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Global optimality conditions for controlled noninertial systems with delays

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A problem of optimal control of noninertial dynamic systems with delays is treated.

When there is no control, a system of such kind can be considered as a particular case of functional equations without free variables [1–3]. On the other hand, they are closely associated with multistage systems, but unlike the latter, the independent variable in them varies continuously.

The sufficient conditions for global optimality of program control and feedback control have been obtained. Necessary conditions are also proved when the phase vector and control vector are finite-dimensional. The first order necessary conditions for local optimality are also investigated.

Difficulties are discussed in transferring the results onto systems of a more general type.

1. Formulation of the problem and applications

Let X, U be arbitrary non-empty sets with elements x, u respectively, $[t_0, t_1] \subset R \triangleq (-\infty, \infty)$, $t_0 < t_1$. Let there be given functions $f:[t_0, t_1] \times \times X \times U \to X, f^0:[t_0, \tau_1] \times X \times U \to \bar{R} \triangleq [-\infty, \infty]$ and $\tau:[t_0, t_1] \to R$. Assume that $t_0 \leqslant \tau(t) \leqslant \tau_1$ for all $t \in [t_0, t_1]$, where $\tau_1 \triangleq \tau(t_1) \geqslant t_1$. Denote $\tau_0 \triangleq \tau(t_0)$. Define also the subset $B(t)$ of X for any $t \in [t_0, t_1]$ and the subsets $Q(t, x)$ of U for any pair (t, x), $t \in [t_0, \tau_1]$, $x \in B(t)$.

Give the system described by the equation

$$x(\tau(t)) = f(t, x(t), u(t)), \quad t \in [t_0, t_1], \tag{1}$$

the name of noninertial controllable dynamic system with delays.

Then interpret the variable t as time, the function $u(\cdot):[t_0, \tau_1] \to U$ as control, $x(\cdot):[t_0, \tau_1] \to X$ as trajectory, the pair $(x(\cdot), u(\cdot))$ as controlled process, and the element $x \in X$ as state.

For all $t \in [t_0, \tau_1]$ impose the following restrictions on the trajectory $x(\cdot)$ and the control $u(\cdot)$

$$\begin{aligned} &x(t) \in B(t), \\ &u(t) \in Q(t, x(t)). \end{aligned} \tag{2}$$

The restriction (2) includes, in particular, a terminal restriction. If, for example, the set $B(t)$ for all $t \in [t_1, \tau_1]$ consists of only one element, it means that the trajectory $x(\cdot)$ must coincide with the given function in the interval $[t_1, \tau_1]$.

Besides, the initial condition for equation (1) is given by

$$x(t) = \alpha(t), \qquad t \in [t_0, \tau_0), \tag{3}$$

where $\alpha:[t_0, \tau_0) \to X$ is a given function such that $\alpha(t) \in B(t)$, $t \in [t_0, \tau_0)$. Assume here that $[t_0, t_0) \triangleq \{t_0\}$. Restriction (3) as well as the terminal one, could be included in (2), but it is convenient to separate it.

Denote by D_0 a set of control processes $v \triangleq (x(\cdot), u(\cdot))$, satisfying the above requirements and such that the function $t \to f^0(t, x(t), u(t)):[t_0, \tau_1] \to \bar{R}$ is summable (i.e. Lebesgue integrable). Define the functional

$$v \to J(v) \triangleq \int_{t_0}^{\tau_1} f^0(t, x(t), u(t))\, dt : D_0 \to \bar{R}. \tag{4}$$

Here and further the symbol $\int$ denotes the Lebesgue integral.

In the general case it is required to find the lower bound

$$d_0 \triangleq \inf_{v \in D_0} J(v),$$

and a sequence $\{(x_s(\cdot), u_s(\cdot))\}$ minimizing J on D_0, i.e. a sequence such that

$$\lim_{s \to \infty} J(x_s(\cdot), u_s(\cdot)) = d_0, \quad (x_s(\cdot), u_s(\cdot)) \in D_0.$$

The problem is also solved if a process $\bar{v} \in D_0$ satisfying the condition $J(\bar{v}) = d_0$ is found. Let us call such a control process, as it is accepted, an optimal control process.

The equation of the same kind as (1), containing no control, have been studied in detail in [1–3] and are called functional equations without free variables. One of the well known equations of this type is the Abel equation: $x(\tau(t)) = x(t) + 1$.

The equation (1) is characterized by the fact that the initial condition is given by the function $\alpha(t)$, defined on $[t_0, \tau_0)$ as is true for differential equations with delays [4]. If the function $\tau(t)$ is continuous and $\tau(t) > t$ for $t \in [t_0, t_1]$ then each initial function $\alpha(t)$, with the control $u(t)$ fixed, defines the only one solution $x(t)$ (the trajectory) of the equation (1), defined in the whole interval $[t_0, \tau_1]$. Unlike the differential equations, there is no question here about the existence of the solution as a whole. Yet, this question becomes non-trivial when the condition $\tau(t) > t$ is violated.

The mathematical model (1) is applicable to a somewhat broader class of situations than a situation with a delay, for when $t \in (t_0, t_1)$, not only the inequality $\tau(t) \geqslant t$ is permitted but also the inequality $\tau(t) < t$.

A dynamic system of the kind under consideration resembles a multi--stage dynamic system, but the argument t varies here continuously. Therefore, the optimality conditions thus obtained resemble Krotov's sufficient conditions [5] and corresponding conditions of the dynamic programming method (see, e.g. [6]), but for the systems with discrete argument there are substantial differences because the optimality criterion (4) is written in integral form.

