Control and Cybernetics

VOL. 17 (1988) No. 2–3

A Technique of Global Bounds in Optimal Control Theory

by

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The paper contains a survey of theoretical and practical results connected with sufficient conditions for global optimality. Both discrete and continuous time systems are discussed, with emphasis on the former.

0. Introduction

In this paper we give a survey of results connected with development and application of an approach where sufficient conditions for global minimum of functionals in variational calculus and optimal control theory are used. The first results were given at the beginning of sixties [1–5], see also [6, 7]. The main element which is looked for in this approach is a so called solving function depending on the state and the argument (time) of the process under consideration. Having properly chosen this function an optimal solution is found through maximization of some scalar function of state, control, and time with respect to first two variables.

Development of this approach went in the following directions:

(i) Elaboration of new analytical methods for particular classes of problems, mainly singular ones, and those which have no solution in the admissible set. Then finding new classes of solutions and their describing equations. It was possible both because of flexibility of the approach caused by the essential nonuniqueness of the solving functions and because of lack of the necessity of the solution admissibility which is required in variational calculus.

(ii) Elaboration of broad destination optimization algorithms which use non-local operations for improving controls or solving functions.

(iii) Extension of the class of solving functions in order to extend the sufficient conditions of optimality to necessary ones or to broaden the possibilities of using the approach.
(iv) Adjustment of the approach to new classes of optimal control problems (with multidimensional argument, delayed argument, incomplete information etc.) and to other similar problems like global bounds for functionals in the feasible regions or synthesis of invariant controls.
(v) Application of the elaborated methodology to real problems of automatic control, mechanics, applied physics, and mathematical economy which led to an extensive experience.

This survey does not cover all of these directions. It is oriented mainly either directly on results connected with the multistage processes or on some of their extensions. But it does not close with it. Two kinds of models are considered, with discrete, and continuous (scalar) argument. No results on multidimensional argument or incomplete information are included.

1. Problem statement

Let us consider a triple of variables $t \in T$, $x \in X$, $u \in U$ and a pair of functions $v = (x(t), u(t))$, $v: T \to X \times U$. The latter is called *an admissible controlled process* if the following conditions are satisfied:

1. $$(t, x(t), u(t)) \in V, \ \forall t \in T, \tag{1.1}$$
   where V is a given subset of the direct product $T \times X \times U$;
2. A process equation or equation of movement is satisfied. Two kinds of those equations are considered:

   a) The multistage (discrete argument) process. Here the set T is a real sequence $\{t_0, t_0+1, t_0+2, \dots, t_1\}$ and the process equation has the form:
   $$\begin{gathered} z(t) \equiv x(t+1) - f[t, x(t), u(t)] = 0 \\ t = t_0, t_0+1, \dots, t_1-1, t_1, \quad x(0) = x_0, \end{gathered} \tag{1.2}$$
   where $f(t, x, u)$ is a given function, $f: T \times X \times U \to X$ and x_0 is a given elements from the set X.

   b) The continuous argument process. Here T is an interval $[t_0, t_1]$ of the real line, X, U are real vector spaces R^n and R^r, respectively, and the process equation has the form:
   $$\begin{gathered} z(t) \equiv \dot{x}(t) - f[t, x(t), u(t)] = 0, \\ t \in (t_0, t_1), \ x(t_0) = x_0, \end{gathered} \tag{1.3}$$
   where $f(t, x, u)$ is a given vector function.

   In the latter case some additional requirements must be imposed on v to satisfy the equation (1.3). Usually this is piecewise continuity of the function $u(t)$ and continuity and piecewise differentiability of the function $x(t)$ or measurability of $u(t)$ and absolute continuity of $x(t)$.

The set of all admissible processes is denoted by D, t is called the process *argument*, x *the state*, and u *the control*. The functions $x(t)$ and $u(t)$ are called *the trajectory* and the *program control*, respectively. To unify the notation we use the sign S which will stand either for the sum $\sum_{t=t_0}^{t_1-1}$ or for the integral $\int_T dt$. Moreover, we use the notation like $[t_0, t_1]$, (t_0, t_1) etc. not only for continuous time but also for discrete sequences with included or excluded initial and end points, respectively. We also admit the following notation: the superscript denotes the intersection of a given set by the set of constant values of a given variable, and the subscript denotes the projection of the set on the subset of this variable. For example V_x^t is the projection of the intersection of V and the set $t=$ constant on X. D_x is the set of the admissible process trajectories. The same letters used in multiplication will denote the summation over the repeated argument.

On the set D we define the functional:

$$J(v) = Sf^0(t, x(t), u(t)) + F(x(t_1)), \tag{1.4}$$

where $f^0(t, x, u)$ and $F(x)$ are given real functions which are continuous when the continuous case is considered.

We introduce also a set E of processes v with the following properties:

1. $D \leqslant E$;
2. There exists a sufficiently simple algorithm for construction of processes $v \in E$;
3. A functional $\varrho(v)$ is defined which satisfies the conditions:

$$\varrho(v) = 0 \quad \text{for} \quad v \in D,$$

$$\varrho(v) > 0 \quad \text{for} \quad v \in E \backslash D,$$

and is called the distance between the element $v \in E$ and the set D;
4. A functional $J'(v)$ satisfying $J'(v) = J(v)$ for $v \in D$ is defined.

In this paper we define the set E as follows. The pair of functions $v = (x(t), u(t)) \in E$ satisfies restrictions (1.1) but not necessarily (1.2) or (1.3). Although this is enough for the discrete time, one has to say more about the properties of the functions $x(t)$ and $u(t)$ in the continuous time case. Namely, we assume that both functions are piecewise continuous. The distance ϱ is defined as:

$$\varrho(v) = S\,|z(t)| + \sum_{t \in \beta, i} |x^i(t)| \Big|_{t-0}^{t+0}, \tag{1.5}$$

where $z(t)$ is the discrepancy in the equation (1.2) or (1.3), and β is the set of arguments in which the function $x(t)$ jumps. The second element in the sum (1.5) is taken into account only in the continuous case.

Two subproblems may be associated with the problem introduced above.
(I) Construction of an admissible process. This consists in finding $\{v_s\} \subset E$ which converge to D:

$$\varrho(v_s) \to 0. \tag{1.6}$$

This is one of the basic problems in control theory: find a control strategy and the trajectory resulting from it which satisfy the given constraints. In the mathematical language, it is the problem of finding a solution of the open system of differential (difference) equations which satisfies the constraints.

It is also often required that during the convergence some criterion is minimized.

(P) Construction of an optimal admissible process. Beside the conditions of subproblem (I) it is required that the sequence $\{v_s\} \subset E$ satisfy:

$$J(v_s) \to d = \inf_D J(v). \tag{1.7}$$

In particular, when a minimum of the functional $J(v)$ exists on D, then it is required to find a solution $\bar{v} \in D$ such that:

$$J(\bar{v}) = \min_{v \in D} J(v). \tag{1.8}$$

Also approximate variants of those subproblems will be considered. Let us introduce a set $D_\varepsilon(\varrho)$ of processes which satisfy the following conditions:

$$v \in E,\ \varrho(v) < \varepsilon,\ \varepsilon > 0. \tag{1.9}$$

We call $D_\varepsilon(\varrho)$ an ε-extension of the set D in the metric ϱ.

The approximate variant of the subproblem (I) is: find an ε-extended solution $\bar{v} \in D_\varepsilon(\varrho)$. In the approximate variant of the subproblem (P) it is required that the solution be also η-optimal on $D_\varepsilon(\varrho)$:

$$J(v) - d_\varepsilon(\varrho) < \eta,\ \eta > 0,\ d_\varepsilon(\varrho) = \inf_{D_\varepsilon(\varrho)} J(v). \tag{1.10}$$

Also two other variants can be considered independently: $\varepsilon = 0$ i.e. η-optimality on strict D, and $\eta = 0$ i.e. strong optimality on ε-extension of D. In connection with the above definitions there arises a question of the problem correctness, that is whether $d_\varepsilon(\varrho) \to d$ when $\varepsilon \to 0$.

We call a solution to the subproblem (P) an *optimal program control* in agreement with one of its parts i.e. program control $u(t)$.

In the control theory an important role is also played by a solution in the feedback form. Let us assume that there is given a function $u(t, x)$, $T \times X \to U$. Let us further assume that there exists a solution $x(t)$ of the system (1.2) or (1.3) with $u(t, x)$ inserted for $u(t)$ to form a program control $u(t) = u(t, x(t))$ such that $v = (x(t), u(t)) \in D$. We say that the process v is *associated* with the control (or feedback policy) function $u(t, x)$.

