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Models and optimality conditions for discrete-continuous processes

by

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In order to describe control systems with variable structure, a hierarchical two-level model with a discrete time dynamical model at the upper level and a continuous time differential model with homogeneous structure at the lower level is considered. New sufficient optimality conditions which generalize those developed earlier by the author for such systems are proposed. In these conditions at each level the Krotov function is replaced by a resolving system which consists of a family of Krotov-type functions and a functional over this family. A brief review of earlier theoretical and applied results is given.

1. Introduction

Many real systems, continuous by nature, may exhibit different properties in different situations. For that reason they are described with difficulties or are not described at all in terms of classical differential equations. As examples we may consider multioperational technological processes, interplanetary space flights, walking robots, etc. They can be conveniently described in terms of hierarchical dynamical models that use discrete models at the upper level, and continuous differential models with the homogeneous structure at the lower level.

The discrete model is described by the following relations

$$x(t+1)=f\big(t, x(t), u(t)\big),\ t\in T=\{t_i, t_i+1, \dots, t_f\}, \tag{1}$$

$$\big(x(t), u(t)\big)\in B(t)\subset X_0(t)\times U_0(t), \tag{2}$$

$$f(t,\cdot): X_0(t)\times U_0(t)\to X_0(t+1),$$

$$\gamma = (t_i, x(t_i), t_f, x(t_f)) \in \Gamma, \tag{3}$$

where X_0, U_0 are some arbitrary sets and B, Γ are given sets. Introducing the projection X of B on X_0 and its section at given $x(t)$ we can write the condition (2) in the following form

$$x(t) \in X(t),\ u(t) \in U(t, x(t)).$$

The chain (1) can be also represented by

$$x(t+1) \in V(t, x(t)) = f(t, x(t)),\ V(t, x(t)).$$

Let us call V the set of possible transitions from the state x or briefly the transition set.

Let us denote by D the set of elements (triplets) $m = (T, x(t), u(t))$, $t \in T$, which satisfy the conditions (1)–(3).

The continuous differential model is described by the following relations

$$\dot{y} = h(\tau, y, w),\ \tau \in T_c = [\tau_i, \tau_f] \in R, \tag{4}$$

$$(y, w) \in B_c(\tau) \subset Y_0 \times W_0, \tag{5}$$

$$\gamma_c = (\tau_i, y(\tau_i), \tau_f, y(\tau_f)) \in \Gamma_c \subset (R \times Y_0)^2, \tag{6}$$

where Y_0 is the Euclidean n-dimensional space, W_0 is an arbitrary set, $h: (R^{n+1} \times W_0) \to Y_0$ is an operator, $B_c(\tau)$, $\Gamma_c(\tau)$ are given sets.

Let us denote by D_c the set of triplets $m_c = (T_c, y(\tau), w(\tau))$, $\tau \in T_c$, where $y(\tau)$ is an absolute continuous or piecewise smooth function, satisfying the conditions (5)–(6) and (4) (almost everywhere on T_c) and call it the admissible set. It is supposed that it is not empty.

The condition (5) can be written in the form

$$y \in Y(\tau), \qquad w \in W(\tau, y),$$

and the equation (4) in the form

$$\dot{y} \in V_c(\tau, y) = h(\tau, y, W(\tau, y)),$$

where $Y(\tau)$, $W(\tau, y)$ are the projection of $B_c(\tau)$ on Y_0 and the intersection of $B_c(\tau)$ with given $y \in Y(\tau)$, respectively.

2. The model of the composite process

Let $T_* \subset T \backslash t_i \backslash t_f$ be some subset of T and assume that there holds $U_0(t) = U_{0d}(t) \times U_{0c}$ on this subset, where U_{0d} is an arbitrary set, U_{0c} is an arbitrary set of triplets m_c and the intersection of the set $B(t) \subset \subset X_0(t) \times U_{0d}(t) \times U_{0c}(t)$ of the discrete model with given $x(t) \in X_0(t)$, $u_d(t) \in U_{0d}(t)$ is the admissible set $D_c(t, x(t), u_d(t))$ of the continuous model.