A distinctive feature of the mathematical apparatus used in this paper is the use of the one-sided (lower) Perron integral [7, p. 297], defined not only for measurable integrand. This allows to avoid a contradiction between bad analytic features of trajectories $x(\cdot)$ of equation (1) and the integral form of optimality criterion (4), demanding the measurability of the integrand. This way we also obtain the main results under weaker assumptions than when using the Lebesgue integral. However, the one-sided lower Perron integral plays only an instrumental role in argumentation and is lacking, as a rule, in the final results. Practically, the same results may be obtained using the one-sided Lebesgue integral although the more general Perron integral is built more naturally.

The use of the one-sided Perron integrals proved convenient in the classical optimal control problem [8]. A similar construction (the upper measure) is used in [9], however, it does not have some features important here.

A noninertial controllable dynamic system with delays can serve as a model for remote control process in cases when a delay of the controlling signal is big in comparison with a transient period in the controlled object. Such a situation can occur when executing from the Earth control of the robot operating on the surface of a planet. On Earth this takes place when conducting remote control using a control signal carrier less rapid than electromagnetic waves, such as, for example, acoustic waves.

Mathematical models similar to (1) or more general (e.g., with several delays) may be used in the problems of controlling the dynamics of biological populations instead of traditional multistage models [10]. The value $\tau(t) - t = \text{const} > 0$ in these models can represent delay between the moment of reproduction ability of a being and the moment of its birth.

2. Sufficient optimality conditions

Let us assume that the following conditions are satisfied throughout the paper.

1°. The function $\tau(t)$ is continuous, and nondecreasing, its upper derivative [7, p. 293] may take the value ∞ only in a countable set of points of the interval $[t_0, t_1]$ and $t_0 < t_1$.

Denote by Φ_0 a set of functions $\varphi: [\tau_0, \tau_1] \times X \to \bar{R}$, satisfying the following condition.

2°. For any controlled process $(x(\cdot), u(\cdot)) \in D_0$ a summable function $t \to \gamma(t): [\tau_0, \tau_1] \to \bar{R}$ may be found as

$$|\varphi(t, x(t))| \leqslant \gamma(t), \; t \dot{\in} [\tau_0, \tau_1].$$

Here and in the sequel the notation $t \dot{\in} [a, b]$ means that almost every element of the interval $[a, b]$ is under consideration.

The condition 2° holds, in particular, if for all $(x(\cdot), u(\cdot)) \in D_0$ the function $t \to \varphi(t, x(t))$ is bounded on the interval $[\tau_0, \tau_1]$.

Using the function φ let us make up the following constructions

$$S(t, x, u) \triangleq \varphi(\tau(t), f(t, x, u))\, \tau'(t) - \\ - \varphi(t, x) + f^0(t, x, u), \; t \in [\tau_0, t_1]; \tag{5}$$

$$G_0(t, w) \triangleq \varphi\big(\tau(t), f(t, \alpha(t), w)\big)\, \tau'(t) + \\ + f^0\big(t, \alpha(t), w\big), \; t \in [t_0, \tau_0); \tag{6}$$

$$G_1(t, x, u) \triangleq -\varphi(t, x) + f^0(t, x, u), \; t \in [t_1, \tau_1]; \tag{7}$$

where $x \in B(t)$, $u \in Q(t, x)$, $\tau'(t) \triangleq d\tau(t)/dt$, $w \in Q\big(t, \alpha(t)\big)$.

THEOREM 1. *Let the function $\varphi \in \Phi_0$ and the function $t \to \mu(t): [t_0, \tau_1] \to \bar{R}$ be such that:*

1) *The function μ is summable,*

2) $S(t, x, u) \geqslant \mu(t)$, $t \dot{\in} [\tau_0, t_1]$, $x \in B(t)$, $u \in Q(t, x)$,

3) $G_0(t, u) \geqslant \mu(t)$, $t \dot{\in} [t_0, \tau_0)$, $u \in Q\big(t, \alpha(t)\big)$,

4) $G_1(t, x, u) \geqslant \mu(t)$, $t \dot{\in} [t_1, \tau_1]$, $x \in B(t)$, $u \in Q(t, x)$.

Then

a) *For all $(x(\cdot), u(\cdot)) \in D_0$ the inequality*

$$J\big(x(\cdot), u(\cdot)\big) \geqslant l(\varphi) \triangleq \int_{t_0}^{\tau_1} \mu(t)\, dt, \tag{8}$$

holds.

b) *If a sequence of controlled processes $\{(x_s(\cdot), u_s(\cdot))\} \subset D_0$ exists, and it is such that*

$$\lim_{s \to \infty} J\big(x_s(\cdot), u_s(\cdot)\big) = l(\varphi), \tag{9}$$

then $l(\varphi)$ *is the lower bound of the functional* J *on* D_0 $(l = d_0)$, *and the sequence* $\{(x_s(\cdot), u_s(\cdot))\}$ *is a minimizing sequence. Besides, any other minimizing sequence satisfies the condition* (9).

Before passing to the proof of Theorem 1, let us try to get auxilliary results, which will be important in further considerations.

Let the function $t \to g(t): [a, b] \to \bar{R}$ be given. We call a continuous function $t \to \eta(t): [a, b] \to R$ satisfying the condition $\eta(a) = 0$ and the inequalities $\bar{D}\eta(t) < \infty$, $\bar{D}\bar{\eta}(t) \leqslant g(t)$ for all $t \in [a, b]$ a subfunction. Here

$$\bar{D}\eta(t) \triangleq \lim_{\tau \to t} (\eta(t) - \eta(\tau))/(t - \tau),$$

is the upper derivative of the function $\eta(t)$.

