value of the goal function. Several different criteria of termination may be applied. For example the algorithm performs a predefined number of steps, or if in a given number of steps no change in the goal function appears then the algorithm stops.

For applying this method to a problem there must exists the possibility, to generate randomly any feasible solution in the neighbourhood of the current solution with the same probability.

If not, then the algorithm may work in a quite pathological way. The main application field are large problems for which exact algorithms are not applicable from the viewpoint of computational complexity and no provably good heuristics exist. The Monte-Carlo method is suitable for problems where a new feasible solution and the difference in goal function between the new and current solution can be determined with low computational effort.

Unfortunately, if the problem has some locally optimal solutions, significantly worse than the global one, then this method can easily fail to find a good approximate solution.

In [8] and [2] a version of the Monte-Carlo method for combinatorial minimization problems was proposed. It looks that such an algorithm can seriously overcome this drawback of the usual Monte-Carlo procedure.

Because the idea of this method comes from the thermodynamical statistics it was called thermodynamically motivated heuristic. The thermodynamical background of this method can be shortly described as follows.

If we consider a system of atoms forming liquid or solid matter then there are N atoms and N positions for them. Any atom any occupy an arbitrary position. Because the number of atoms is of order 10^{23} per cubic centimeter, only the most probable behaviour of the system in thermal equilibrium at a given temperature is observed in experiments.

A fundamental question in thermodynamical statictics is: What happens to the system at the limit of low temperature, how to find the state of substance with the minimum energy?

In practical contexts, low temperature is a neccessary but not a sufficient condition for finding the minimum energy state of matter. Experiments that determine the minimum energy state of a substance are done by careful annealing, first melting the substance, then lowering the temperature slowly, and spending a long time at temperatures in the vicinity of the freezing point. If it is not done, and the substance is allowed to get out of equilibrium it can form only metastable, locally optimal structures.

A simple algorithm for efficient simulating the behaviour of a collection of atoms in equilibrium at a given temperature was introduced in the earliest days of scientific computing. In each step of this algorithm, an atom is given a small random displacement and the resulting change ΔE in the energy of the system is computed. If $\Delta E \leqslant 0$, the displacement is accepted, and the configuration with the displaced atom is used as the starting point of the next step. The case $\Delta E > 0$ is treated probabilistically: the probability, that the configuration is accepted, is $P(\Delta E) = \exp(-\Delta E/(k_B T))$ (k_B — Boltzmann constant). Random numbers uniformly distributed in the interval (0, 1) are a convenient means for implementing the random part of the algorithm. One such number is selected and compared with $P(\Delta E)$. If it is less than $P(\Delta E)$, the new configuration is retained: if not, the original configuration is used to start the next step. By repeating the basic stap many times the thermal motion of atoms in thermal contact with a heat bath at temperature T is simulated. This choice of $P(\Delta E)$ has the consequence that the system evolves into a Bibbs-Boltzmann distribution.

Using the goal function in place of the energy and a feasible solution of some combinatoral minimization problem as configuration of atoms of given matter, it is straightforward with such a procedure to generate a population of feasible solutions at some effective temperature. This temperature is simply a control parameter in the same units as the goal function. The simulated annealing process consists of first "melting" the system being optimized at a high effective temperature, then lowering the temperature by slow stages until the system "freezes" and no further changes occur. At each temperature the simulation must proceed long enough for the system to reach a steady state. The sequence of temperatures and the number of new generated feasible solutions attempted to reach equilibrium at each temperature can be considered an annealing schedule.

The thermodynamically motivated procedure can always omit a local optimum at nonzero temperature, which is the main advantage over the ordinary Monte-Carlo method.

For using such a procedure in solving practical problems we must be able to generate a new feasible solution in a way similar to a small random displacement of an atom in the annealing process of matter. For each of the previously stated combinatoral optimization problems it

can be done in the following way. Assuming the case of a p-partite graph $G = (U_1, ..., U_p; E)$, $p \geq 2$, and a set of reduced adjacency matrices $A_1, ..., A_{p-1}$, respectively, we select $s \in \{1, ..., p\}$ at random. Then $i, j \in \{1, ..., n_s\}$, $i \neq j$, $n_s = |U_s|$ are randomly choosen. Every of these random choices must be performed such that any of the possibilities can be selected with the same probability. Interchanging now the i-th and j-th vertex of the set U_s we obtain a new feasible solution and the corresponding change $\underline{\Delta}F$ in the goal function F is calculated. This interchange in U_s corresponds to a simultaneous change of the i-th and j-th column in the matrix A_{s-1} and the i-th and j-th row in A_s, respectively, where $A_0 = 0$ and $A_p = 0$.