Let B be a set of initial conditions (t_0, x_0) and let there exist a family of optimization problems with an initial condition $(t_0, x_0) \in B$ and the rest of the problem conditions fixed. We shall include the dependence on the initial conditions in the notation like $D(t_0, x_0)$, $d(t_0, x_0)$, $v(t_0, x_0)$ etc. Let there, for every $(t_0, x_0) \in B$, exist the unique process $v(t_0, x_0) \in D(t_0, x_0)$ associated with $u(t, x)$. Then we call the latter *a control synthesising function* or simply *a control synthesis.*

Let a synthesis $\bar{u}(t, x)$ satisfy the following conditions:

$$J(\bar{v}(t_0, x_0)) - d(t_0, x_0) \leqslant \varepsilon,\ \varepsilon > 0,\ \forall t_0, x_0 \in B.$$

The function $\bar{u}(t, x)$ will be called an ε-optimal control synthesis and for $\varepsilon = 0$ an optimal synthesis. That means that a construction of an optimal synthesising function is equivalent to the solution of a family of optimal program control problems with an initial condition (t_0, x_0).

2. Bounding and solving functions, sufficient conditions of optimality

Let us introduce a class Π of real functions $\varphi(t, x)$: $T \times X \to R^1$ (continuously differentiable in the continuous case) such that the following exist:

$$R(t, x, u) = \begin{cases} \partial\varphi/\partial x^i \cdot f^i(t, x, u) - f^0(t, x, u) + \partial\varphi/\partial t \text{ (continuous)} \\ \varphi[t+1, f(t, x, u)] - \varphi(t, x) - f^0(t, x, u) \text{ (discrete)} \end{cases} \tag{2.1}$$

$$G(x) = \varphi(t_1, x) + F(x), \tag{2.2}$$

$$\mu(t) = \sup_{(x, u) \in \Gamma^t} R(t, x, u), \quad m = \inf_{x \in V_x^{t=t_1}} G(x), \tag{2.3}$$

$$L(\varphi, v) = G(x(t_1)) - SR(t, x(t), u(t)) - \varphi(t_0, x_0), \tag{2.4}$$

$$l(\varphi) = m - S\mu(t) - \varphi(t_0, x_0), \tag{2.5}$$

$$\tilde{V}^t = \operatorname{Arg} \max_{(x, u) \in \Gamma^t} R(t, x, u), \tag{2.6}$$

$$\tilde{E}: v \in E, (x(t), u(t)) \in \tilde{V}^t, t \in [t_0, t_1), x(t) \in \operatorname{Arg} \min_{V_x^{t=t_1}} G(x), \tag{2.7}$$

$$\tilde{u}(t, x) \in \tilde{U}(t, x) = \operatorname{Arg} \max_{u \in V^{tx}} R(t, x, u), \tag{2.8}$$

$$P(t, x) = \sup_{u \in V_x^t} R(t, x, u). \tag{2.9}$$

The functions $\varphi(t, x) \in \Pi$ will be called *bounding functions.*

The values and variables introduced above have some properties which are useful when analysing the problems considered in the paper. They will be reviewed below.

1. $L(\varphi, v) = J(v)$, $\forall \varphi \in \Pi$, $v \in D$. This equation defines a family of functionals $J(v)$ on D. For a given (nonoptimal) process v_0 it is possible to choose $L(\varphi, v_0)$ in such a way that it is obvious how to *improve* the process v_0,

i.e. how to choose a $v \in D$ such that $L(\varphi_0, v) = J(v) < J(v_0) = L(\varphi_0, v_0)$ and also how to guess a rule of choosing it. These bounding functions will be called *improving*. Repeating the improving operation it is possible to construct methods for approximate solution of the problem (P) in D by a sequential improvements. Besides, $L(\varphi, v)$ is used to define $J(v)$ on E (outside D) as $J'(v) = L(\varphi, v)$. This is used in the proof of sufficient optimality conditions and in some algorithm.

2. $J(v) \geqslant l(\varphi)$, $\forall v \in D$, $\varphi \in \Pi$, i.e. to any bounding function φ corresponds a lower bound for the functional J on D. This inequality can be used to obtain sufficient optimality conditions and directly to obtain global bounds for the criterion. From these bounds it is possible to find the best:

$$\bar{l} = \sup l(\varphi), \ \varphi \in \Pi. \tag{2.10}$$

3. A sufficient condition of optimality. Assume that there are given: a function $\bar{\varphi} \in \Pi$ and a process $\bar{v} = \{\bar{x}(t), \bar{u}(t)\} \in D$, such that $\bar{v} \in \tilde{E}(\bar{\varphi})$, or more explicitly:

$$\begin{aligned} R\left(t, \bar{x}(t), \bar{u}(t)\right) &= \max_{x, u \in V'} R(t, x, u) \equiv \mu(t), \ t \in [t_0, t_1), \\ G\left(\bar{x}(t)\right) &= \min_{x \in V'_x} G(x) = m, \ t = t_1, \end{aligned} \tag{2.11}$$

Then:

$$J(\bar{v}) = \min_{D} J(v) = l(\bar{\varphi}) = \max_{\Pi} l(\varphi). \tag{2.12}$$

More generally: assume that there exist sequences $\{\varphi'_s\} \subset \Pi$ and $\{v_s\} \subset D$ such that:

$$S\left[R_s\left(t, x_s(t), u_s(t)\right) - \mu_s(t)\right] \to 0, \tag{2.13}$$

$$G\left(x_s(t_1)\right) - m_s \to 0. \tag{2.14}$$

Then:

$$J(v_s) \to \inf_{D} J(v) = \bar{l} = \lim l(\varphi_s). \tag{2.15}$$

These conditions of optimality are the basis of the following approach to solution of the variational problems (the principle of optimality, [2]): for different bounding functions find solutions $\tilde{x}(t)$, $\tilde{u}(t)$ of the family of extremal problems (2.11) with the parameter t and then $\varphi = \bar{\varphi}$, such that the process $\tilde{v} = \left(\tilde{x}(t), \tilde{u}(t)\right)$ satisfies the equations (1.2) or (1.3) and the earlier mentioned properties of the functions $x(t)$, $u(t)$ in the continuous time (the satisfaction of the condition (1.1) is looked for after construction of $\tilde{v}$). The function $\bar{\varphi}$ is in general essentially nonunique and when specified for different subclasses of Π it leads to different methods of solutions. The function $\bar{\varphi}(t, x) \in \Pi$ is called *a solving function* and the set of all these functions is called $\bar{\Pi}$. Finding a pair $\bar{v} \in D$, $\bar{\varphi} \subset \Pi$ means that the pair of the dual problems (2.12) has been solved.

4. According to (2.7) for any function $\varphi \in \Pi$ there exists a process $\tilde{v} =$

$=(\tilde{x}(t), \tilde{u}(t)) \in \tilde{E}$ (perhaps nonunique) whose distance from the set D is $\varepsilon = \varrho(\bar{v})$. It is a solution of the ε-extended problems (I) and (P) where:

$$\min_{v \in D_\varepsilon} J'(v) = l(\varphi), \; J'(v) = L(\varphi, v). \tag{2.16}$$

If a sequence $\{\varphi_s\} \subset \Pi$ is a solution of the dual problem (2.10), then for some sufficiently weak conditions there hold $\varrho(\tilde{v}_s) \to 0$, $l(\varphi_s) \to d$, i.e. the sequence $\{\tilde{v}_s\}$ is a solution of the problems (I) and (P). From that numerical algorithms for computing admissible and optimal processes can be built. They will be considered in the sequel.

5. Let a function $\tilde{u}(t, x)$ implied by $\varphi \in \Pi$ through (2.8) be a synthesising function on the set B of initial conditions (t_0, x_0). For example this is always true in the discrete case if there are no constraints on states: $V_x^t = X$, $t = t_0+1, \dots, t_1$. Then for all (t_0, x_0) the following is true, see [4]:

$$J(\tilde{v}(t_0, x_0)) - d(t_0, x_0) \leqslant \Delta(\varphi) =$$
$$= S\,[\sup_{V_x^t} P(t, x) - \inf_{V_x^t} P(t, x)] + \sup_{V_x^{t_1}} G(x) - \inf_{V_x^{t_1}} G(x) \tag{2.17}$$

i.e. synthesis $\tilde{u}(t, x)$ is ε-optimal, $\varepsilon = \Delta(\varphi)$. Minimizing the functional $\Delta(\varphi)$ it is possible to have it sufficiently small.