If a set of subfunctions is not empty, then the value

$$(\mathbf{P}) \int_a^b g(t)\, dt \triangleq \sup_{\eta(\cdot)} \eta(b) < \infty,$$

where sup is calculated over the whole set of subfunctions $\eta(\cdot)$ is called a lower Perron integral (**P**-integral) [7, p. 297]. Similarly a superfunction and an upper Perron integral may be defined.

Use then some properties of this integral. Enumerate the most important of them, assuming, in contrast to [7], the case when the integrand has the subfunctions only, as a result of which the **P**-integral can turn into $+\infty$.

Let the functions $t \to g(t): [a, b] \to \bar{R}$, $t \to h(t): [a, b] \to \bar{R}$ be given.

1*. If the function g is Lebesgue integrable, **P**-integrable, and

$$(\mathbf{P}) \int_a^b g(t)\, dt = \int_a^b g(t)\, dt.$$

2*. If the function h is Lebesgue integrable and $|g(t)| \leqslant h(t)$, $t \dot{\in} [a, b]$, then the function g has a finite **P**-integral.

3*. If $g(t) \geqslant h(t)$, $t \dot{\in} [a, b]$, and the function h is **P**-integrable, then the function g is **P**-integrable and

$$(\mathbf{P}) \int_a^b g(t)\, dt \geqslant (\mathbf{P}) \int_a^b h(t)\, dt. \tag{10}$$

4*. If the assumptions of item 3* are fulfilled and $g(t) > h(t)$ on a set of positive measure, then the inequality (10) is strict.

5*. If the function g is **P**-integrable, and h is Lebesgue integrable, then their sum is **P**-integrable and

$$(\mathbf{P}) \int_a^b (g(t) + h(t))\, dt = (\mathbf{P}) \int_a^b g(t)\, dt + \int_a^b h(t)\, dt.$$

6*. If the function h has a finite **P**-integral and the difference $g - h$ is **P**-integrable, then the function g is also **P**-integrable and

$$(\mathbf{P})\int_a^b (g(t)-h(t))\,dt \leqslant (\mathbf{P})\int_a^b g(t)\,dt-(\mathbf{P})\int_a^b h(t)\,dt. \tag{11}$$

7*. If, in addition to the condition of item 6*, the difference $g-h$ is Lebesgue — integrable, then an equality sign holds in (11).

8*. If the functions g and $(-g)$ are **P**-integrable, then

$$-(\mathbf{P})\int_a^b (-g(t))\,dt \geqslant (\mathbf{P})\int_a^b g(t)\,dt.$$

9*. If the function g is **P**-integrable (on $[a, b]$), then it is **P**-integrable in any interval $[\xi, \eta]\subset[a, b]$ and for any $c\in[a, b]$

$$(\mathbf{P})\int_a^b g(t)\,dt=(\mathbf{P})\int_a^c g(t)\,dt+(\mathbf{P})\int_c^b g(t)\,dt.$$

10*. (Substitution of the variable). If the function g is **P**-integrable and finite almost everywhere on $[a, b]$, while the function $\xi:[\alpha, \beta]\to\bar{R}$ is continuous and nondecreasing, its upper derivative may take the value $+\infty$ only in a countable set of points of the interval $[\alpha, \beta]$ and $\xi(\alpha)=$ $=a, \xi(\beta)=b$, then

$$(\mathbf{P})\int_a^b g(t)\,dt=(\mathbf{P})\int_\alpha^\beta g(\xi(s))\,\frac{d\xi(s)}{ds}\,ds. \tag{12}$$

In the equality (12) and everywhere in the sequel let us suppose that the integrand is equal to zero if it is not defined.

The properties 1*—9* of **P**-integrals are proved in [7] or easily follow directly from the definitions, the properties of lower and upper derivatives [7, p. 293] and well known theorems of analysis. The possibility of substituting the variable in the Perron integral is proved in [7, p. 316, Theorem 4], the result 10* required here has been obtained while proving the above mentioned theorem from [7].

LEMMA 1.

1). *For any* $\varphi\in\Phi_0$ *and* $v\triangleq(x(\cdot), u(\cdot))\in D_0$ *the following inequality is valid*

$$J(v)\geqslant L_\varphi(v)\triangleq(\mathbf{P})\int_{t_0}^{\tau_0} G_0(t, u(t))\,dt+$$
$$+(\mathbf{P})\int_{\tau_0}^{t_1} S(t, x(t), u(t))\,dt-(\mathbf{P})\int_{t_1}^{\tau_1}\left[-G_1(t, x(t), u(t))\right]dt. \tag{13}$$

2). *If, in addition, function* $t\to S(t, x(t), u(t)):[\tau_0, t_1]\to\bar{R}$ *is summable, then*

$$J(v)=L_\varphi(v). \tag{14}$$

Proof. Let $\varphi \in \Phi_0$ and $v \in D_0$, then the equality

$$J(v) = (\mathbf{P}) \int_{t_0}^{t_1} \varphi\left(\tau(t), x\left(\tau(t)\right)\right) \tau'(t)\, dt - \\ -(\mathbf{P}) \int_{\tau_0}^{\tau_1} \varphi\left(t, x(t)\right) dt + \int_{t_0}^{\tau_1} f^0\left(t, x(t), u(t)\right) dt, \tag{15}$$

holds true.

The conditions 1°, 2° and the properties 2*, 10* of the P-integral are used here. The existence and finiteness of the second addend in (15) follow from 2° and the property 2*, as well as the existence and finiteness of the first addend follow from 1° and 10*, and also the validity of equality (15) itself.