For applying the thermodynamically motivated procedure to these problems we need an appropriate annealing schedule. This can be obtained as follows. Let q be the number of new solutions generated at temperature T. Then the new values of T and q can be obtained by $T := a \cdot T$, $q := b \cdot q$, where a and b are some parameters. For determining an annealing schedule it is now enough to define starting values of T and q, and values of a and b. The procedure terminates when either at some temperature no acceptance of a new feasible solution occurs or temperature becomes too small.

5. Computational experience

A lot of examples were considered using a series of different goal functions. We observed that in general any of the goal functions: BCN, bandwidth, and linear 2-arrangement lead to results of almost the same quality. This is valid also for $p > 2$. However, from the viewpoint of computational complexity they differ significantly. Let $n = \max \{n_i | 1 \leq i \leq p\}$. The calculation of $\underline{\Delta}F$ can be done for p-partite crossing number in $0(n^3)$, for bandwidth in $0(n^2)$, and for linear 2-arrangement in $0(n)$. It indicates that for large sized problems linear 2-arrangement is the most promising criterion for all considered problems. Obviously, the starting values of T and q must depend on both n and p. We used $T = c \cdot (n+p)$ and $q = 2 \cdot (n+p)$. The best among all tested values of the parameters a and b were $0.8 < a < 0.9$ (e.g. $a = 0.85$) and $b = 1.1$. This ensures slow and careful annealing of the system. The best value of the parameter c must be established experimentally for every concrete problem. It ought to quarantee a careful annealing process and prevent the solution from moving too far from the optimal solution during the first iterations. Thus the quality of the solution and the computational effort depend highly on a proper parameter setting.

Though there exists no performance guarantee for this procedure it

exhibits very well in practice and shows a stable behaviour. If we assume that h, the number of all solutions generated by the algorithm, is a good measure of the computational complexity of this algorithm, then all our computational experiments show that $h \leqslant 5 \cdot (n+p)^2$. So the algorithm with the linear 2-arrangement goal function has in practice the complexity of $0\,(n \cdot (n+p)^2)$. Another important observation is that in average 90% of the possible improvement in the goal function is reached after one half of the running time. That means if a good approximate solution is acceptable we may interrput the procedure much earlier, before the stop criterion is reached.

We tested examples up to a size of $p = 7$ and the size 66×66 of the reduced adjacency matrix. The program was written in FORTRAN and is running on a 16-bit minicomputer (K 1630 robotron). The examples were taken partly from the literature and practical applications. Applied to the bandwidth minimization problem the algorithm, using all of the above-mentioned goal functions, was able to produce the best known solution for the test examples among them those from [3], [5] and [13]. For matrices exceeding the size of 30×30 the BCN criterion becomes too expensive from the computational point of view. The linear 2-arrangement provides the best approximate solutions to the BCN problem and in most cases is proved to be an even stronger criterion than bandwidth.

We note that several deterministics algorithms for the BCN problem can be found in [10] and [12].

For illustration we give an example where $p = 3$, $n_1 = 10$, $n_2 = 15$, and $n_3 = 12$. We present the starting matrices A_1 and A_2 and the result A_1' and A_2' obtained by the algorithm using 3-partite crossing number criterion. The non-zero entries are indicated by x. Figure 1 shows the initial 3-partite graph G corresponding to A_1 and A_2 and the resulting graph G' corresponding to A_1' and A_2'. Here the number of crossings was reduced from $N_3(G) = 612$ to $N_3(G') = 92$.

$A_1 =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1		x				x		x	x				x		
2					x										x
3			x	x					x		x		x		
4	x				x		x			x		x		x	
5							x							x	
6		x						x	x						
7		x				x		x	x						
8			x	x						x	x				x
9				x							x		x		x
10							x					x		x	