A group of numerical algorithms for an approximate optimal solution is based on this idea. Let there exist a function $\varphi(t, x)$ which satisfies the conditions:

$$P(t, x) = c(t), \; \forall t; \; \varphi(t_1, x) = -F(x) + C_1, \tag{2.18}$$

where $c(t)$ is a function and C_1 a constant. Then, according to (2.17) the synthesis $u(t, x)$ is optimal. If we take $c(t) \equiv 0$ and $C_1 = 0$, then the function $\varphi(t, x)$ which satisfies (2.18) is the dynamic programming return (or optimal value) function with a negative sign. The equation (2.18) is then the Hamilton-Jacoby or the dynamic programming equation in the respective cases.

6. The transformation $\varphi' = \varphi + c(t)$ where $c(t)$ is a differentiable function does not change the values of the functionals $l(\varphi)$ and $L(\varphi, v)$ (when v is fixed) nor the sets $\tilde{E}$, $\tilde{U}(t, x)$, $\tilde{V}^t$. From this it is seen that the bounding function $\varphi(t, x)$ can be defined in such a way that $\mu(t) \equiv 0$, $m = 0$. Then the function φ is called *normalized.*

7. Along the admissible trajectory $x(t) \in D_x$ a normalized function $\varphi(t, x)$ is nonincreasing and thus:

$$\varphi(t, x(t)) \leqslant \varphi(t_0, x_0), \; \forall t, \; \forall x(t) \in D_x.$$

8. If the function $\varphi = \bar{\Pi}$ is normalized, then all optimal trajectories $\bar{x}(t)$ are situated on the surface $\varphi(t, x) = \text{constant} = \varphi(t_0, x_0)$. In particular it means that the optimal trajectories can not cross this surface neither in the upper nor in the lower direction.

The mathematical facts mentioned above are elementary but they imply nontrivial corollaries. Sufficient conditions of optimality include the basic equalities and inequalities of variational calculus and optimal control theory like the maximum principle equations, the Jacoby conditions, and the Hamilton-Jacoby-Bellman equations. This means that they are quite close to necessary conditions. And in fact, after some natural additional assumptions they become necessary. This observation allowed to find new classes of solutions for the variational calculus problems and also methods of finding them. These facts were the basis for new ideas of constructing numerical algorithms for computing optimal or simply admissible processes. The mathematical methods which use the bounding functions and related constructions were found efficient not only for problems formulated in this paper but also in many other problems in analysis and synthesis of dynamic system control. It seems that they are as much adequate for solving global problems as the methods which use adjoint equations for local problems.

The equations (2.1)–(2.9), the presented mathematical facts, and resulting new possibilities and nontraditional directions in solving variational calculus and optimal control problems were developed in the papers surveyed here, starting from [1–5]. But also earlier papers containing some elements of this theory should be mentioned. The Hamilton-Jacoby method in variational calculus and analytical mechanics can be regarded as first applications of solving functions. We can also consider that they are used in Bellman dynamic programming [8] which is a generalization of the Hamilton-Jacoby method to modern problems of control and, in particular, to problems of optimal control of multistage processes. However, these are solving functions of special types, defined by equation (2.18). They do not cover all possible applications of this theory. Functions of the solving type were used by Caratheodory [8, p. 335] for examining local conditions of variational calculus. To those results we can also add the second Lyapunov method for analysis of stability of motion. In this method [7], the bounding functions were defined and extensively used.

3. Relation to other optimality conditions

The relation of the described optimality conditions to Pontryagin's maximum principle [10] is obvious from the following necessary extremum conditions (2.11), see [1, 6, 7]:

$$R_x\left(t, \bar{x}(t), \bar{u}(t)\right) \equiv \dot{\psi} + H_x\left(t, \psi(t), \bar{x}(t), \bar{u}(t)\right) = 0, \tag{3.1}$$

$$\bar{u}(t) \in \operatorname{Arg}\max_u R\left(t, \bar{x}(t), u\right) = \operatorname{Arg}\max_u H\left(t, \psi(t), \bar{x}(t), u\right), \tag{3.2}$$

$$H(t, \psi, x, u) = \psi_i f^i(t, x, u) - f^0,$$

$$\psi(t) = \varphi_x\left(t, \bar{x}(t)\right). \tag{3.3}$$

In the discrete variant the analogous conditions have the following form, see [5]:

$$R_x\left(t, \bar{x}(t), \bar{u}(t)\right) \equiv H_x\left[t, \psi(t+1), \bar{x}(t), \bar{u}(t)\right] - \psi(t) = 0,$$

$$t \in [0, T-1], \tag{3.4}$$

$$R_u\left(t, \bar{x}(t), \bar{u}(t)\right) \equiv H_u\left[t, \psi(t+1), \bar{x}(t), \bar{u}(t)\right] = 0, \tag{3.5}$$

i.e. the maximum principle equations coincide with the given earlier maximum conditions for the function $R(t, x, u)$. Together with the process equations (1.2), (1.3) they form a closed system of equations where the solving function $\varphi(t, x)$ is represented only by its gradient on an optimal trajectory. The analogous coincidence of equations is true for appropriate extensions of the maximum principle (Dubovitzky-Milyutin conditions) and for state constraints, see Khrustalev [29] and [7, p. 120–136].

Equations (2.11) extend this necessary optimality conditions to the global sufficient conditions which depend on the functions $\varphi(t, x)$ such that $\varphi_x\left(t, \bar{x}(t)\right) = \psi(x)$. Simple conditions of this type can be obtained taking a linear solving function $\varphi(t, x) = \psi(t)\, x$. They were considered in [11]. In [3, 7] differential equalities for the matrix $\sigma(t) = \left\|\varphi_{x^i x^j}\left(t, \bar{x}(t)\right)\right\|$ were given. Their satisfaction guarantees the strong or weak relative minimum of the functional. The (necessary and sufficient) Jacoby conditions of variational calculus are equivalent to the existence of the matrix $\sigma(t)$ in the appropriate cases. Development of these kinds of conditions for a local optimum is given in the papers by Rozenberg [12] and Ziedan [13].

The Bellman dynamic programming equations [8] and the Hamilton--Jacoby partial differential equations of variational calculus coincide with the equation (2.18) which define the solving function of a special type. Extensive analysis of the relations between the return functions and the solving functions has been done by Girsanov [14].

4. Analytical methods of solving singular and improper problems

As mentioned earlier the solving function is not completely defined by the optimality conditions (2.11). This way considering different classes of problems we can define different methods of finding them. This possibility was used first of all in those problems where the maximum principle equations or more generally the variational methods can not be used or are ineffective, and application of the dynamic programming equation is too difficult. Firstly, it includes the problems where the minimum is outside of the set D. This means that the necessary conditions of the variational methods can not be used. Secondly, it includes so called singular problems which are unsuitable for traditional methods. These two classes of problems have a nontrivial common part. Many typical minimizing sequences like sliding or discontinuous solutions usually originate from singularity. On the other hand singular problems can have solutions inside the set of admissible processes.

Some special methods of dealing with singular problems were developed using the presented approach. This includes "alternative formalism" for the scalar state problems ($n=1$), see [2, 6], and the method of multiple maxima for multidimensional state problems, see [6, 7, 15]. The method of multiple maxima and some general approaches initialized by it in variational problems which are connected with the idea of extensions are the subject of the present intensive studies by Gurman and his associates [16]. An interested reader can find description of those methods and their applications in the references. They were helpful in finding some new classes of solutions: discontinuous solutions of a special type — so called (x, u)-objects [17, 7], cyclic sliding solutions [18, 19], positional control etc. Because these results were mainly considered for continuous time problems it seems that it is not advisable to describe them in details in this special issue which is dedicated to the discrete time problems. Let us note, however, that probably these methods can be effectively transferred to the discrete time.