Using the property 5* of a P-integral and taking into account the inequalities $t_0 \leqslant \tau_0 < t_1 \leqslant \tau_1$ let us divide the integrals in (15) into the integrals in intervals $[t_0, \tau_0]$, $[\tau_0, t_1]$ and $[t_1, \tau_1]$. Grouping by means of 5* and using the equation (1) and the notations of (5)—(7), we get

$$J(v) = (\mathbf{P}) \int_{t_0}^{\tau_0} G_0\left(t, u(t)\right) dt - (\mathbf{P}) \int_{t_1}^{\tau_1} \left[-G_1\left(t, x(t), u(t)\right)\right] dt + \\ +(\mathbf{P}) \int_{\tau_0}^{t_1} \left[S\left(t, x(t), u(t)\right) + \varphi\left(t, x(t)\right)\right] dt - (\mathbf{P}) \int_{\tau_0}^{t_1} \varphi\left(t, x(t)\right) dt. \tag{16}$$

Taking 6* into account, unite the last two addends of the right hand side of (16) and get the inequality (13). If the assumption of item 2) in the lemma is satisfied, the equality (14) is obtained by 7* from (16).

Let us formulate the condition for the function $t \to \mu(t): [t_0, \tau_1] \to \bar{R}$ which is different from the assumption 1) of Theorem 1.

3°. The function μ on the interval $[t_0, \tau_1]$ and the function $t \to \delta(t) \triangleq -\mu(t): [t_1, \tau_1] \to \bar{R}$ are **P**-integrable.

LEMMA 2. *Let $\varphi \in \Psi_0$, the condition 3°, and the assumptions 2)–4) of Theorem 1 hold.*

Then for all $v \triangleq \left(x(\cdot), u(\cdot)\right) \in D_0$ the inequality

$$J(v) \geqslant l_1(\varphi) \geqslant l_2(\varphi), \tag{17}$$

is satisfied, where

$$l_1(\varphi) \triangleq (\mathbf{P}) \int_{t_0}^{t_1} \mu(t)\, dt - (\mathbf{P}) \int_{t_1}^{\tau_1} \left(-\mu(t)\right) dt, \tag{18}$$

$$l_2(\varphi) \triangleq (\mathbf{P}) \int_{t_0}^{\tau_1} \mu(t)\, dt. \tag{19}$$

Proof. Let the conditions of the lemma hold and let v be a process from D_0. For function φ and process v the inequality (13) holds true due to Lemma 1 from which, the conditions 2)–4) of Theorem 1, 3* and 9* taken into account, follows the validity of the first of the inequalities in (17). Considering 3* it is easy to observe that $l_1(\varphi) \geqslant l_2(\varphi)$ and so the second inequality in (17) is true. Since values l_1 and l_2 are not dependent of the choice of the element $v \in D_0$, the inequalities (17) hold for all $v \in D_0$. This proves the assertion of the lemma. ■

Proof of Theorem 1. The conditions of the theorem and 1* guarantee the applicability of Lemma 2. Moreover, $l_2(\varphi) = l(\varphi)$. So, the assertion a) of the theorem is true.

Since the inequality (8) holds true for all $(x(\cdot), u(\cdot)) \in D_0$ then $d_0 \geqslant l(\varphi)$, from which, taking into account the definition of the lower bound, we obtain

$$J(x_s(\cdot), u_s(\cdot)) \geqslant d_0 \geqslant l(\varphi). \tag{20}$$

From (9) and (20) follows

$$\lim_{s \to \infty} J(x_s(\cdot), u_s(\cdot)) = d_0 = l(\varphi),$$

which makes the assertion b) of the theorem obvious. ■

The result described below follows directly from item b) of Theorem 1.

Corollary 1. *If the conditions of Theorem 1 hold and the process* $(\bar{x}(\cdot), \bar{u}(\cdot)) \in D_0$ *satisfies the condition* $J(\bar{x}(\cdot), \bar{u}(\cdot)) = l(\varphi)$, *then this control process is optimal.*

Remark 1. It is easy to see that Theorem 1 remains true when the assumption 1) of the summability of μ is substituted by condition 3° and $l(\varphi)$ by $l_1(\varphi)$ or $l_2(\varphi)$.

Such substitution has not been done directly in Theorem 1 because of the desire to prevent the inclusion of **P**-integral in the final results. However, such a variant of Theorem 1 is necessary to prove Theorem 2, which is considered below.

Theorem 2. *Let the function* $\varphi \in \Phi_0$ *and the process* $\bar{v} \triangleq (\bar{x}(\cdot), \bar{u}(\cdot)) \in D_0$ *satisfy the conditions*:

1) $S(t, \bar{x}(t), \bar{u}(t)) = \min\limits_{x \in B(t),\, u \in Q(t,x)} S(t, x, u)$, $t \dot{\in} [\tau_0, t_1]$.

2) $G_0(t, \bar{u}(t)) = \min\limits_{u \in Q(t, \alpha(t))} G_0(t, u)$, $t \dot{\in} [t_0, \tau_0)$.

3) $G_1(t, \bar{x}(t), \bar{u}(t)) = \min\limits_{x \in B(t),\, u \in Q(t,x)} G_1(t, x, u)$, $t \dot{\in} [t_1, \tau_1]$.

4) *The function* $t \to S(t, \bar{x}(t), \bar{u}(t)) : [\tau_0, t_1] \to \bar{R}$ *is summable.*

Then the process $\bar{v}$ *is optimal and any other optimal control process satisfies the conditions* 1)–4).

