Introductory remarks.
Some connections between mathematical programming, continuous
and discrete time optimal control theory

by

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The paper provides an introduction to the special issue of "Control and Cybernetics" on discrete time optimal control theory. It outlines some connections and points out the benefits which the discrete time optimal control theory takes by assimilating ideas and results from mathematical programming and continuous time optimal control theory.

1. Introduction

The motivation for this special issue of "Control and Cybernetics" is basically that discrete time optimal control theory is useful for solving real problems in a variety of situations. To mention a few: management of energy and other resources, development of economic systems, determination of design parameters for technical equipment, control of production process. Many more specific applications can be mentioned.

The reason for this is that many problems are naturally formulated, communicated or interpreted as discrete time optimal control problems. The control structure comes naturally when considering systems that evolve in time, but also other types of problems can be formulated as such,

even if the parameter which is varying is not "time" in the natural sense, but something else.

Some problems are more naturally formulated as continuous time optimal control problems. However, for finding the optimal control one often has to use numerical methods, and some of these consist essentially in solving a discrete time optimal control problem. Moreover, often implementation of the optimal control will be performed as discrete time process. Thus, also in the context of continuous time problems, discrete time theory is important.

A discrete time optimal control problem may be interpreted as a mathematical programming problem (with some particular structure), and there exist theoretical results and computational methods for solving such problem. However, we think that stronger theoretical results and better numerical methods may be obtained by exploiting directly the control structure. Thus, there are good reasons for developing optimal control theory for discrete time.

However, it is our impression that such theory is relatively undeveloped. This statement should be seen in the light of the rapid development of particularly mathematical programming over the last 20 years or so.

This is then the specific motivation for this special issue; to present new trends in the development of discrete time deterministic optimal control.

In the next section we shall give some historical remarks on the development of continuous time optimal control theory — the basic ideas for the discrete time case originated here. Then we indicate some of the ideas from mathematical programming which we think could give inspiration to development of discrete time optimal control, and finally we formulate some direction of development, and place the contributions in this special issue in relation to these.

2. Continuous time optimal control

The prototype continuous time optimal control problem is

$$\max \int_0^1 r\left(x\left(t\right), u\left(t\right), t\right) dt$$

$$x\left(t\right) = f\left(x\left(t\right), u\left(t\right), t\right)$$

$$u\left(t\right) \in U, \text{ some prescribed set}$$

$$x\left(0\right) \text{ given, } x\left(1\right) \text{ given or free.}$$

The modern control theory for continuous time systems have provided the basic results of the maximum principle and dynamic programming.

The main idea of the maximum principle is the construction of a special function, the Hamiltonian

$$H\left(x\left(t\right), u\left(t\right), p\left(t\right), t\right) = r\left(x\left(t\right), u\left(t\right), t\right) - p\left(t\right) f\left(x\left(t\right), u\left(t\right), t\right)$$

The Hamiltonian is maximized by the optimal control $u^*\left(t\right)$ at the optimal state, and the optimal costate vector $p\left(t\right)$ satisfies a differential equation $p\left(t\right) = -\nabla_x H\left(x\left(t\right), u\left(t\right), p\left(t\right), t\right)$ along the optimal trajectory.

The maximum principle contains as special cases the classical first order necessary condition for optimality of the calculus of variations: the Euler-Lagrange equations, the Lagrange multiplier rule, Legendre's necessary conditions and the Weierstrass inequality.

Dynamic programing is the other essential analytical technique applicable to the optimal control problem. The problem is solved "backwards", and the maximal attainable criterion value from time t ($0 \leqslant t \leqslant 1$) to time 1 is constructed and expressed as a function V of the state x at time t. The value is $V\left(x\left(t\right), t\right)$.

Under some rather strong assumptions, particularly that V be sufficiently smooth, some nice relationships obtain. Particularly we have that along the optimal path

$$-\nabla_t V\left(x\left(t\right), t\right) = \max_{u \in U}\left[r\left(x\left(t\right), u\left(t\right), t\right) + \nabla_x V\left(x\left(t\right), t\right) f\left(x\left(t\right), u\left(t\right), t\right)\right]$$

Here in this Hamilton-Jacobi equation we see a clear relationship between dynamic programming and the maximum principle, since $-p\left(t\right)$ and $\nabla_x V\left(x\left(t\right), t\right)$ play the same role.

Continuous time system of small dimensions may be solved analytically by applying the optimality conditions of the maximum principle or dynamic programming; particularly the former can often yield a solution. The solution may be quantitative, or it may give qualitative information, e.g. concerning bang-bang control.

With more then one, or maybe two dimensions of the state vector it is seldom possible to find an analytical solution. To produce a solution a numerical method must therefore be applied. Since this typically involves numerical integration of the transformation and adjoint equation, we see that many practical continuous time problems are actually solved numerically by treating them as discrete time systems.

3. Mathematical programming

Mathematical programming has experienced rapid development over the last 20 years. We shall relate the discussion to the prototypical mathematical programming problem

$$\max r\left(x\right)$$
$$g\left(x\right) \leqslant a$$
$$h\left(x\right) = b.$$

The first basic results on necessary and sufficient condition were the Karush-Kuhn-Tucker conditions. They apply to problems with smooth functions r, g and h. If the constraint function g and h satisfy some qualification, then there exist Lagrange multipliers λ and μ with certain relations to the gradients of r, g and h at the optimal point x. If further some convexity properties are fulfilled, then these KKT conditions are also sufficient to yield a global optimum. From linear programming and the KKT-theory it is known that the simplex- or Lagrange multipliers may be given some "shadow price" interpretation. Where the optimal solution therefore gives information on one problem, through shadow prices approximate information is provided about problems that are close to the original problem.