In other words, D_c is formed at a given t by the parametric family of the ordinary differential systems (4)–(6) with $x(t)$, $u_d(t)$ as the parameters.

Let the operator $f(t, \cdot)$ be given in the following form

$$f(t, x(t), u(t)) = g(t, x(t), u_d(t), \gamma_c),$$

where γ_c is the bound of the corresponding differential system and $g(t, \cdot): X_0(t) \times U_{0d}(t) \times (R \times Y_0)^2 \to X(t+1)$.

Let us consider the set $(T_*, x(t), u_d(t), m_c(t))$ (or more explicitely $(T_*, x(t), u(t), T_c(t), y(t, \tau), w(t, \tau)),\ t \in T_*,\ \tau \in T_c(t)))$ as the solution of the above combined system. In this case the element $m = (T, x(t), u(t)) \in D$ on the whole we call the discrete-continuous or alternatively the composite process.

3. Sufficient optimality conditions

Let us consider the optimal control problem for the model developed above in some standard form as the problem of the minimization of a functional $I = F(\gamma)$ (of the upper level model) with the given t_i, t_f. Sufficient optimality conditions for this problem can be obtained on the basis of the principle of extensions [1, 2], which is expressed by the following abstract lemma.

LEMMA 1 (V. F. Krotov, M. M. Khrustalev). *Assume that for a given functional $I: M \to R$ there exist a sequence $\{m_s\} \subset D \subset M$, a functional $L: M \to R$, a set $E \subset M$ and a number l such that* 1) $m \in D \subset E$; 2) $L(m) \leqslant I(m) \forall m \in D$; 3) $l \leqslant L(m), m \in E$, 4) $I(m_s) \to l$.

Then $l = \inf_E L = \inf_D I$, $\{m_s\}$ minimizes I on D and $I(m_s) \to l$ for any minimizing sequence of I on D.

Proof. The first assertion of the lemma is evident. Consider the second one. If $\{m_s\}$ is a minimizing sequence, then $I(m_s) \to \inf_D I = l$. Assume that $L(m_s) \not\to l$. Then by the definition of l a number $\varepsilon > 0$ can be found such that $L(m_s) > l + \varepsilon$. But this contradicts the condition $I(m_s) \to l$. Therefore there exists a number s such that $I(m_s) < l + \varepsilon$ and consequently $I(m_s) < L(m_s)$. But this is impossible by the definition of $L(m)$. ■

The principle of extensions consists in replacing the initial problem (D, I) by a similar problem (E, L) which is simpler in some sense but also gives a solution to the initial problem.

Let us call the problem (E, L) with the conditions 1, 2 an extension of the initial problem (D, I) and a resolving extension if the conditions 3, 4 of lemma 1 are satisfied.

The second assertion of the lemma means that if a resolving extension is found for some solution of the initial problem then it is resolving for any other solution i.e. it permits obtaining all solutions of the initial problem.

For the problem considered we shall obtain an extension (L, E) in the following way. Exclude the recurrent chain (1) and differential constraint (4) from the restrictions of the sets D and D_c. Introduce functionals $\omega \in \Omega$, $\varphi(t, x(t), \alpha)$ $(\varphi(t, \cdot)): X_0(t) \times A \to R$ and a parametric family

$$(\varkappa(z) \in \Omega, \theta(z, \tau, y, \delta) \in \Xi),$$

where $z = (t, x(t), u_d(t))$, $\theta(z, \cdot): R^{n+1} \times \Delta(z) \to R$, $\Xi = \left\{\theta : \frac{d}{d\tau}(\theta(\tau, y(\tau)) = \theta_\tau + \right.$ $\left. + \theta'_y \dot{y}, \forall m_c \in D_c \right\} \left(\theta_y = \operatorname{colon}\left\{\frac{\partial \theta}{\partial y^1}, \ldots, \frac{\partial \theta}{\partial y^n}\right\}, \theta_\tau = \frac{\partial \theta}{\partial \tau}\right)$, $\Omega = \{\omega : R \to R,\ R =$ $= \{\tau : Q \to R\},\ \tau(q) \leqslant 0 \Rightarrow \omega \leqslant 0,\ \omega(0) = 0\}$, and Q is an arbitrary set, $Q = A$ or $Q = \Delta(z)$.

