Necessary conditions for Pareto optimality in nondifferentiable discrete control problems

by

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A class of multiobjective discrete control problems described only by locally Lipschitz functions is studied from the point of view of necessary optimality conditions in the Pareto sense. It is assumed that possible state and control constraints are given implicitly as general sets being approximated by the respective generalized tangent cone. To investigate this type of nondifferentiable optimization problems some basic facts of the currently developed nonsmooth analysis have to be applied. The crucial role is played by a generalized gradient of a locally Lipschitz function. Using these concepts together with the available results for multiobjective optimization problems one is able to formulate the corresponding necessary optimality conditions in a fairly general form. A special case of a discrete control problem is studied separately to postulate these conditions in a more familiar form.

1. Introduction

The existing results in the field of nondifferentiable optimization are of sufficiently constructive character to be applied to the related areas. In particular, the developed theory in nondifferentiable mathematical programming contains a number of deep results, e.g. see [3–5], [14], [15], [21], [23], which can be directly used when dealing with discrete optimal control problems.

Also in the differentiable case many achievements in discrete optimal control theory are due to development of the mathematical programming theory, as shown in [1], [2], [7], [22]. In nondifferentiable case necessary optimality conditions were investigated mainly in locally Lipschitz setting [9], [11], [21], [25].

At present time there exist several attempts to treat the so-called multi-objective (vector) optimization problems of mathematical programming removing the differentiability assumption. Let us recall at least the conclusions for convex problems [16], [19]. Fundamental results for locally Lipschitz problems of this kind are included in [10], [12], [17], [18] and [20]. All these contributions deal primarily with various forms of necessary conditions for Pareto optimality in mathematical programming problems.

This paper aims to apply these results to the case of multiobjective optimization of discrete-time systems in order to generalize the existing conditions [6], [8] to the locally Lipschitz case. The alternative idea of "isoperimetric" and "max-type" reduction of multiobjective optimization problems introduced in [12] is briefly outlined and discussed. Few necessary facts from nondifferentiable calculus are included for the reader's convenience. Then the necessary conditions for Pareto optimality of discrete control problems are presented assuming only locally Lipschitz formulation.

2. Multiobjective optimization problems

Consider a vector function $f : R^n \to R^s$ with components $f_1, f_2, ..., f_s$ and a set $\Omega \subset R^n$. The function f represents s objectives (criteria) according to which the choice of an element (decision) from the admissible set Ω is controlled. The solution concept in this case is the so-called Pareto point (vector minimum, noninferior point, etc.).

DEFINITION 1. A point $\hat{x} \in \Omega$ is called a Pareto point for vector performance index $f = (f_1, f_2, ..., f_s)$ if and only if for every $x \in \Omega$ either $f_i(x) = f_i(\hat{x})$ for all $i = 1, ..., s$, or there exists at least one $i \in \{1, 2, ..., s\}$ such that $f_i(x) > f_i(\hat{x})$.

Thus Pareto point is nothing else than a minimal element of $f(\Omega)$ with respect to the partial order $\leqslant$ in R^s. In fact, there is a variety of possible equivalents of Definition 1, which are responsible for a number of alternative approaches to the problems of necessary conditions for Pareto points. This is illustrated by the following two propositions which can be verified by straightforward contradictions.

PROPOSITION 1. Let f and Ω be as in Definition 1. Then $\hat{x}$ is a Pareto point of f on Ω if and only if $\hat{x}$ is a solution to the problem of minimizing $f_i(x)$ subject to the constraints $x \in \Omega_i = \{y \in \Omega | f_j(y) \leqslant f_j(\hat{x}), \ j \neq i, \ j = 1, 2, ..., s\}$ for all $i = 1, 2, ..., s$.

This is the mentioned "isoperimetric" reduction of multiobjective optimization problems to problems with scalar objective only. For convex problems in mathematical programming it was applied in [16] and for locally Lipschitz problems in [12]. Thus one has to use some existing

necessary optimality conditions to the scalar minimization problems given in Proposition 1. The only prerequisite is that these optimality conditions include inequality type constraints as required for handling the constraining sets Ω_i.