We shall illustrate this approach on the following problem:

$$J(v) = \int_{t0}^{t_1} [g(t, x, u) + h(t, x)\, u]\, dt \to \min,$$

$$\dot{x} = u, \; x(0) = x_0, \; x(t_1) = x_1,$$

$n = r = 1$, the function $g(t, x, u)$ is bounded with respect to u, the function $h(t, x)$ is differentiable. We have:

$$R(t, x, u) = [\varphi_x - h(t, x)]\, u - g(t, x, u) + \varphi_t.$$

We define the function $\varphi(t, x)$ by the following equality:

$$\varphi_x = h(t, x).$$

Then:

$$\varphi(t, x) = \int_k^x h(t, \xi)\, d\xi + c(t),$$

$$\varphi_t = \int_k^x \partial h(t, \xi)/\partial t\, d\xi + \dot{c}(t),$$

where $c(t)$ and k are a function and a constant, respectively. Inserting the above in the expression for $R(t, x, u)$ and taking $c(t) \equiv 0$ we get:

$$R(t, x, u) = -g(t, x, u) + \int_k^x \partial h(t, \xi)/\partial t\, d\xi.$$

Let there exist a pair of piecewise continuous functions $(\tilde{x}(t), \tilde{u}(t))$ such that:

$$R\left(t, \tilde{x}(t), \tilde{u}(t)\right) \equiv \mu(t),$$

and such that $\tilde{x}(t)$ is piecewise differentiable. Let us take an integer $s > 0$ and construct the pair of functions $x_s(t), u_s(t)$ in the following way. The interval $[t_0, t_1]$ is divided by the points:

$$t_0 < \tau_1 < \tau_2 < \dots < \tau_s < t_1,$$

which contain all the points in which $\left(x(t), u(t)\right)$ jumps. On each subinterval $[\tau_p, \tau_{p+1}]$, $p = 1, 2, \dots, S-1$, the function $x_s(t)$ is defined by the following formula:

$$x_s(t) = \begin{cases} \tilde{x}(\tau_p+0) + \tilde{u}(\tau_p-0)(t-\tau_p) & \text{for} \quad \tau_p \leqslant t < \tau'_p, \\ \tilde{x}(\tau_{p+1}+0) + \dfrac{t-\tau_{p+1}}{\tau_{p+1}-\tau'_p}[\tilde{x}(\tau_{p+1}+0) - \tilde{x}(\tau_p+0) - \\ \qquad \tilde{u}(\tau_p+0)(\tau'_p-\tau_p)] & \text{for} \quad \tau'_p \leqslant t < \tau_{p+1}, \end{cases}$$

where:

$$\tau'_p = \tau_{p+1} - (\tau_{p+1} - \tau_p)/s^2,$$

and on the subintervals $[t_0, \tau_1]$ and $[\tau_s, t_1]$ by:

$$x_s(t) = x_0 + \frac{\tilde{x}(\tau_1) - x_0}{\tau_1 - t_0}(t - t_0) \quad \text{for} \quad t_0 \leqslant t < \tau_1,$$

$$x_s(t) = x_1 + \frac{\tilde{x}(\tau_1) - x_0}{\tau_s - t_1}(t - t_1) \quad \text{for} \quad \tau_s \leqslant t < t_1,$$

The function $u_s(t)$ is defined by:

$$u_s(t) = \dot{x}_s(t).$$

It is not difficult to check that the sequence of functions defined above, with an additional condition:

$$\Delta_s = \max(\tau_1 - t_0, \tau_2 - \tau_1, \dots, \tau_s - \tau_{s-1}, t_1 - \tau_s) \to 0,$$

satisfies the sufficient conditions (2.13) and (2.14) for the function $\varphi(t, x)$ defined above. The sequence of the state trajectories $\{x_s(t)\}$ converges to the function $\tilde{x}(t)$ while the sequence of its derivatives $\{\dot{x}_s(t) = u_s(t)\}$ converges (in measure) not to $\dot{\tilde{x}}(t)$ but to the function $\tilde{u}(t) \neq \dot{\tilde{x}}(t)$. The intervals where $\dot{x}_s(t)$ is close to $\tilde{u}(t)$ alterate infinitely often with impulses of infinite height. This kind of sequences we call *(x, u)-policy* or *impulse sliding policy*. It is completely given by the pair of function $\tilde{x}(t), \tilde{u}(t)$. The former, $\tilde{x}(t)$, will be called the function of zero closure, and the latter, $\tilde{u}(t)$, the basic control. An algorithm for solving the problem consists in finding the functions $\tilde{x}(t)$ and $\tilde{u}(t)$ from maximization of the function $R(t, x, u)$ for fixed t.

Example.

The following functional is to be minimized:

$$I = \int_0^1 (1 - e^{-(t-u)^2} - txu)\, dt.$$

Here:

$$g \equiv 1 - e^{-(t-u)^2}, \quad h \equiv -tx.$$

We have:

$$\varphi_t = \int_0^x \partial h / \partial t \, dx = -x^2/2.$$

Maximizing this function we get:

$$\tilde{x}(t) = 0, \quad \tilde{u}(t) = t.$$

The above discontinuous solutions were given in [17]. They have interesting application in mechanical and geometrical problems [6, 7].

These kinds of solutions have even more interesting and diverse constructions and realizations for $n > 1$, like, for example, cyclic sliding policies or positional controls. They have intuitive realization in jet propulsion control problems. If the amount of fuel is taken as the process argument, then vertical segments in the state diagram correspond to the motion with the engine turned off. In these points the trajectory is discontinuous. The above optimal solutions can be interpreted as the sequence of infinitely short inclusions of the engine in the given points of the state space with the given thrust and then the cyclic bendings of the trajectory corresponding to the motion of the system with the engine turned off. As shown, the optimal change of space orbits, Gurman [19], and the control for stopping rotation of a solid body with a fixed point, Ioslovich [20, 21], are of this kind.

5. Extension of the class of solving functions and optimality conditions

Function-theoretical properties of admissible processes were established to satisfy two goals which are generally contradictory. These are the simplicity and sufficient completeness of the class D. To satisfy the latter it would be the best to define D in such way that the minimum of the functional $J(v)$ exists. However, firstly, this is not always possible, and secondly, even if it were possible, then the problem formulation might be too bulky. To achieve simplicity it would be convenient to assume that the functions $x(t)$ and $u(t)$ are continuous or sufficiently smooth. But then the solution would very often be in the form of the minimizing sequence. The properties of the admissible processes which are taken here form a known compromise between these concepts.

A similar situation exists with the properties of solving functions $\varphi \in \bar{\Pi}$.

The continuous differentiability is assumed entirely for simplicity and conciseness. However, even in linear problems this is burdensome. The solving function in these problems has the form $\varphi = \psi(t)\,x$, where $\psi(t)$ is the adjoint vector function which is defined by the maximum principle equations. The latter is, as a rule, piecewise differentiable. To include this kind of functions in the class Π we have to weaken the assumptions connected with it, changing the assumption of continuous differentiability to that of continuity on $T \times X$ and to continuous differentiability everywhere on $T \times X$ except for finite number of cross-sections for fixed t's. This was done already in [1].

The problem of extension of the class Π and extension of the sufficient optimality conditions to necessary ones is not reduced, however, only to weakening the function-theoretical restrictions on Π. There the following aspects can be chosen:

1. Existence of an extending sequence $\{\varphi_s\} \in \Pi$ which satisfies (2.15),
2. Weakening requirements for Π to such an extent that the equality (2.15) can be strengthened to:

$$\inf_D J(v) = \max_\Pi l(\varphi), \tag{5.1}$$

3. Extension of the class of solving and bounding functions either by weakening the function-theoretical restrictions in such a way that 2. holds or going over to other representations in order to extend the possibilities of choosing the optimization tools.

It was supposed in [2] that the sufficient optimality conditions above can also be necessary under some additional assumptions. This was formulated as an optimality principle.

There are two approaches to prove this principle. They use quite different mathematical tools. In one a proof of existence of the Hamiltonian-Jacoby-Bellman equation under given conditions and in a given class of functions is used. In the other the direct existence of the maximum of the functional $l(\varphi)$ and its equivalence with the minimum of the functional $J(v)$, $v \in D$, is shown.

The first approach was applied for proving necessity of sufficient conditions (2.11) of the discrete time processes [5]. It was also applied for continuous time processes without state constraints and end conditions [22]. In this case it was found necessary to take the class Π as the class of locally Lipschitzian functions. Then (5.1) is satisfied.

The second approach, that is a proof of the existence of $\max l(\varphi)$, $\varphi \in \Pi$, was used by Ioffe [23].