$A_1' =$

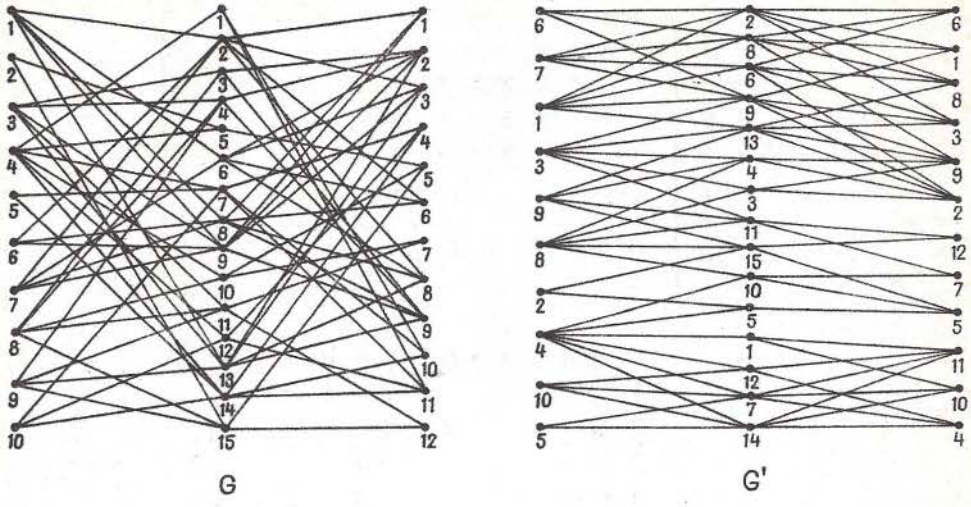
	2	8	6	9	13	4	3	11	15	10	5	1	12	7	14
6	x	x		x											
7	x	x	x	x											
1	x	x	x	x	x										
3			x	x	x	x	x								
9				x	x		x	x							
8					x	x	x	x	x						
2								x		x					
4									x	x	x	x	x	x	x
10												x	x	x	
5														x	x

$A_2 =$

	1	2	3	4	5	6	7	8	9	10	11	12
1										x	x	
2	x		x			x		x				
3		x							x			
4		x							x			
5					x							
6		x	x			x		x				
7				x						x	x	
8	x					x		x	x			
9	x	x	x						x			
10					x		x					
11							x					x
12				x							x	
13		x	x					x	x			
14				x						x	x	
15					x							x

$A_2' =$

	6	1	8	3	9	2	12	7	5	11	10	4
2	x	x	x	x								
8	x	x	x		x							
6			x	x		x						
9		x		x	x	x						
13			x	x	x	x						
4					x	x						
3					x	x						
11							x	x				
15							x		x			
10								x	x			
5									x			
1										x	x	
12										x		x
7										x	x	x
14										x	x	x



6. Applications and conclusions

The BCN problem and the p-partite crossing number problem, respectively arise in a number of applications, ranging from matrix permutation problems over transpostation problems to layout problems in the automation of design and documentation.

In general, the optimal representation of a matrix according to the BCN yields an optimal or near-optimal solution to the bandwidth minimization problem and vice versa, though there exist some pathological cases. Those conversion problems for sparse matrices appear in solving large systems of linear equations and differential equations as well as in network analysis. Furthermore in the physical layout of digital systems the minimum p-partite crossing number or any criterion from chapter 3 can be used in placing modules on special multi-row structures. This situation is met for instance in standard cell and gate erray technology. The algorithm developed was tested in the placement of several permutation networks. Both criteria the minimization of the longest signal and minimum total signal length were applied. Comparing our results with the placement done by a skilled designer we were able to produce about the same quality of solution according to the total channel width in the channel routing, succeeding the placement procedure [14]. It has turned out that minimum BCN is an appropriate measure for a good module placement subject to channel routing. Another interesting application is in the field of automatic layout of schemtics [12], which can be considered a spacial branch of two-dimensional computer graphics. The p-partite crossing number criterion is especially

suited for multi-row schematics [9] which represent a broad class of schematic drawings such as logic diagrams. It was succesfully applied in laying out logic schematics in order to fulfil certain functional and aesthetical conditions. Several results and additional algorithms are presented in [11] and [12]. Potential applications lie in the field of transportation and communication, where certain streets or communication lines are required to intersect each other in as few crossings as possible.

Hence, there are good reasons both from graph theory and from the application viewpoint to continue the research in the BCN problem. The considerations made in this paper can for instance be extended to arbitrary graphs presented on parallel consecutive rows as described in chapter 1.

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O liczbie przecięć w grafach dwudzielnych

W pracy przedstawiono pewne wyniki dotyczące liczby przecięć w grafach dwudzielnych. Jest to NP — trudne zadanie optymalizacji dyskretnej. Uzyskano oszacowania rozwiązania optymalnego. Zaprezentowano szereg zbliżonych sformułowań zagadnień optymalizacyjnych na grafach w postaci macierzowej. W pracy przedstawiono algorytm termodynamiczny dla odnajdywania minimalnej liczby przecięć w grafach *p*-dzielnych oraz wyniki eksperymentu obliczeniowego.

О числе пересечений в двудольных графах

В работе представлены некоторые результаты касающиеся числа пересечений в двудольных графах. Это является NP-трудной задачей диекретной оптимизации. Получены оценки оптимального решения. Представлен ряд аналогичных формулировок оптимизационных задач на графах в матричном виде. В работе представлен термодинамический алгоритм для нахождения минимального числа пересечений в р-дольных графах и результаты численного эксперимента.

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