Proof. Let the conditions of the theorem hold true. Define the function $\mu : [t_0, \tau_1] \to \bar{R}$ by means of the equalities: $\mu(t) \triangleq G_0(t, \bar{u}(t))$ when $t \in [t_0, \tau_0)$,

$\mu(t) \triangleq S(t, \bar{x}(t), \bar{u}(t))$ when $t \in (\tau_0, t_1)$, $\mu(t) \triangleq G_1(t, \bar{x}(t), \bar{u}(t))$ when $t \in [t_1, \tau_1]$ and $\mu(\tau_0) \triangleq 0$ if $\tau_0 > t_0$.

The function μ satisfies the condition 3° due to the assumption 2°, summability of the function $t \to f^0(t, \bar{x}(t), \bar{u}(t))$ on the interval $[t_0, \tau_1]$ and 1*. Hence, Theorem 1 and Remark 1 taken into account, we can prove the optimality of the process $\bar{v}$ by showing that

$$J(\bar{v}) = l_1(\varphi), \tag{21}$$

where $l_1(\varphi)$ is defined by equality (18).

Thus, let us prove the validity of (21). Due to the assumption 4) it follows from item 2) of Lemma 1 that the equality

$$J(\bar{v}) \doteq L\varphi(\bar{v}), \tag{22}$$

holds true.

Thereby, considering the definition of function μ

$$J(\bar{v}) = (\mathbf{P})\int_{t_0}^{\tau_0} \mu(t)\,dt + (\mathbf{P})\int_{\tau_0}^{t_1} \mu(t)\,dt - (\mathbf{P})\int_{t_1}^{\tau_1} [-\mu(t)]\,dt. \tag{23}$$

Now, following 9* and (18) it is easy to show that the equality (23) is equivalent to (21). Hence, the optimality of process $\bar{v}$ is proved.

Let now $v \triangleq (x(\cdot), u(\cdot)) \in D_0$, $v \neq \bar{v}$, be some other optimal control process such that at least one of the equalities 1)–3) is violated on the set of moments t of positive measure after substituting $\bar{v}$ by v.

The inequality (13) holds true for the process v by Lemma 1. Taking 4* into account and comparing (13) with (22) it becomes easy to show that $J(v) > J(\bar{v})$. But this contradicts the assumption that the process v is optimal. Therefore, the conditions 1)–3) for the process v cannot be violated on the set of positive measure, i.e. the conditions 1)–3) of the theorem hold true for the process v. This implies, in particular, that $S(t, x(t), u(t)) = S(t, \bar{x}(t), \bar{u}(t))$, $t \dot{\in} [\tau_0, t_1]$, and thus the condition 4) of the theorem also holds true for the process v. ■

3. Dynamic programming

Theorems 1 and 2 contain sufficient optimality conditions for the case when the initial condition (3) is fixed. One can pose the problem of finding the universal function φ, which satisfies all possible initial conditions simultaneously.

In this part of the paper we shall assume that the following condition is satisfied.

4°. For all $t \in [t_0, t_1]$ the inequality $\tau(t) \geqslant t$ holds.

Let B_0 be a non-empty set of initial elements $w_\alpha \triangleq (t_\alpha, \alpha(\cdot))$ such that

$t_\alpha \in [t_0, t_1)$, $\tau_\alpha \triangleq \tau(t_\alpha) \leqslant t_1$, for the function $t \to \alpha(t): [t_\alpha, \tau_\alpha) \to X$ for all $t \in [t_\alpha, \tau_\alpha)$ the inclusion $\alpha(t) \in B(t)$ holds true. Assume, as was also done in Section 1, that $[t_\alpha, t_\alpha) \triangleq \{t_\alpha\}$. Besides, any additional restrictions may be imposed on the elements w_α. The set of all those restrictions defines B_0.

Consider an optimal control problem made from the initial problem by substituting everywhere in Section 1.1 the moments t_0, τ_0 by t_α, τ_α, correspondingly, and in particular the initial condition (3) by the condition

$$x(t) = \alpha(t),\ t \in [t_\alpha, \tau_\alpha),\ w_\alpha \in B_0, \tag{24}$$

and a family of such problems with different $w_\alpha \in B_0$. Taking this into account, denote a set of admissible pairs $(x(\cdot), u(\cdot))$ of functions $u: [t_\alpha, \tau_1] \to U$, $x: [t_\alpha, \tau_1] \to X$, similar to D_0, by $D(w_\alpha)$, and the lower bound of the functional by $d(w_\alpha)$.

Because of the assumption 4° the family of problems with initial elements from B_0 is defined correctly.

Denote by $\mathscr{U}$ a set of various functions $(t, x) \to u^0(t, x): [t_0, \tau_1] \times X \to U$ such that $u^0(t, x) \in Q(t, x)$ for all $t \in [t_0, \tau_1]$, $x \in B(t)$.

Definition 1. Call the function $u^0 \in \mathscr{U}$ a *feedback control*, if for any w_α from B_0 there exists a solution $x_\alpha(\cdot): [t_\alpha, \tau_1] \to X$ of the equation

$$x_\alpha(\tau(t)) = f\left(t, x_\alpha(t), u^0(t, x_\alpha(t))\right),\ t \in [t_\alpha, t_1], \tag{25}$$

with the initial condition (24) and

$$\left(x_\alpha(\cdot), u_\alpha(\cdot)\right) \in D(w_\alpha), \tag{26}$$

where

$$u_\alpha(t) \triangleq u^0\left(t, x_\alpha(t)\right),\ t \in [t_\alpha, \tau_1]. \tag{27}$$

Remark 2. To verify the condition (26) we must prove the inclusion $x_\alpha(t) \in B(t)$, $t \in [t_\alpha, \tau_1]$, and summability of the function

$$t \to \xi_\alpha(t) \triangleq f^0\left(t, x_\alpha(t), u_\alpha(t)\right): [t_\alpha, \tau_1] \to \bar{R}.$$

Remark 3. The verification of (26) reduces to a verification of summability of the function ξ_α when a set $Q(t, x)$ is not empty for all $t \in [t_0, t_1]$, $x \in B(t)$ and for any $t \in [t_0, t_1]$

$$M(t) \triangleq \{y \in X: y = f(t, x, u),\ x \in B(t),\ u \in Q(t, x)\} \subset B\left(\tau(t)\right).$$

Definition 2. Call the function $u^0 \in \mathscr{U}$ an *optimal feedback control*, if u^0 is a feedback control and for any w_α from B_0 the control process $(x_\alpha(\cdot), u_\alpha(\cdot))$, satisfying the conditions (26), (27), is optimal, i.e. it minimizes the functional J on the set $D(w_\alpha)$.