Let us suppose that φ is normalized and the following inequality, equivalent to (2.3):

$$R(t, x, u) \leqslant 0, \ \forall (t, x, u) \in V,$$

traditional way presented for example by Kelley [39], Eneev [40], Krylov & Chernousko [41], Bryson & Ho [42]. However, the choice of an improving function allows to optimize not in a local (gradient) direction but in a global one.

(ii) Dual methods which are connected with a construction of sequences of solving functions $\{\varphi_s\} \subset \Pi$ maximizing the functional $l(\varphi)$ given by (2.5). This way we get the increasing sequence of lower bounds for the functional J on the set D which converges under appropriate conditions to $\inf_D J(v)$. Yet a solution to the problem is not this sequence but the sequence $\{v_s\} \subset E$ which satisfies (1.6) and (1.7). The role of this sequence is played by $\{\tilde{v}_s\} = \{\tilde{x}_s(t), \tilde{u}_s(t)\} \subset \tilde{E}_s$ which is related to $\{\varphi_s(t, x)\}$ through (2.7). This way we get an approximation to $\inf J(v)$, $v \in D$, by an "outside" approximation of an admissible process. Thus we solve not only the problem (P) but also the problem (I).

(iii) Methods where the ε-optimal feedback control $u(t, x)$ is constructed using the bound (2.17). This leads to minimization of the functional $\Delta(\varphi)$ until it is not greater than a given ε.

6.1. The methods of successive improvements of control

We start by description of the mentioned methods of local improvement of control in terms of the improving function. Let us assume that we know an admissible process $v_0 = (x_0(t), u_0(t)) \in D$. We want to improve it, i.e. to find a $v = (x(t), u(t)) \in D$ such that $J(v) < J(v_0)$. We replace optimization of the functional $J(v)$ by optimization of $L(v, \varphi)$ given by (2.4) with a suitably chosen function φ. We shall look for v which is sufficiently close to v_0 in such a way that the sign of $\Delta J = J(v) - J(v_0)$ is the same as its main linear part:

$$\begin{gathered}\delta J = \delta L = G_x(x_0(t_1))\, \delta x(t_1) - S\left(R_x\, \delta x(t) + R_u\, \delta u(t)\right), \\ \delta x = x - x_0, \qquad \delta u = u - u_0.\end{gathered} \tag{6.1}$$

It is tacitly assumed above that the functions $R(t, x, u)$ and $G(x)$ are differentiable. The formula for δL is given with accuracy down to the function $\varphi(t, x)$. We require that it complies with the equalities:

$$R_x(t, x_0(t), u_0(t)) = 0, \tag{6.2}$$

$$G_x(x(t_1)) = \psi(t_1) + F_x(x(t_1)) = 0. \tag{6.3}$$

These equations contain only the gradient of the function $\varphi(t, x)$ in the points of the trajectory $x_0(t)$. The value of $\psi(t) = \varphi_x(t, x_0(t))$ and the values of (3.1) and (3.4) are determined after replacing $\bar{x}(t), \bar{u}(t)$ by $x_0(t), u_0(t)$. This means that the equations (6.2) and (6.3) are satisfied by functions of the form $\varphi(t, x) = \psi_i(t)\, x^i$, where the vector $\psi(t) = \{\psi_i(t)\}$ is determined by (6.2) and (6.3). This function we call local improving for control. Then:

$$\delta J(v_0) = \delta L(v_0, \varphi) = SR_u(t, x_0(t), u_0(t))\,\delta u(t), \tag{6.4}$$

where $R_u(t, x_0(t), u_0(t))$ equals $H_u(t, \psi(t), x_0(t), u_0(t))$ or $H_u(t, \psi(t+1), x_0(t), u_0(t))$ for the continuous and discrete variants of the problem, respectively.

Let there be given a function $\delta u(t)$ and an arbitrarily small parameter $\varepsilon > 0$ such that:

1. The right-hand side of (6.4) is positive,
2. $u(t, \varepsilon) = u_0 + \varepsilon \delta u \in V_u^t$, $t \in T$,
3. $x(t, \varepsilon) \in V_x^t$, where $x(t, \varepsilon)$ is the trajectory determined by the program control $u(t, \varepsilon)$, the equation of motion, and the initial conditions.
4. $v(\varepsilon) = (x(t, \varepsilon), u(t, \varepsilon)) \in D$.

Then there exists $\varepsilon > 0$ such that:

$$J(v) < J(v_0), \; v = v(\varepsilon). \tag{6.5}$$

Without state constraints, i.e. for $V_x^t = X$, $t \in (0, T]$, the improvement of the given program control $u_0(t)$ reduces to the following steps:

(i) Find the trajectory $x_0(t)$ by solving the Cauchy problem (1.2) or (1.3) with $u = u_0(t)$, $x(0) = x_0$. The program control $u_0(t)$ should satisfy $v_0 = (x_0(t), u_0(0)) \in D$.

(ii) Find $\psi(t)$ and $R_u(t, x_0(t), u_0(t))$ by solving the linear Cauchy problem (6.2) with the initial condition (6.3) which determines a local improving function $\varphi = \psi(t)\, x$.

(iii) Set a variation of the program control $\delta u(t)$ which makes the right-hand side of (6.4) positive.

(iv) For different $\varepsilon > 0$ solve the problem (I) with $u = u_0 + \varepsilon \delta u$. The value of ε should be taken in such a way that (6.5) holds.

The basic part of this algorithm is the "sweeping" solution of the pair of Cauchy problems: the equation of motion from t_0 to t_1 and the adjoint equation from t_1 to t_0. The consecutive repetition of these operations allows to find the improving sequence $\{v_s\} \subset D$.

The expression (6.4) gives the gradient of the functional $J(u)$ in the space of control functions $u(t)$. The presented method can then be considered as an application of the gradient techniques to the above class of problems. A weak point of it is the local character of improvement which is guaranteed only for small variations of the control $u(t)$. This is not only troublesome because the convergence is slow but also because the small variations can be unrealizable, for example when the set V_u^t is finite. This deficiency can be avoided when the globally improving functions are used.

It was shown in [43] that the function $\varphi(t, x)$ is globally improving for a given process $v_0 = (x_0(t), u_0(t)) \in D$ if it satisfies the following conditions:

$$R\left(t, x_0(t), u_0(t)\right) = \min_x R\left(t, x, u_0(t)\right), \quad t \in T,$$
$$G\left(x_0(t)\right) = \max_x G(x), \quad t = t_1. \tag{6.6}$$

A process $v = \left(x(t), u(t)\right)$ which is determined by the control $\tilde{u}(t, x) = \arg\max_u R(t, x, u)$ satisfies the inequality $J(v) < J(v_0)$ if the process v_0 is not an optimal one. For continuous processes it also holds that $\tilde{u}(t, x) = \arg\max_u H[t, \varphi_x(t, x), x, u]$. That is when previously the local improvement was realized by a small variation of control in order to increase the function $R\left(t, x_0(t), u\right)$, then now the new control is chosen as a global maximum of R with respect to u. The condition (6.6) which is satisfied by an improving function φ can be slightly weakened:

$$R\left(t, x_0(t), u_0(t)\right) \leqslant R\left(t, x(t), u_0(t)\right), \quad t \in \bar{T},$$
$$G\left(x_0(t_1)\right) \geqslant G\left(x(t_1)\right), \tag{6.7}$$

where $x(t)$ is the trajectory determined by $u(t, x)$.