Specify the functional L in the following way

$$L = G - \omega\Big(\sum_{T \setminus T_* \setminus t_f} K(t, \cdot) - \sum_{T_*} G_c(t, \cdot) - \varkappa\Big(\int_{T_c(z)} K_c(z, \delta, \cdot)\, d\tau\Big),$$

where

$$G(\gamma) = F(\gamma) + \omega\big(\varphi(t_f, x(t_f), \alpha) - \varphi(t_i, x(t_i), \alpha)\big),$$

$$K(t, x(t), u(t), \alpha) = \varphi\big(t+1, f(t, x(t), u(t), \alpha)\big) - \varphi(t, x(t), \alpha),$$

$$G_c(z, \gamma_c, \alpha) = -\varphi(t+1, g(z, \gamma_c), \alpha) + \varphi(t, x(t), \alpha) +$$
$$+ \varkappa\big(\theta(z, \tau_f, y(\tau_f), \delta) - \theta(z, \tau_i, y(\tau_i), \delta)\big),$$

$$K_c(z, \tau, y, w, \delta) = \theta'_y(z, \tau, y, \delta)\, h(z, \tau, y, w) + \theta_\tau(z, \tau, y, \delta).$$

It is easily seen that $L = I$ when $m \in D$. Actually when $m \in D$ $(m_c(z) \in$ $\in D_c(z))$ and the restrictions (1) and (4) are satisfied, then

$$L = F(\gamma) + \omega(\varphi_f - \varphi_i)_\alpha - \omega\Big(\sum_{T \setminus T_* \setminus t_f} (\varphi(t+1, x(t+1), \alpha) -$$
$$- \varphi(t, x(t), \alpha) + \sum_{T_*} (\varphi(t+1), x(t+1), \alpha) - \varphi(t, x(t), \alpha) +$$
$$+ \varkappa(z, (\theta_f - \theta_i)_\delta) - \varkappa\big(z, \int_{T_c(z)} (\theta'_y \dot{y} + \theta_\tau)_\delta\, d\tau\big)\Big) =$$
$$= F(\gamma) + \omega(\varphi_f - \varphi_i)_\alpha - \omega\Big(\sum_{T \cdot t_f} (\varphi(t+1, \cdot) - \varphi(t, \cdot))_\alpha\Big) = F(\gamma) = I.$$

From this fact and the principle of extensions (lemma 1) we come to the following statement.

THEOREM 1. *Assume that we have a sequence of composite processes* $\{m_s\} \subset D$ *and functionals* $\varphi, \theta \in \Xi$, $\omega \in \Omega$, $\varkappa \in \Omega$ *such that*

1) $\sup\limits_{(y,w) \in B_c(z)} K_c(z, \tau, y, w, \delta) = 0\ \ \forall t \in T_*,\ \delta \in \Delta(z),$

$\tau \in T_c(z),\ x(t) \in X(t),\ u_d(t) \in U_d(t, x(t));$

2) $\theta\Big(\int\limits_{T_{cs}} K_c(z_s, \tau, y_s(\tau), w_s(\tau), \delta)\, d\tau\Big) \to 0;$

3) $\inf G_c(z, \gamma_c, \alpha) = 0\ \ \forall t \in T_*,\ \alpha \in A;$

$\gamma_c \in \Gamma_c(z) \cap \{\gamma_c : y_i \in Y(z, \tau_i),\ y_f \in Y(z, \tau_f)\},$