Introduce the set Φ of functions $\varphi: [t_0, \tau_1] \times X \to \bar{R}$, satisfying the condition 2° for any of the sets $D(w_\alpha)$, $w_\alpha \in B_0$.

REMARK 4. Everywhere in the sequel assume for the function $u \to g(u): Q \to \bar{R}$ that

$$\min_{u \in Q} q(u) \triangleq \infty,$$

if $q(u) = \infty$ for all $u \in Q$.

THEOREM 3. *Let B_0 be a set of initial elements, let the function $\varphi \in \Phi$ satisfy the conditions*:

1) $\varphi(t, x) = \min_{u \in Q(t, x)} [\varphi(\tau(t), f(t, x, u))\tau'(t) + f^0(t, x, u)]$, $t \dot{\in} [t_0, t_1]$, $x \in B(t)$,

2) $\varphi(t, x) = \min_{u \in Q(t, x)} f^0(t, x, u)$, $t \dot{\in} [t_1, \tau_1]$, $x \in B(t)$,

and let the function $u^0 \in \mathscr{U}$ satisfy the conditions:

3) $u^0(t, x) \in \operatorname{Arg} \min_{u \in Q(t, x)} [\varphi(\tau(t), f(t, x, u))\tau'(t) + f^0(t, x, u)]$, $t \dot{\in} [t_0, t_1]$, $x \in B(t)$,
4) $u^0(t, x) \in \operatorname{Arg} \min_{u \in Q(t, x)} f^0(t, x, u)$, $t \in [t_1, \tau_1]$, $x \in B(t)$,
5) $u^0(t, x)$ *is the feedback control.*

Then

a) *the function $u^0(t, x)$ is the optimal feedback control,*
b) *for any w_α from B_0 the optimal value of the functional* (4) *is*

$$d(w_\alpha) = (\mathbf{P}) \int_{t_\alpha}^{\tau_\alpha} \varphi(t, \alpha(t))\, dt. \tag{28}$$

Proof. To verify the assertion a) of the theorem it is sufficient for any $w_\alpha \in B_0$ to apply Theorem 2 to the process $v_\alpha \triangleq (x_\alpha(\cdot), u_\alpha(\cdot))$ which satisfies the conditions (26), (27). In doing this we take as the function φ of the Theorem 2 a restriction of a function $\varphi \in \Phi$ on $[t_\alpha, {}^{\tau}{}_1] \times X$ which satisfies the conditions 1) and 2) of Theorem 3. Satisfaction of the conditions 1)–3) of Theorem 2 follows directly from the conditions of Theorem 3. The condition 4) of Theorem 2 is also satisfied because due to the conditions 1) and 3) of Theorem 3 and (27) the equality

$$S(t, x_\alpha(t), u_\alpha(t)) = 0, \ t \dot{\in} [\tau_\alpha, t_1], \tag{29}$$

holds. Equality (29) and item 2 of Lemma 1 imply the equality $J(v_\alpha) = L_\varphi(v_\alpha)$, which by (29), the conditions 2) and 4) of Theorem 3, the initial condition (24) and notation (7) can be written

$$J(x_\alpha(\cdot), u_\alpha(\cdot)) = (\mathbf{P}) \int_{t_\alpha}^{\tau_\alpha} G_0(t, u_\alpha(t))\, dt.$$

Hence, by (6) and condition 1) of Theorem 3 we get the equality (28). ■

It is interesting to note that function $\varphi \in \Phi$ satisfying the conditions of Theorem 3 is not an analogue of the Bellman function of the classical optimal control problem. The role of the Bellman function here is played by the functional

$$w_\alpha \to I(w_\alpha) \triangleq (\mathbf{P}) \int_{t_\alpha}^{\tau_\alpha} \varphi(t, \alpha(t))\, dt : B_0 \to R.$$

The element $w_\alpha \in B_0$ plays for the system (1) the role of the initial position.

In this connection we shall give the following definition.

DEFINITION 3. Call a function $\varphi \in \Phi$ satisfying the conditions 1), 2) of Theorem 3 and for which there exists at least one function $u^0 \in \mathscr{U}$ satisfying the conditions 3)–5) of this theorem an *originative function*.

A pleasant feature of the problem under consideration is that values of the functional $I(\cdot)$ can be calculated constructively with the known originative function.

EXAMPLE. Let the controlling and controlled objects be placed in the same point at the moment $t = 0$ and let them move apart at a speed half of that which the control signal spreads. Thereby, $\tau = 2t$. Taking this into account, let the equation (1) have the following appearence

$$x(2t) = x(t) + u(t), \quad x, u \in R^1, \quad t \in [0, t_1].$$

A method of dynamic programming kept in mind, consider not only the trajectories beginning at $t_0 = 0$ but also these with other $t_\alpha \in [0, t_1/2]$. Therefore, for $t \in [t_\alpha, 2t_\alpha)$ the function $x(t)$ is defined as $x(t) = \alpha(t)$ (condition (24)). The criterion (4) is like

$$J = \int_{t_\alpha}^{2t_1} \left(x^2(t) + u^2(t)\right) dt.$$

Apply Theorem 3. Taking advantage of the condition 2) of the theorem, it is easy to show that $u(t) = 0$, $\varphi(t, x) = x^2$ on the interval $[t_1, 2t_1]$.