DEFINITION 2. A point $\hat{x} \in \Omega$ is called a weak Pareto point for vector performance index $f = (f_1, f_2, ..., f_s)$ if and only if there exists no $x \in Q$ such that $f_i(x) < f_i(\hat{x})$, $i = 1, 2, ..., s$.

Also in this case one can find an alternative characterization of a weak Pareto point using the auxiliary minimization problem with the scalar "max-type" objective function. Thus this reduction scheme assumes the ability to deal with nondifferentiable problems, as the max operation does not preserve differentiability, in general.

PROPOSITION 2. Let f and Ω be as in Definition 2. Then $\hat{x}$ is a weak Pareto point of f on Ω if and only if $\hat{x}$ is a solution to be problem of minimizing function $F(x) = \max\{f_i(x) - f_i(\hat{x}) | i = 1, 2, ..., s\}$ subject to $x \in \Omega$.

It is easy to see that every Pareto point is at the same time also a weak Pareto point.

3. Necessary conditions for Pareto points

Let us briefly summarize some basic facts concerning the calculus of locally Lipschitz functions, i.e. functions being almost everywhere differentiable. More exactly, $f: R^n \rightarrow R^1$ is locally Lipschitz iff for any bounded set $B \subset R^n$ there exists a constant L such that for all $x, y \in B$ one has $|f(x) - f(y)| \leq L \|x - y\|$. The reader can consult [3], [4], [14], [15] for further details. A comprehensive theory of nonsmooth optimization is included in [5]. In what follows all functions are assumed to be locally Lipschitz (in a vector case component-wise).

DEFINITION 3. The generalized gradient of a function $f: \dot{R}^n \rightarrow R^1$ at the point x, denoted $\partial f(x)$, is the set (co denotes the convex hull)

$$\partial f(x) = \operatorname{co}\{\lim_{i \to \infty} \nabla f(x_i) | x_i \to x\},$$

with f differentiable at x_i for each i.

It can be shown that $\partial f(x)$ is a nonempty and compact set. If f is continuously differentiable at x, than $\partial f(x) = \nabla f(x)$, and convex f implies that $\partial f(x)$ is a subdifferential of f at x.

The following properties of the generalized gradient relate to the subsequent exposition. Let $f, g: R^n \rightarrow R^1$ and $c \in R^1$. Then

$$\partial(cf(x)) = c\,\partial f(x), \tag{1}$$

$$\partial(f+g)(x) \subset \partial f(x) + \partial g(x), \tag{2}$$

$$f \text{ attains a local minimum at } x \Rightarrow 0 \in \partial f(x). \tag{3}$$

Let $f(x) = \max \{f_i(x) | i = 1, 2, ..., s\}$, where each f_i is locally Lipschitz. Then f is locally Lipschitz and

$$\partial f(x) = \mathrm{co} \{\partial f_i(x) | i \in I(x)\}, \tag{4}$$

where $I(x)$ is the set of indices i such that $f_i(x) = f(x)$.

If now $Q \subset R^n$ is non-empty and closed, denote as $d_Q(x)$ the real function giving the distance of x to Q, i.e. $d_Q(x) = \inf \{\|x - q\| \,|\, q \in Q\}$. As the function d_Q is a Lipschitz function, it is possible to introduce the generalized normal cone to Q at the point x as the set (cl denotes the closure)

$$N(Q_{jx}) = \mathrm{cl} \{\gamma q | \gamma > 0, q \in \partial d_Q(x)\}. \tag{5}$$

The polar cone to $N(Q_{jx})$ is the so-called generalized tangent cone $T(Q; x)$, which plays the role of a conical approximation to the set Q.

Let us remark that the listed properties remain valid, in general, also in Banach space context. For the necessary modification of Definition 3 see [3], [4]. Further consider the problem of minimization of the function f subject to the constraints $x \in Q \subset R^n$, $h_i(x) = 0$, $i = 1, ..., p$ and $g_i(x) \leqslant 0$, $i = 1, ..., q$, with f, h_i, g_i being real locally Lipschitz on R^n. The following result can be found in [15, Corollary 3.2]. Only the sign of multipliers is reversed to conform with later studied discrete optimal control problems, where such usage is common.