$u_d(t) \in U_d(t, x(t)),\ x(t) \in X(t),$

4) $\sup\limits_{(x(t), u(t)) \in B(t)} K(t, x(t), u(t), \alpha) = 0\ \ \forall t \in T \backslash T_*,\ \alpha \in A;$

5) $\omega\Big(\sum\limits_{T \backslash T_* \backslash t_f} K(t, x_s(t), u_s(t), \alpha) - \sum\limits_{T_*} G_c(z_s, \gamma_{cs}, \alpha)\Big) \to 0;$

6) $G(\gamma_s) \to l = \inf G(\gamma),\ \gamma \in \Gamma \cap \{\gamma : x(t_i) \in X(t_i),\ x(t_f) \in X(t_f)\}.$

Then the sequence $\{m_s\}$ is minimizing for the functional I on D and any minimizing sequence for this problem satisfies the conditions 2, 5, 6.

This theorem gives general optimality conditions for composite processes.

They can be specified according to a resolving extension defined by functionals ω, φ, $\varkappa$, θ. This makes it possible to generate different approaches to given problems under investigation. One of these approaches leads to the well-known dynamic programming procedure.

4. Sufficient conditions in the Bellman form

Let $x(t_i)$ and $X(t) = X_0(t)$, $t \in T \backslash t_i, \tau_i, y(\tau_i)$ be given for the given z:

$$\tau_i = \tau_i(z),\ y(\tau_i) = y_i(z),\ (\tau_f, y(\tau_f)) \in \Gamma_{cf}(z),\ g(z, \gamma) = g(z, \tau_f, y(\tau_f)),$$

where $\Gamma_{cf}(z)$ is a surface in R^{n+1} not containing (τ_i, y_i), $F(\gamma) = F(x(t_f))$.

Define ω, $\varkappa$ to be identical and define φ, θ by the following conditions: for $t \in T_*$

$$\sup_{w \in W(z, \tau, y)} K_c(z, \tau, y, w) = 0,$$

$$-\varphi(t+1, g(z, \tau_f, y_f)) + \theta(z, \tau_f, y_f) = 0 \text{ when } (\tau_f, y_f) \in \Gamma_{cf}(z);$$

$$\inf_{u_d \in U'_d(t, x(t))} G_c(z, \gamma_c) = 0;$$

for $t \in T \backslash T_*$

$$\sup_{u(t) \in U(t, x(t))} K(t, x(t), u(t)) = 0,\ G(x(t_f)) = -\varphi(t_i, x(t_i)).$$

Considering the left parts of these equations we come to the following recurrence chain for φ, θ:

$$\varphi(t_f, x(t_f)) = -F(x(t_f));$$

$$\varphi\left(t, x(t)\right) = \sup_{u \in U(t, x(t))} \varphi\left(t+1, f\left(t, x(t), u(t)\right)\right), \ t \in T \backslash T_*,$$

$$\theta_\tau + \sup_{w \in W(z, \tau, y)} \left(\theta_y' \, h(z, \tau, y, w)\right) = 0, \tag{7}$$

$$\theta(z, \tau_f, y_f) = \varphi\left(t+1, g(z, \tau_f, y_f)\right),$$

when $(\tau_f, y_f) \in \Gamma_{cf}(z)$, $t \in T_*$,

$$\varphi\left(t, x(t)\right) = \sup_{u_d \in U_d(t, x(t))} \theta\left(z, \tau_i(z), y_i(z)\right), \ t \in T_*,$$

which is solved from t_f to t_i.