To satisfy the inequalities (6.6) it is enough to consider improving functions of the form:

$$\varphi(t, x) = \psi_i(t)\, x^i + \sigma_{ij}(t)\left(x^i - x_0^i(t)\right)\left(x^j - x_0^j(t)\right),$$

where the coefficient $\psi(t) = \{\psi_i\}$, $\sigma_{ij}(t)$, $i, j = \overline{1, n}$, have to be found. It is easy to see that the equations for $\psi(t)$ implied by (6.6) are the same as (6.2) and (6.3). Determination of the matrix $\sigma(t)$ is not unique. One possibility is to consider additionally the equations:

$$R_{x^i x^j}\left(t, x_0(t), u_0(t)\right) = \delta_{ij}\,\eta,\; G_{x^i x^j}\left(x_0(t)\right) = -\delta_{ij}\,\alpha, \quad i, j = \overline{1, n}, \tag{6.8}$$

Here δ_{ij} is the Kronecker delta: $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ii} = 1$; η and α are positive constants. The equations (6.8) form the system of $(n+1)\,n/2$ linear differential (or difference) equations with unknowns $\sigma_{ij}(t) = \sigma_{ji}(t)$ and the given boundary condition at $t = t_1$. These equations together with (6.2), (6.3), and arbitrary positive η, α determine the coefficients of the function $\varphi(t, x)$ such that $x = x_0(t)$ is a relative minimum of $R\left(t, x, u_0(t)\right)$ and minimum of $G(x)$. Appropriately choosing η we can satisfy inequalities (6.7) and therefore (6.5). This way we obtain the following algorithm for improving a solution:

(i) Set $\eta \geqslant 0$, $\alpha \geqslant 0$ and find $\psi(t)$, $\sigma(t)$, $\varphi(t, x)$, $\tilde{u}(t, x)$ by solving the linear Cauchy problem (6.2), (6.3), (6.8) from t_1 to t_0,

(ii) Find the process $v = \left(x(t), u(t) = \tilde{u}[t, x(t)]\right) \in D$ by solving the Cauchy problem for the equation of motion with $u = \tilde{u}(t, x)$, $x(t_0) = x_0$, from t_0 to t_1 and verify the inequality $J(v) < J(v_0)$. If it is not satisfied, then choose another η, α and repeat the calculations.

This procedure improves any process which does not satisfy the maximum principle equations or its discrete equivalent.

Consecutively repeating the above algorithm we find an improving sequence $\{v_s\} \subset D$. However, in general it does not converge to $\inf J(v)$, $v \in D$.

EXAMPLE 1 [44]

The problem is:

$$J = -x^2(2) \to \min,$$

s.t.

$$x^1(t+1) = x^1(t) + 2u(t),$$
$$x^2(t+1) = -(x^1(t))^2 + x^2(t) + u^2(t), \quad t = 0, 1,$$
$$x^1(0) = 3, \quad x^2(0) = 0, \quad |u(t)| \leqslant 5.$$

The optimal solution is $\bar{u}(0) = -2$, $\bar{u}(1) = \mp 5$, $J = -19$. For this problem the Pontryagin maximum principle does not hold. The hamiltonian $H(t, u) = \psi_i(t) f^i(t, \bar{x}(t), u)$ has for $t = 0$ at $\bar{u}(0) = -2$ not the maximum but the minimum. We take $\sigma_{12}(t) = \sigma_{22}(t) = 0$, $\forall t$, $\sigma_{11} = \sigma$ and thus:

$$\varphi(t, x) = \psi_1(t) x^1 + \psi_2(t) x^2 + \sigma(t)(x^1 - x_0^1(t))^2/2.$$

The functions R and G take the form:

$$R(t, x, u) = \psi_1(t+1)[x + 2u] + \psi_2(t+1)[-x^2 + x + u^2] +$$
$$+ 0.5\sigma(t+1)[x + 2u - x_0^1(t+1)]^2 - \psi_i(t) x^i - 0.5\sigma(t)[x^1 - x_0^1(t)]^2,$$
$$G(x) = -x^2 + \psi_i(2) x^i + 0.5\sigma(2)[x^1 - x_0^1(2)]^2.$$

The adjoint equation and the equation for $\sigma(t)$ are as follows:

$$\psi(t) = \psi_1(t+1) - 2x^1(t)\psi_2(t+1), \quad \psi_1(2) = 0,$$
$$\psi_2(t) = \psi_2(t+1), \quad \psi_2(t) = 1,$$
$$\sigma(t) = -2\psi_2(t+1) + \sigma(t+1) - \eta, \quad \sigma(2) = \alpha.$$

See Table 1 for the results.

Table 1

The numerical results of example 1

It. no.	$u(0)$	$u(1)$	$x^1(1)$	$x^2(1)$	$x^2(2)$	$\psi_1(1)$	$\psi_2(1)$	J
1	0	0	3	-9	-18	-6	1	18
2	-2	-5	-1	-5	-19	2	1	19

We choose a class of nonlinear optimal control problems for which the global improving function satisfying (6.6) has the form $\varphi(t, x) = \psi_i(t) x^i$. In this case the algorithm presented above substantially simplifies because there is no need to adjust the coefficients η, α nor to solve the system of equations (6.8). The problem functions have now the form:

$$f(t,x,u)=A(t,u)\,x+B(t,u),\quad F(x)=\lambda x,\quad u\in V_u^t,$$
$$f^0(t,x,u)=a^0(t,u)\,x+b^0(t,u). \tag{6.9}$$

An interesting subclass of these problems is connected with the control of quantum systems by means of a laser radiation. It was investigated and algorithmized using the method described above in [45]. In a simulated experiment a good convergence and effectiveness of the method was obtained for very big dimensions of the state vector which reached some ten thousands.

We mention also a class of so called knapsack multivariate problems where the above method seems to be effective:

$$J(v)=\sum_{t=0}^{N} c_t\,u_t\to\min u_t\in[0,\beta_t], \tag{6.10}$$

$$\sum_{t=0}^{N} a_t^i\,u_t\leqslant b^i\leftrightarrow i=\overline{1,n}, \tag{6.11}$$

and u_t is integer. Application of the above method to these problems was considered by the present author together with Feldman. Introducing a sequence $\{x_t\}\subset R^n$, $t=0,1,\dots,N$:

$$x_{t+1}^i=x_t^i+a_t^i\,u_t,\quad x(1)=0,\quad x^i(N+1)\leqslant b^i,\quad i=\overline{1,n}, \tag{6.12}$$

we can transform the problem (6.10)–(6.11) to the multistage optimization problem where:

$$t_0=0,\quad t_1=N+1,\quad X=R^n,\quad U\text{ — set of integers},$$
$$f(t,x,u)=x+a(t)\,u,\quad f^0=c(t)\,u,\quad V_x^t=R^n\text{ for }t<t_1,$$
$$V_x^t=\{x:x^i\leqslant b^i\}\text{ for }t=t_1,\quad V^{tx}=[0,\beta_t].$$

We have:

$$\varphi(t,x)=\psi_i(t)\,x^i+\sigma_{ii}(t)\left(x^i-x_0^i(t)\right)^2/2,\quad \sigma_{ij}=0,\quad i\neq j,$$
$$R(t,x,u)=\psi_i(t+1)\left(x^i+a^i(t)\,u\right)+\sigma_{ii}(t+1)\left(x^i+a^i(t)\,u-x_0^i(t+1)\right)^2-$$
$$-\psi_i(t)\,x^i-\sigma_{ii}(t)\left(x^i-x_0^i(t)\right)^2-c(t)\,u,$$
$$G(x)=\psi_i(t_1)\,x^i+\sigma_{ii}(t_1)\left(x^i-x_0^i(t_1)\right)^2.$$

Taking $\eta=0$ and solving the equations (6.2), (6.3), and (6.8) we get:

$$\psi_i(t)=\text{constant}=\psi_i=0\text{ if }x_0^i(t_1)<b^i\text{ and}$$
$$\psi_i(t)\geqslant 0\text{ if }x_0^i(t_1)=b^i,\quad i=\overline{1,n};$$

$$\sigma_{ii}(t) = \text{constant} = -\alpha_i, \quad \alpha_i > 0$$

$$R(t, x, u) = -A(t)\, u^2/2 + B(t, x)\, u + C(t, x),$$

$$A(t) = \alpha_i \left(a^i(t)\right)^2 > 0,$$

$$B(t, x) = -\alpha_i\, a^i(t) \left(x^i - x_0^i(t+1)\right) + \psi_i\, a^i(t) - c(t),$$

$$C(t, x) = \alpha_i \left[x_0^i(t+1) - x_0^i(t)\right] + \alpha_i \left[x_0^{i2}(t+1) - x_0^{i2}(t)\right].$$

The expression for $R(t, x, u)$ satisfies (6.6). The control $u(t, x)$ is taken as an integer from the interval $[0, \beta_t]$ which is the closest to the value $u^*(t, x) = B(t, x)/A(t)$. The values α_i are chosen in such a way that the improved trajectory satisfies the inequalities (6.11).