PROPOSITION 3. If $\hat{x}$ is a minimizing point in the above problem, then there exist multipliers $\mu_0 \leqslant 0$, λ_i, $i = 1, ..., p$, and $v_i \leqslant 0$, $i = 1, ..., q$, not all zero, such that

$$\partial \left(\mu_0 f + \sum_{i=1}^{p} \lambda_i h_i + \sum_{i=1}^{q} v_i g_i\right)(\hat{x}) \cap N(Q; \hat{x}) \neq \phi. \tag{6}$$

and $v_i g_i(\hat{x}) = 0$, $i = 1, ..., q$.

As shown in [5, Thm. 6.1.1], this result is valid also in Banach space setting. The condition (6) implies with respect to (2) the weaker "separated" condition, the analogy of which will be used later

$$\mu_0 \,\partial f(\hat{x}) + \sum_{i=1}^{p} \lambda_i \,\partial h_i(\hat{x}) + \sum_{i=1}^{q} v_i \,\partial g_i(\hat{x}) \cap N(Q_j \hat{x}) \neq \phi. \tag{7}$$

Now let $f: R^n \to R^s$, $f = (f_1, f_2, ..., f_s)$ be locally Lipschitz, and let

$$\Omega = \{x \in Q \subset R^n | h_i(x) = 0, i = 1, ..., p, g_i(x) \leqslant 0, i = 1, ..., q\}, \tag{8}$$

where Q is closed and h_i, g_i are real locally Lipschitz functions. On applying Proposition 1 it is possible to derive the following result [12].

THEOREM 1. *If $\hat{x}$ is a Pareto point of problem (8), then there exist multipliers*

$\mu_i \leqslant 0$, $i = 1, \ldots, s$, λ_i, $i = 1, \ldots, p$, and $v_i \leqslant 0$, $i = 1, \ldots, q$, *not all zero, such that*

$$\partial \left(\sum_{i=1}^{s} \mu_i f_i + \sum_{i=1}^{r} \lambda_i h_i + \sum_{i=1}^{q} v_i g_i \right) (\hat{x}) \cap N(Q; \hat{x}) \neq \phi. \tag{9}$$

with $v_i g_i(\hat{x}) = 0$, $i = 1, \ldots, q$.

When using suggested max-type reduction scheme, Proposition 2 is used together with (4) to obtain necessary optimality conditions for weak Pareto points in the "separated" form only [12].

THEOREM 2. *If $\hat{x}$ is a weak Pareto point of problem* (8), *then there exist multipliers* $\mu_i \leqslant 0$, $i = 1, \ldots, s$, λ_i, $i = 1, \ldots, p$ *and* $v_i \leqslant 0$, $i = 1, \ldots, q$, *not all zero, such that*

$$\sum_{i=1}^{s} \mu_i \, \partial f_i(\hat{x}) + \sum_{i=1}^{p} \lambda_i \, \partial f_i(\hat{x}) + \sum_{i=1}^{q} v_i \, \partial g_i(\hat{x}) \cap N(Q; \hat{x}) \neq \phi. \tag{10}$$

with $v_i g_i(\hat{x}) = 0$, $i = 1, \ldots, q$.

The indicated indirect approach can be an alternative one when deriving necessary optimality conditions for multicriterial problems. However, the available stronger general result [5, Thm. 6.6.3] implies that the formulation of Theorem 1 is valid also for weak Pareto points. From this point of view the analyzed reduction schemes do not give equivalent results. Therefore Theorem 1 can be used to study both types of solutions. These results valid in the Banach space formulation generalize the respective theory developed for differentiable [24] and convex [18], [20] cases.

An important question of the regularity, i.e. not all μ_i in (9) or (10) equal zero, can be investigated along the lines followed in [12]. It is necessary to assume additional calmness of the pertinent scalar-valued optimization problem.