When $t \in [0, t_1)$ we find the function φ in the form $\varphi(t, x) = \sigma(t)\, x^2$. Then, using the condition 1) of Theorem 3, we obtain the following equation for the function $\sigma(t)$

$$\sigma(t) = 1 + \sigma(2t)\left(1 + \sigma(2t)\right)^{-1}.$$

Since $\sigma(t) = 1$ at $t \in [t_1, 2t_1]$, one can see that function $\sigma(t)$ is constant, $\sigma(t) = \sigma_k$, on each of the intervals $[2^{-k} t_1, 2^{1-k} t_1)$, $k = 1, 2, \ldots$, the number of which is infinite and accumulates in approaching the point $t = 0$. A few of the first values of σ_k are as follows: $\sigma_1 = 1.5$, $\sigma_2 = 1.6$, $\sigma_3 = 1.615, \ldots$. If k becomes infinite, $\sigma_k \to (1 + \sqrt{5})/2$. An optimal feedback control is the linear regulator $u(t) = -\sigma(2t)\left(1 + \sigma(2t)\right)^{-1} x(t)$ when $t \in [0, t_1]$ and $u(t) = 0$ when $t \in [t_1, 2t_1]$. It is interesting to note that the feedback factor in the regulator is piecewise-constant, which is not obvious from the formulation of the problem.

4. The existence of originative function. Necessary and sufficient optimality conditions

Let us study the question on the existence of originative function and necessity for conditions of the theorem 2.

Naturally, the question cannot be answered at such an abstract level, at which the theorems 1—3 have been proved. Therefore, assume in this part of the paper that X, U are finite-dimensional Euclidean spaces, Q is a non-empty compact subset of U independent of the point (t, x), the functions f, f^0 are defined on $[t_0, \tau_1]\times X\times Q$ only (earlier they were defined on $[t_0, \tau_1]\times X\times U$). The following assumptions will be required with respect to the functions f, f^0, the sets $B(t)$ and the initial elements $w_\alpha \triangleq (t_\alpha, \alpha(\cdot))\in B_0$.

5°. The set $V\triangleq\{(t, x): t\in[t_0, \tau_1], x\in B(t)\}$ is closed.

6°. The function f is continuous.

7°. The function f^0 is lower semi-continuous (l.s.c.) and finite.

8°. The function τ is continuously differentiable.

9°. For all $t\in[t_0, t_1]$ the inequality $\tau(t)>t$ holds.

10°. The function α is Borel-measurable.

11°. There exists a constant $c\in R$ such that $\|\alpha(t)\|\leqslant c$ for all $t\in[t_\alpha, \tau_\alpha)$.

12°. There exists at least one process $(x(\cdot), u(\cdot))$ with the initial element w_α satisfying phase restrictions

$$x(t)\in B(t),\ t\in[t_\alpha, \tau_1]. \tag{30}$$

THEOREM 4. *If the assumptions 5°–9° hold true and for any $w_\alpha\in B_0$ the assumptions 10°–12° hold. then an originative function exists.*

THEOREM 5. *Let the assumptions 5°–9° and 10°–12° hold for the fixed initial element $(t_0, \alpha(\cdot))$.*

In order for the control process $(\bar{x}(\cdot), \bar{u}(\cdot))\in D_0$ to minimize the functional J on D_0 it is necessary and sufficient that there exists a function $\varphi\in\Phi_0$ satisfying the conditions 1)–4) of Theorem 2.

Before turning to a proof of Theorems 4, 5 let us set some auxilliary assertions associated with the properties of l.s.c. functions.

In what follows we assume that A, K are non-empty closed subsets of finite-dimensional Euclidean spaces.

LEMMA 3. *Let the function $g: K\to\bar{R}$ be l.s.c. and the function $h: A\to K$ be continuous.*

Then the function $x\to\Gamma(x)\triangleq g(h(x)): A\to\bar{R}$ is l.s.c.

Proof. Let $x\in A$, then $\bar{y}\triangleq h(\bar{x})\in K$. Since $h(A)\subset K$ and $h(x)\to\bar{y}$ for $x\to\bar{x}$,

$$\varliminf_{x\to\bar{x}}\Gamma(x)\geqslant\varliminf_{y\to\bar{y}} g(y), \tag{31}$$

if $x \in A$ and $y \in K$.
But it follows from the definition of the l.s.c. function g that

$$\underline{\lim}_{y \to \bar{y}}\, g(y) \geqslant g(\bar{y}) = \Gamma(x). \tag{32}$$

Comparing (31) and (32) one can see that the function Γ is l.s.c. $\blacksquare$

LEMMA 4. *Let K be a compact set and the function $h: A \times K \to \bar{R}$ l.s.c. and not attaining the value $-\infty$.*

Then the function

$$x \to g(x) \triangleq \min_{u \in K} h(x, u): A \to \bar{R}, \tag{33}$$

is l.s.c.

Proof. As K is a compact set, the equality (33), by Remark 4, holds true. Let the sequence $\{x_s\} \subset A$ converge to an element $\bar{x} \in A$. Choose a subsequence $\{x_{s'}\} \subset \{x_s\}$ such that

$$\underline{\lim}_{s \to \infty}\, g(x_s) = \lim_{s' \to \infty} g(x_{s'}). \tag{34}$$

For any s' choose an element $u_{s'}$ such that $g(x_{s'}) = h(x_{s'}, u_{s'})$.