EXAMPLE 2 [46]

The problem is:

$$J = -[6u_1 + 4u_2 + u_3] \to \min,$$

s.t.

$$u_1 + 2u_2 + 3u_3 \leqslant 5,$$

$$2u_1 + u_2 + u_3 \leqslant 4,$$

$$u_t = \{0, 1\},$$

and the optimal solution, see Table 2:

Table 2

The numerical results of example 2

Iter. number	Control			Vector α		Functional
	u_1	u_2	u_3	α_1	α_2	
0	0	0	0	1	1	0
1	1	0	0	1	1	−6
2	1	1	0			−10

$$\bar{u}_1 = \bar{u}_2 = 1, \quad \bar{u}_3 = 0, \quad J = -10.$$

EXAMPLE 3 [47]

The problem is:

$$J = -[3u_1 + 3u_2 + 13u_3] \to \min,$$

s.t.

$$-3u_1 + 6u_2 + 7u_3 \leqslant 8,$$

$$6u_1 - 3u_2 + 7u_3 \leqslant 8,$$
$$0 \leqslant u_t \leqslant 5, \quad u_t \text{ — integer,}$$

and the optimal solution, see table 3:

Table 3

The numerical results of example 3

Iter. number	Control			Vector α		Functional
	u_1	u_2	u_3	α_1	α_2	
0	0	0	0	0.3	0.3	0
1	0	0	1			−13

$$\bar{u}_1 = \bar{u}_2 = 0, \quad \bar{u}_3 = 1, \quad J = -13.$$

In the paper [43] the version with global improving function quadratic in x is disscussed. There exist other versions of this method which are presented in [48].

6.2. The methods of successive improvement of the bounding function

The method is presented according to [49, 50]. Let there exist a function $\varphi_0(t, x) \in \Pi$. We lay down the operation of its improving, i.e. finding a function $\varphi \in \Pi$ such that $l(\varphi) > l(\varphi_0)$. We assume that it has the form: $\varphi = \varphi_0 + \lambda\gamma$, where λ, $\gamma(t, x)$ are a coefficient and a function which should be determined. We introduce a functional:

$$\delta(v) = Sr\left(t, x(t), u(t)\right) + \gamma\left(t, x(t)\right)\Big|_{t=0}^{t_1}, \qquad r(t, x, u) = \gamma_{x^i} f^i. \tag{6.13}$$

We denote by $R(t, x, u, \lambda)$, $\tilde{E}(\lambda)$ etc. the appropriate constructions associated with $\varphi = \varphi_0 + \lambda\gamma$, and also $R_0(t, x, u) = R(t, x, u, 0)$ etc. Taking into account (6.13) and (2.5) the increment $\Delta l = l(\lambda) - l_0$ can be written in the form:

$$\Delta l = \lambda\delta(v) + [L_0(v) - l_0], \quad v \in \tilde{E}(\lambda). \tag{6.14}$$

From this it follows that $l(\varphi) > l(\varphi_0)$ if at least for one $v \in \tilde{E}(\lambda)$:

$$\lambda\delta(v) > 0. \tag{6.15}$$

Fitting λ and γ which satisfy the above inequality will be called *an elementary bound improving operation.* In the sequel we consider for simplicity only the case when the set $\tilde{E}$ contains only one element $\tilde{v}(\lambda) = (\tilde{x}(t, \lambda), \tilde{u}(t, \lambda))$, i.e. the function $R(t, x, u, \lambda)$ has only one maximum. We also denote $\delta(\lambda) = \delta(\tilde{v}(\lambda))$. Under sufficiently general conditions the function $\delta(\lambda)$ is lower semicontinuous at $\lambda = 0$. Thus if we define the function γ to satisfy:

$$\delta(0) = S r\left(t, \tilde{x}_0(t), \tilde{u}_0(t)\right) + \gamma\left(t, \tilde{x}_0(t)\right)\Big|_0^{t_1} > 0, \tag{6.16}$$

then for a sufficiently small $\lambda > 0$ the inequality (6.15) holds and therefore $l(\varphi) \geqslant l(\varphi_0)$. We see then that the elementary operation can be done in two steps. In the first $\gamma(t, x)$ is chosen according to (6.16) and in the second a $\lambda > 0$ is taken.

It is easier to interpret the idea of elementary operation when the improving component is taken in the form $\gamma = v_i(t)\, x^i$ and the functional $\delta(v)$ in the form:

$$\begin{aligned} \delta(v) &= \int_{t_0}^{t_1} v(t)\, \tilde{z}(t)\, dt + \sum_{t \in \beta} v(t)\, \tilde{x}(t)\Big|_{t-1}^{t+0}, \\ &\sum_{t=0}^{t'-1} v(t+1)\, \tilde{z}(t), \end{aligned} \tag{6.17}$$

where $\tilde{z}(t)$ is related to the process $\tilde{v}$ through (1.2) or (1.3), β is the set of points of discontinuity of the function $\tilde{x}(t)$. The first and the second expressions correspond to the continuous and the discrete processes, respectively.

It follows from (6.17) that if there exists a value $t = \tau$ such that $\tilde{z}(\tau) \neq 0$ or in the continuous time $\tilde{x}(\tau+0) - \tilde{x}(\tau-0) \neq 0$, then the improvement of the function $\varphi_0(t, x)$ can be achieved by adding a linear term $\gamma(t, x) = v_i(t)\, x^i$ where the function $v(t)$ is taken to keep the right hand side of (6.17) positive for $\tilde{v} = \tilde{v}_0\left(\tilde{x}_0(t), \tilde{u}_0(t)\right)$.

The use of $\gamma(t, x)$ in more complicated cases is necessary only when a maximum of the function $R_0(t, x, u)$ is not unique.

A weak point in this method is the necessity to maximize the function $R(t, x, u)$ for every t in order to form the process $\tilde{v} = \left(\tilde{x}(t), \tilde{u}(t)\right)$ or more general, the set $\tilde{E}$. Therefore the method can be applied only to problems where this maximization can be performed analytically or there exist efficient numerical procedures for doing it.

Repeating consecutively the elementary improving operations we get the sequence $\{\varphi_s\}$ for which the value $l(\varphi)$ increases.

There exist theorems where it is shown that under some stronger conditions for γ and λ above sequences ensure a solution to the problem (P) for a wide class of systems. Namely, the sequence $\{\tilde{v}_s\} = \{\tilde{x}_s(t), \tilde{u}_s(t)\}$ corresponding to $\{\varphi_s\}$ by (2.7) is a generalized solution to the problem (I) in the sense of (1.5) and (1.6), and to the problem (P) in the sense $l(\varphi_s) \to \lim\limits_{\varepsilon \to 0} d_\varepsilon(\varrho)$ where $d_\varepsilon(\varrho)$ is given by (1.10).

EXAMPLE 4

Find a solution to the system:

$$\dot{x} = u,$$
$$x(0) = x_0 > 0,$$
$$x(1) = 0,$$

which minimizes the functional:

$$J = \int_0^1 (u^2 + x^2)\, dt.$$

Here x and u are scalar functions. We look for a solution in the form of a sequence:

$$\varphi_s = \psi_s(t)\, x, \quad \gamma_s = v_s\, x.$$

We have:

$$R_s(t, x, u) = \psi_s u - u^2 - x^2 + \dot{\psi}_s x,$$
$$\tilde{x}_s = \dot{\psi}_s/2,$$
$$u = \psi_s/2,$$
$$\mu_s(t) = R_s(t, \tilde{x}_s, \tilde{u}_s) = (\psi_s^2 + \dot{\psi}_s^2)/4,$$
$$l_s = -\psi_s(0)\, x_0 - \int_0^1 (\psi^2 + \dot{\psi}^2)\, dt,$$
$$\tilde{z} = \hat{x}_s - \tilde{u}_s,$$
$$\Delta_s = \Delta_s^1 + \Delta_s^2,$$
$$\Delta_s^1 = \int_0^1 |\tilde{z}_s|\, dt,$$
$$\Delta_s^2 = |\tilde{x}(1)| + |\tilde{x}_s(0) - x_0|,$$
$$\delta_s(x, u) = -v_s(1)\, \tilde{x}_s(1) - v_s(0)\left(x_0 - \tilde{x}(0)\right) - \int_0^1 v_s(t)\, \tilde{z}_s\, dt,$$
$$\tilde{\delta}_s = \delta_s(\tilde{x}_s, \tilde{u}_s),$$
$$\Delta\tilde{x}_s = \dot{v}_s/2,$$
$$\Delta\tilde{u}_s = v_s/2,$$
$$R_s(t, x, u, \lambda) = R_s(t, x, u) + \lambda r_s(t, x, u),$$
$$r_s(t, x, u) = v_s u + \dot{v}_s x,$$
$$\tilde{x}_s(\lambda) = \tilde{x}_s + \lambda\Delta\tilde{x}_s,$$
$$\tilde{u}_s(\lambda) = \tilde{u}_s + \lambda\tilde{u}_s.$$