Finally, for $Q \subset R^n$ denote by int Q the interior of Q in R^n. Now let Q_1 and Q_2 be non-empty and closed sets in R^n, and let $x \in Q_1 \cap Q_2$ be a point for which $T(Q_1; x) \cap \text{int } T(Q_2; x) \neq \phi$. Then according to [23] the following relation holds.

$$N(Q_1 \cap Q_2; x) \subset N(Q_1; x) + N(Q_2; x). \tag{11}$$

Observe the apparent analogy of the above requirement to the "regularity" condition of [13] postulated when dealing with general extremum problems with constraints.

Further details can be found in the listed references. Having in mind the later applications let us mention also certain drawbacks of the existing theory with respect to the definition of a partial generalized gradient [4], [23]. To do this let $f : R^n \times R^m \to R^1$ be locally Lipschitz. For each $x \in R^n$ define the generalized gradient of a function $f(x, \cdot)$ by $\partial_y f(x, y)$ and in a similar way also for $\partial_x f(x, y)$. Such definition of a partial generalized

gradient seems to be quite natural and reasonable. However, there is no relationship between the sets $\partial f(x, y) = \partial_{x,y} f(x, y)$ and $\partial_x f(x, y) \times \partial_y f(x, y)$. Simple examples show that neither of these sets is contained in the other, as would be desirable in our applications.

This fact is also the main reason that in the studied general case of discrete control problem in the next section the obtained necessary conditions exhibit more formal character. On the other hand, for a particular case of discrete systems, which are additive in state and control variables, a familiar structure of necessary optimality conditions can be maintained.

4. Multiobjective discrete control problem

Let us consider a discrete dynamical system described by the following relations

$$x_{k+1} = f_k(x_k, u_k), \quad k = 0, 1, \ldots, K-1, \tag{12}$$

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} \in M_k \subset R^n \times R^m, \quad k = 0, 1, \ldots, K-1, \tag{13}$$

$$x_K \in A_K \subset R^n. \tag{14}$$

Here K denotes the prescribed number of stages, $x \in R^n$ the state, and $u \in R^m$ the control. The aim is to minimize the function

$$J = g(x_K) + \sum_{k=0}^{K-1} h_k(x_k, u_k), \tag{15}$$

where $f_k : R^n \times R^m \to R^n$, $h_k : R^n \times R^m \to R^s$ and $g : R^n \to R^s$. It is assumed that all these functions are locally Lipschitz and all introduced sets non-empty and closed.

It is not very difficult to realize that the stated control problem (12)–(15) represents a multiobjective mathematical programming problem of type (8) in the space of dimension $mK + n(K+1)$, i.e. one has to work with a variable $z = (x_0, x_1, \ldots, x_K, u_0, u_1, \ldots, u_{K-1})$. A special structure of such mathematical programming problem enables to decompose it with respect to the discrete time variable k.

To derive the further formulated theorem one has to understand that if function f does not depend on a certain variable, the corresponding component of all vectors in ∂f is zero. Moreover, if $Q \subset R^n$ and $x \in Q$, the generalized normal cone $N(Q; x)$ is in the same time also a generalized normal cone to the set $Q \times R^m \subset R^n \times R^m$ at (x, y), $y \in R^m$. Then the relations (9) and (11) are repeatedly applied together with the properties (1) and (2). There are no principal difficulties in this construction, which is a straight-

forward one. The only complication is somewhat extensive notation. Let us therefore omit the particular details and formulate only the final result using vector notation for convenience.