One may therefore solve the problem for various right hand sides (a, b) and express the optimal criterion value as a function $V(a, b)$, the pertubation function, optimal value function or marginal function. A major point now is, that there is a strong relation between the Lagrange multiplier λ, μ and the derivative of $V(a, b)$. The nice case is when $(\lambda, \mu) = \nabla V(a, b)$, the gradient of V. It turns out that it is necessary to make fairly strong assumptions to get this relationship. What then is the relationship when these assumptions do not hold? The clarification of this question has been subject to much research over the last 10 years, concerning for instance second order conditions, nonsmooth function and concepts of generalized gradients.

It is worth noting at this place that the idea of perturbation function is well known in optimal control. The dynamic programming optimal value function is in this perspective nothing but the perturbation function calculated stagewise.

The clarification of the role of Lagrange multipliers for the characterization of $V(a, b)$ is very important for answering whether the problem above can be solved by Lagrangian relaxation, i.e. whether x can be found as the solution to

$$\max \left[r(x) - \lambda g(x) - \mu h(\lambda) \right].$$

In the convex case, with contraint qualifications fulfilled, it can. In the general case it also can, but then it may be necessary to use non-linear penalty function $\lambda(g(x))$ and $\mu(h(x))$. For smooth functions one may try to use quadratic functions, in which case we get the so called augmented Langrangian.

In optimal control the idea of relaxation by Lagrange multipliers is also well established. It is precisely what is captured in the maximization of the Hamiltonian function, that is, performing a relaxation with a (linear) penalty function.

The "best" multipliers in some sense are those that solve the dual problem

$$\min_{\lambda, \mu} \Big(\max_{x} \big(r(x) - \lambda \left(g(x) - a \right) - \mu \left(h(x) - h \right) \big) \Big), \ \lambda \geqslant 0.$$

Duality theory has proved very rich and fruitful as a theoretical subject and as a guide for computational algorithms.

4. On this issue

It is our opinion that discrete time optimal control can benefit greatly by assimilating ideas and results from other areas of optimization, particularly mathematical programming. As we have seen, some of the basic ideas are part of the standard optimal control theory: the perturbation function, Lagrangian relaxation. But while these ideas have been further developed and applied in mathematical programming, the development of discrete time optimal control apparently stopped at an early stage.

While the discrete time optimal control problem can be seen as a mathematical programming problem, and treated as such, we think that the special structure could advantageously be exploited. The very elegant and effective results of the maximum principle and dynamic programming shows this. We would therefore like to see the development taken much further in this direction. The present issue of Control and Cybernetics contributes to this.

The first three papers to some extent survey the areas where the research has been continued for some time. The paper by Krotov presents works which have started already in the early sixties. Although mainly continuous time problems were developed, the approach taken allows considering both continuous and discrete time cases under an almost unified notation. The paper exemplifies how the discrete time theory can profit from the continuous time ideas and solutions.

Unification of the basic methods of solving the control problems, which is evident in Krotov's paper, is also one of the main threads of the next paper by Nahorski & Ravn. They developed the set of geometrical concepts which are close in spirit to the mathematical programming constructions of the perturbation or marginal functions and show that it is used either implicitly or explicitly in all basic methods.

The idea of differential dynamic programming, which is the subject of the paper by Yakowitz, reach as far back in time as Krotov's ideas. The paper surveys, however, recent developments of which the author is the main animator. The assumed high smoothness of the problem is the main element which contributes to construction of powerful algorithms.

An interesting feature is the apparent convergence of the approaches of all these papers. This fact is pointed out in the paper by Nahorski & Ravn but it seems to be still far fuil exploration.

As opposed to the paper by Yakowitz, the next two papers, by Vinter and by Doležal, deal with the nonsmooth problems. Particularly the former is of the "summing up" type because it drives the state of art in this field to the level comparable with this of the classical maximum principle, i.e. necessary conditions with the stagewise hamiltonian maximum under (a little weakened) directional convexity assumption.

The paper by Doležal considers the similar problem in the presence of multiple optimality criteria.

In the paper by Rockafeller an advantage of converting the discrete time control problem to the mathematical programming formulation is taken. Under convexity assumptions the author derives very elegant duality results.

The paper by Gurman adresses a two level control problem. It consists of a sequence of continuous time subproblems. The end point of the system at each intermediate stage is the starting point for the next stage. Thus the problem consists of the optimal continuous time control at the lower level and the optimal discrete time control at the higher. Obviously, both level subproblems are related which calls for a common treating.

Finally, Khrustalev in his paper considers a system whose state depends on the previous values of state and control. The delay is varying in time.

Together the papers demonstrate that there are many possible ways to treat the discrete time optimal control problems. The results presented here cover a broad range, and provide deep insight into the problem. This indicates that it will be possible to reach further in the analysis of the mathematical properties of the problem, and also that it will be possible to apply the presented and future results in practical problem solving.

Uwagi wstępne. Związki między programowaniem matematycznym oraz ciągłą i dyskretną teorią sterowania

Praca jest wstępem do specjalnego numeru "Control and Cybernetics" poświęconego teorii sterowania optymalnego systemów z czasem dyskretnym. Przedstawiono w niej powiązania i wskazano na korzyści jakie płyną dla dyskretnej teorii sterowania optymalnego z asymilacji pomysłów i wyników z programowania matematycznego i ciągłego sterowania optymalnego.

Предварительные замечания. Взаимосвязи между математическим программированием и непрерывной а также дискретной теорией управления

Статья является введением к специальному номеру „Control and Cybernetics", посвященному теории оптимального управления системами дискретными по времении. Представлены в ней взаимосвязи и указана польза вытекающая для дискретной теории оптимального управления из применения идей и результатов математического программирования и непрерывного оптимального управления.