Assume that the solution of this chain $\varphi(t, x(t))$, $\theta(z, \tau, y)$ exists and moreover that the functions $\tilde{u}(t, x(t))$, $\tilde{u}_d(t, x(t))$, $\tilde{w}(z, \tau, y)$ which are obtained as results of maximization can be supplied. After inserting these functions into the right-hand parts of the initial discrete and continuous time relations we obtain

$$x(t+1) = f\left(t, x(t), \tilde{u}(t, x(t))\right), x(t_{II}) = x_{II}, \ t \in T \backslash T_*;$$

$$x(t+1) = g\left(\tilde{z}, \tau_f\left(t, x(t)\right)\right), y\left(\tau_f\left(t, x(t)\right)\right),$$

$$\dot{y} = h\left(\tilde{z}, \tau, y, \tilde{w}(\tilde{z}, \tau, y)\right), y(\tau_i) = y_i(\tilde{z}), \ t \in T_*,$$

$$\tilde{z} = \left(t, x(t), \tilde{u}_d\left(t, x(t)\right)\right).$$

Then the solution of this discrete-continuous chain $\tilde{m} \in D$ is an optimal composite process satisfying all the conditions of theorem 1, because conditions (7) hold.

Actually the solution of the problem is obtained for any combination of initial condition with the same functions $\left(\tilde{u}(\cdot), u_d(\cdot), \tilde{w}(\cdot)\right)$ and such result we shall call a traditionally optimal control synthesis.

5. Conclusions

The hierarchical description of the class of processes considered, with the proper interpretation of the elements of the abstract discrete process, was previously proposed in [3]. In this approach sufficient optimality conditions were developed as a combination of known Krotov's conditions for continuous and discrete processes. These conditions were then used for developing the first order control improving algorithms with applications in some problems of space flight and walking robot control [4–7]. Advantages of the proposed approach become apparent particularly in the latter work which dealt with an area of control problems which was new at that time. Initial control strategies defined intuitively from practical considerations were improved from 4 to 5 times with respect to some typical criteria.

In this work new generalized sufficient optimality conditions are proposed.

The resolving Krotov functions (φ, θ) for "discrete" and "continuous" levels are replaced accordingly by resolving systems. Eeach of them consists of some family of Krotov-type functions φ_α, θ_α and a functional over this family [2]. In [8] the new technique of investigation of continuous systems connected with new conditions is demonstrated on some clear examples. But on the whole the algorithmic development of these conditions is at the beginning.

Notice that a continuous process can be represented as a particular case of a discrete-continuous process by dividing the given time interval into some final number of stages. As a result the conditions of Th. 1 become sufficient for optimality of continuous processes. They are more general than the conditions [8] because they permit the change of the resolving system $(\varkappa, \theta)$ in time and state as opposed to the constant resolving system in [2, 8]. This feature is advantageous when optimal sliding regimes are investigated.

Th. 1 can be generalized practically without any change to the case when Y_0 is an arbitrary linear normed space.

In some practical cases it is convenient to use models, describing so-called composite discrete processes where the continuous model in the lower level is replaced by the discrete one. A modification of theorem 1 in such cases is not difficult.

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Modele i warunki optymalności dla systemów dyskretno-ciągłych

Do opisania sterowanych systemów ze zmienną strukturą rozważono model hierarchiczny o dwóch poziomach. Na wyższym poziomie znajduje się model dynamiczny z czasem dyskretnym, a na niższym ciągły model różniczkowy o jednorodnej strukturze. Zaproponowano nowe dostateczne warunki optymalności dla takich systemów. Uogólniają one warunki wyprowadzone wcześniej przez autora. Występujące poprzednio na każdym poziomie funkcje Krotova zamieniono systemami rozwiązującymi, które składają się z rodzin funkcji typu Krotova i funkcjonału określonego na tej rodzinie. Podano krótki przegląd wcześniejszych wyników teoretycznych i praktycznych.

Модели и условия оптимальности дискретно-непрерывных процессов

Для описания управляемых систем с переменной структурой рассматривается иерархическая двухуровневая модель, на верхнем уровне которой используется дискретная динамическая модель, а на нижнем — непрерывная дифференциальная система однородной структуры. Предлагаются новые достаточные условия оптимальности таких систем, обобщающие ранее выведенные автором. В них вместо функций Кротова на каждом уровне фигурируют разрешающие системы, состоящие из семейства функций типа Кротова и функционала над этим семейством. Дается краткий обзор предшествующих теоретических и прикладных результатов.