THEOREM 3. Let $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{K-1}$ and the corresponding $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K$ be a Pareto (weak Pareto) point of the stated multiobjective discrete optimal control problem (12)–(15), and assume that

$$\text{int } T\left(M_k; (\hat{x}_k, \hat{u}_k)\right) \neq \phi, \; k = 0, 1, \ldots, K-1.$$

Then there exists a vector $\mu \in R^s$, $\mu_i \leq 0$, $i = 1, \ldots, s$, and vectors $\lambda_k \in R^n$, $k = 1, \ldots, K$, not all zero, such that

$$\partial_{x,u} H_{k+1}(\hat{x}_k, \hat{u}_k) \cap \binom{\lambda_k}{0} + N\left(M_k; (\hat{x}_k, \hat{u}_k)\right) \neq \phi, \; k = 0, 1, \ldots, K-1, \tag{16}$$

with $\lambda_0 = 0$, and

$$\partial_x \mu^T g(\hat{x}_K) \cap \lambda_K + N(A_K; \hat{x}_K) \neq \phi. \tag{17}$$

As usual

$$H_{k+1}(x, u) = \mu^T h_k(x, u) + \lambda_{k+1}^T f_k(x, u), \; k = 0, 1, \ldots, K-1. \tag{18}$$

One can see an overall analogy of the obtained necessary conditions with those known for the differentiable case [6], [8]. The imposed assumption concerning the respective generalized tangent cones ensures the application of (11) in order to be able to decompose also the overall constraining set of the resulting mathematical programming problem. On the other hand, the mentioned property of partial generalized gradients ∂_x and ∂_u does not allow to decompose the adjoint conditions (16) in simple way to separate the relations for x and u. Therefore the indicated composed form of a generalized gradient $\partial_{x,u}$ must be preserved, in general. This was also the motivation to assume general implicit constraints of the "mixed" type in (13). Otherwise, such generalization seems not to be of great practical importance. The presented fairly general form of necessary optimality conditions, although interesting from a theoretical point of view, does not possess too much practical impact and more concrete form is therefore desirable. Problems of this kind arise in some applications in economy and management science especially in connection with multi-level decision-making.

5. Special class of discrete control problems

One way to overcome the encountered difficulty with generalized gradient formulation is to impose certain additional assumptions on the studied control problem. First, one can simply assume the so-called subdifferential

regularity [23] of all functions f_k, h_k and g. Then one has that, e.g. $\partial_{x,u} f_k(x, u) \subset \partial_x f_k(x, u) \times \partial_u f_k(x, u)$, which makes the required decomposition possible. Otherwise, one can assume the "additive" structure of all f_k and h_k as mentioned earlier and used for a scalar objective in [9]. Then the generalized gradient inclusion is still preserved. Therefore let

$$x_{k+1} = f_k^1(x_k) + f_k^2(u_k), \quad k = 0, 1, ..., K-1, \tag{19}$$

$$x_k \in A_k \subset R^n, \quad k = 0, 1, ..., K, \tag{20}$$

$$u_k \in U_k \subset R^m, \quad k = 0, 1, ..., K-1. \tag{21}$$

The aim is to minimize the functional

$$J = g(x_K) + \sum_{k=0}^{K-1} \left(h_k^1(x_k) + h_k^2(u_k) \right). \tag{22}$$

Here $f_k^1 : R^n \to R^n, f_k^2 : R^m \to R^n, h_k^1 : R^n \to R^s, h_k^2 : R^m \to R^s$, and $g : R^n \to R^s$. Again it is assumed that these functions are locally Lipschitz and all indicated sets non-empty and closed. For this special case Theorem 3 takes the form as follows.

THEOREM 4. *Let* $\hat{u}_0, \hat{u}_1, ..., \hat{u}_{K-1}$ *and the corresponding* $\hat{x}_0, \hat{x}_1, ..., \hat{x}_K$ *be a Pareto (weak Pareto) point of the multiobjective discrete optimal control problem* (19)–(22). *Assume that* int $T(A_k; \hat{x}_k) \neq \phi$ *and* int $T(U_k; \hat{u}_k) \neq \phi$, $k = 0, 1, ..., K-1$. *Then there exists a vector* $\mu \in R^s$, $\mu_i \leqslant 0$, $i = 1, ..., s$, *and vectors* $\lambda_k \in R^n$, $k = 1, ..., K$, *not all zero, such that the following conditions are satisfied.*