The value λ_s is taken to satisfy the condition:

$$\tilde{\delta}_s(\lambda) \equiv \delta_s(\tilde{x}_s + \lambda \Delta \tilde{x}_s, \tilde{u}_s + \lambda \Delta \tilde{u}_s) = 0,$$

which is in this case an elementary improving operation [48]. We have:

$$\tilde{\delta}_s(\lambda) = \delta_s(\tilde{x}_s, \tilde{u}_s) + \lambda \delta_s(\Delta \tilde{x}_s, \Delta \tilde{u}_s),$$

$$\lambda_s = -\delta_s(\tilde{x}_s, \tilde{u}_s)/\delta_s(\Delta \tilde{x}_s, \Delta \tilde{u}_s),$$

$$\psi_{s+1} = \psi_s + \lambda_s v_s.$$

The function $R(t, x, u)$ has the unique maximum at $\tilde{x}(t)$, $\tilde{u}(t)$ for any ψ and thus the elementary operation is solvable in the class of linear functions $\gamma(t, x) = v(t)\, x$ for any ψ which does not ensure the strict optimum. We provide the specific iterations starting from $\psi_0 = 0$.

ITERATION 1

We have:

$$\tilde{x}_0(t) = \tilde{u}_0(t) = 0,$$

$$\Delta_0^1 = 0, \quad \Delta_0^2 = x_0 - \tilde{x}_0(0) = x_0,$$

$$\tilde{z}(t) \equiv 0, \quad l_0 = 0,$$

$$\tilde{\delta} = \delta_0(\tilde{x}_0, \tilde{u}_0) = -v_0(0)\, x_0 + \int_0^1 v_0(t)\, \tilde{z}_0(t)\, dt = -v_0(0)\, x_0.$$

The condition (6.16) is satisfied for $v_0(0) = -1$. For other values of t the function $v_0(t)$ can be defined arbitrary. We define it in a simple way: $v_0(t) = -1$. We have $\tilde{\delta}_0 = x_0$, $\Delta \tilde{x}_0 = 0$, $\Delta \tilde{u}_0 = -1/2$. $\delta_0(\Delta \tilde{x}_0, \Delta \tilde{u}_0) = -1/2$, $\lambda_0 = 2x_0$. Hence $\psi_1(t) = 0 + \lambda_0 v_0 = -2x_0$. Moreover $\tilde{x}_1(t) \equiv 0$, $\tilde{u}_1(t) = -y_0$, $\Delta_1^1 = x_0$, $l_1 = x_0^2 > l_0 = 0$.

This way in the first iteration the value of l increased but the pair $\tilde{x}$, $\tilde{u}$ did not move closer to D neither in the boundary conditions, i.e. in the norm Δ^2, nor in the integral norm Δ^1.

ITERATION 2

We have:

$$\tilde{\delta}_1 = \delta_1(\tilde{x}_1, \tilde{u}_1) = -v_1(0) + x_0 \int_0^1 v_1(t)\, dt.$$

According to (6.16) and requirements of normalization [50] (1st way):

$$v_1(0) = -1, \quad v_1(t) \equiv 1 \text{ for } t \in (0, 1).$$

This function is discontinuous and does not comply with the conditions of the elementary operation. Therefore we take as $v_1(t)$ a continuous function from the approximating sequence $\{1 - 2(t-1)^k\}$, $k = 2, 4, 6, \ldots$. We choose the function which is the simplest for computing, i.e. $v_1(t) = 1 - 2(t-1)^2$.

We have:

$$\tilde{\delta}_1 = 4/3 x_0,$$

$$\Delta x_1 = 2(1-t) \qquad \Delta u_1 = 1/2-(t-1)^2,$$
$$\delta_1(\Delta x_1, \Delta u_1) = -29,$$
$$\lambda_1 = \tilde{\delta}_1/\delta_1(\Delta x_1, \Delta u_1) = 40/87\, x_0,$$
$$\tilde{x}_2(t) = \lambda_1 \Delta u_1 = 80/87\, x_0 (1-t),$$
$$\tilde{u}_2(t) = \tilde{u}_1 + \lambda_1 \Delta u = -1/87\, x_0 [67+40(t-1)^2].$$

The estimate of the distance from D is:

$$\Delta_1^1 = 10/87\, x_0 \approx 1/9\, x_0, \quad \Delta_2^2 = x_0 - \tilde{x}_2(0) = 7/87\, x_0 \approx 7/90\, x_0.$$

The lower bound is $l_2 \approx x_0^2$. Therefore in the second iteration the pair $\tilde{x}$, $\tilde{u}$ was moved substantially closer to D, approximately 10 times in each norm.

The above method was applied for developing algorithms for solving integral assignment, scheduling, traveling salesman problems [51], different optimization problems of space maneuvers [52] and distributed parameters systems [53].

6. 3. Methods of ε-optimal control synthesis

We want to find an ε-optimal control synthesising function $\tilde{u}(t, x)$. We consider the case when there is no state constraints, including in it also boundary constraints, i.e. $V_x^t = X$, $\forall t$, $x \in T \times X$. Other problems can be solved by this method using penalty functions. We showed above that this problem can be solved using the bounding expression (2.17) and minimizing the functional $\Delta(\varphi)$ until it has the value $\Delta(\bar{\varphi}) = \varepsilon$. Then the synthesising function $\tilde{u}(t, x) = \arg\max_{u \in V^{tx}} R(t, x, u)$ is ε-optimal. The problem of finding an optimal control synthesis is therefore reduced to minimization of the functional $\Delta(\varphi)$. The lower bound for the latter is zero. This bound is attained when in the class Π or its above mentioned refinements there exists a solution of the dynamic programming equation (2.18) or a sequence which approximates this solution in the sense of $\Delta(\varphi)$.

There exist numerical algorithms which use this approach. One of these algorithms [4, 6, 54, 55] is the following. The desired function $\varphi(t, x)$ is taken as an interpolating polynomial in the space $X = R^n$. Its parameters depend on t and are determined from the equations:

$$P(t, x_1(t)) = 0, \quad G(x_1(t_1)) = 0, \tag{6.18}$$

where $\{x_1\}$ is a given set of interpolation knots, $P(t, x)$ and $G(t, x)$ are given by (2.9) and (2.2). The equations (6.18) form a system of normal differential (difference) equations in the function $\varphi(t, x)$ parameters, with the given boundary conditions for $t = t_1$. Solving this system we get the function $\varphi(t, x)$, the corresponding control synthesis $\tilde{u}(t, u)$, and the bound $\Delta(\varphi)$. If the latter is too big, then the computations are repeated with a better set $\{x_1\}$. This is reiterated until we get $\Delta(\varphi) < \varepsilon$.

A second algorithm which was used in some interesting applied problems [7, pp. 349–367] consists in solving the problem $\Delta(\varphi) \to \min$ by the Ritz method. Then a class of functions $\varphi(t, x) = \xi(t, x, a)$ depending on a parameter a is taken. The functional $\Delta(a) = \Delta(\xi(t, x, a))$ is computed and the minimal value of $\Delta(\varphi)$ is found using the mathematical programming method in this class.

The possibilities of using the above methods are limited because of the operations $\sup_x P(t, x)$ and $\inf_x P(t, x)$ which are in (2.17). For many specific problems [56, 57, 58] these operations can be performed analytically. In these cases it is much easier to realize and justify the algorithms for solving the problem $\Delta(\varphi) \to \min$ in the class of the bounding functions which are quadratic in x and moreover to get exact solutions in the form of minimizing sequences of control synthesising functions.

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Received. August 1987.

* Avtomatika i Telemehanika is translated to English as Automation and Remote Control.

The paper was translated from Russian by Z. Nahorski.

Metoda ograniczeń globalnych w teorii sterowania optymalnego

Praca zawiera przegląd rezultatów teoretycznych i praktycznych związanych z warunkami dostatecznymi globalnej optymalności. Obejmuje ona zarówno systemy z czasem ciągłym jak i dyskretnym, z położeniem większego nacisku na te drugie.

Техника глобальных оценок в теории оптимального управления

Дается обзор ряда теоретических и прикладных результатов по достаточным условиям абсолютного оптимума. Рассматривается случай непрерывного и дискретного времении, с акцентом на последнем случае.