(a) *The vectors* λ_k *satisfy the adjoint relations*

$$\partial_x(\mu^T h_k^1 + \lambda_{k+1}^T f_k^1)(\hat{x}_k) \cap \lambda_k + N(A_k; \hat{x}_k) \neq \phi, \quad k = 0, 1, ..., K-1, \tag{23}$$

with $\lambda_0 = 0$, *and*

$$\partial_x \mu^T g(\hat{x}_K) \cap \lambda_K + N(A_K; \hat{x}_K) \neq \phi. \tag{24}$$

(b) *The optimal control sequence satisfies the relations*

$$\partial_u(\mu^T h_k^2 + \lambda_{k+1}^T f_k^2)(\hat{u}_k) \cap N(U_k, \hat{u}_k) = \phi, \quad k = 0, 1, ..., K-1. \tag{25}$$

Now one can see more evident analogy with a differentiable case [6], [8], as the "adjoint system equations" and "optimality conditions" are separated — see (23) and (25). A question can arise about the maximum principle formulation of (25). As it is known, such form requires additonal assumptions regarding the convexity of discrete control problems. The results valid for the scalar nondifferentiable case are readily modified to the multiobjective setting — see [9], [25]. Furthermore, one can also derive, without no substantial troubles, more explicit case with all constraining sets given as a system of equalities and inequalities [9].

6. Conclusions

A possible application of recent results in the field of nonsmooth analysis to nondifferentiable (locally Lipschitz) multiobjective discrete control problems was investigated. Necessary optimality conditions for Pareto points were presented for the studied classes of problems. In this connection it was shown that only under the additional assumptions we are able to bring these conditions to familiar form. A special class of a discrete control problem was therefore treated in detail to overcome this difficulty. All results can be in an obvious way generalized to Banach spaces formulation.

The present theory of nondifferentiable optimization also includes the cases when only lower semicontinuity instead of locally Lipschitz continuity is assumed. There exist some results dealing with general type of functions only [3], [23]. On the other hand, it has to be realized that in such general setting several useful properties of the locally Lipschitz case are lost. Then especially applications to discrete optimal control, as concluded also in [9], become more questionable as the used decomposition technique cannot be simply substituted.

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Warunki konieczne optymalności w nieróżniczkowalnych zadaniach sterowania dyskretnego

W pracy rozważa się klasę wielokryterialnych zadań sterowania dyskretnego opisanych funkcjami lokalnie lipschitzowskimi pod kątem wprowadzenia koniecznych warunków optymalności w sensie Pareto. Założono, że ograniczenia na stany i sterowania są zadane w sposób pośredni jako zbiory, które można aproksymować uogólnionymi stożkami stycznymi. Do badania tego typu nieróżniczkowalnych zadań optymalizacji zastosowano pewne podstawowe konstrukcje obecnie rozwijanej analizy nieróżniczkowalnej. Główną rolę gra w nich uogólniony gradient funkcji lokanie lipschitzowskiej. Użycie tego aparatu oraz istniejących wyników dla wielokryterialnych zadań optymalizacji pozwoliło na sformułowanie odpowiednich warunków optymalności w dostatecznie ogólnej postaci. Rozważono dodatkowo specjalny przypadek zadania sterowania dyskretnego pozwalający na przedstawienie tych warunków w bardziej rozpowszechnionym zapisie.

Необходимые условия оптимальности
в недифференцируемых задачах дискретного управления

В работе рассматривается класс многокритериальных задач дискретного управления, описываемых локально липшицовыми функциями с точки зрения введения необходимых условий оптимальности в смысле Парето. Предполагается, что ограничения по состоянию и управлению заданы посредством множеств, которые можно аппроксимировать обобщенными касательными конусами. Для исследования этого типа недифференцируемых задач оптимизации использовались некоторые конструкции развиваемого в настоящие время недифференцируемого анализа. Основную роль играет в них обобщенный градиент покально липшицовых функций. Использование этого аппарата и существующих результатов для многокритериальных задач оптимизации позволило сформулировать соответствующие условия оптимальности в достаточно обобщенном виде. Дополнительно рассмотрен особый случай задачи дискретного управления, позволяющий представить эти условия в более удобном виде.