Since K is a compact set, it is possible to choose a subsequence $\{u_{s''}\} \subset \{u_{s'}\}$ converging to an element $\bar{u} \in K$. Since the function h is l.s.c. on $A \times K$, we have

$$\lim_{s'' \to \infty} h(x_{s''}, u_{s''}) \geqslant h(\bar{x}, \bar{u}). \tag{35}$$

But $h(x_{s''}, u_{s''}) = g(x_{s''})$, and

$$\lim_{s'' \to \infty} g(x_{s''}) = \lim_{s' \to \infty} g(x_{s'}),$$

therefore

$$\lim_{s'' \to \infty} h(x_{s''}, u_{s''}) = \lim_{s' \to \infty} g(x_{s'}). \tag{36}$$

Taking into account that $h(\bar{x}, \bar{u}) \geqslant g(\bar{x})$, by (35) and (36), we obtain

$$\lim_{s' \to \infty} g(x_{s'}) \geqslant g(\bar{x}). \tag{37}$$

Comparing (34) with (37) and taking into account that the sequence $\{x_s\} \subset A$ is arbitrary, we can conclude that the function g is l.s.c. $\blacksquare$

To prove Theorem 4 we have to use the theorem on measurable choice, which is somewhat different from those traditionally used [11, p. 236], [12, p. 59].

THEOREM 6. *Let K be a compact set and let the function $h: A \times K \to R$ be l.s.c.*

Then, there exists a Borel-measurable (Borelian) function $u^*: A \to K$, satisfying the conditions

$$h\left(x, u^*(x)\right)=\min_{u\in K} h(x, u), \; x\in A.$$

Proof. The epigraph epi $h\subset R\times A\times K$ of function h is a closed set [11, p. 19], hence it is Borelian. Then, according to the Novikov theorem [13, Theorem 1.5] the projection H of the set epi h on $R\times A$ is a Borelian set. According to the Louzin-Yankov theorem [13, Theorem 1.2] there exists a Borel-measurable function $(\xi, x)\to u_0(\xi, x): H\to K$*⁾.

By Lemma 4 the function

$$x\to g(x)\triangleq\min_{u\in K} h(x, u): A\to R,$$

is l.s.c. Since for the l.s.c. function g the Lebesgue sets $S(\eta)\triangleq\{y\in A: g(x)\leqslant\eta\}$, $\eta\in R$, are closed [11, p. 19] then g is Borelian.

Thus, the functions u_0 and g are Borelian. Then, their superposition $x\to w(x)\triangleq u_0\left(g(x), x\right): A\to K$ is also Borelian. But

$$w(x)\in\{v\in K: h(x, v)=\min_{u\in K} h(x, u)\},$$

for all $x\in A$. Therefore, we can take $u^*(x)=w(x)$, $x\in A$. ■

Proof of Theorem 4. The outline of the proof for the theorem is as follows. Since the function τ is continuous and $\tau(t)>t$ for all $t\in[t_0, t_1]$, we apply the well-known step method. Using this method, by the conditions 1)–4) of Theorem 3 we build the function φ and the control u^0 of separate pieces, moving from the end of the interval $[t_0, \tau_1]$ to the beginning. We will show later that functions φ and u^0, constructed in such a way, satisfy all conditions of Theorem 3.

We form a finite system of intervals $\Delta_i\triangleq[t^i, t^{i+1}]$, $i=\overline{0, N}$ such that $t^{i+1}\triangleq\tau(t^i)$, $i=\overline{1, N}$ and $t^0\triangleq t_0$, $t^N\triangleq t_1$. Such a system exists since the assumption 9° holds. Note that $t^1=\tau(t_0)$ is not necessary and the left end interval may be shortened.

Introduce the system of sets $W_i^*\triangleq\{(t, x)\in V: t\in\Delta_i\}$, $i=\overline{0, N}$. The sets W_i^* are closed by the assumption 5°. Further, for any $i=\overline{0, N}$ form the sets W_i defined as

$$W_i\triangleq\{(t, x)\in W_i^*: \left(t, f(t, x, Q)\right)\cap W_{i+1}\neq\emptyset\},$$

for $i=\overline{0, N-1}$ and $W^N\triangleq W_N^*$. The sets W_i must be formed recurrently, beginning from W_N. Since the set Q is compact, the function f by the assumption 6° is continuous and W_i^*, $i=\overline{0, N}$ are closed, and the sets W_i, $i=\overline{0, N}$, are also closed.

*⁾ Theorem 1.2 from [13] demands that the set $R\times A$ be compact; however, it is not important here, for $R\times A$ can be presented as a union of not more than a countable number of compact sets.

The set $W \triangleq \cup_{i=0}^{N} W_i$ has the following property: for any initial element $(t_\alpha, \alpha(\cdot))$ such that

$$(t, \alpha(t)) \in W, \ t \in [t_\alpha, \tau_\alpha], \tag{38}$$

the condition 12° holds. Violation of the condition (38) at least at one point of the interval $[t_\alpha, \tau_\alpha)$ contradicts 12°. Violation of the condition (38) on a set of positive measure leads to a violation of phase constraints (30) also on a set of positive measure of the interval $[t_\alpha, \tau_1]$.

Denote $Y_i \triangleq \Delta_i \times X$. For any $i = \overline{0, N}$ and for any $k = \overline{1, N}$ build the functions $\varphi_i : Y_i \to \bar{R}$ and $\chi_k : Y_k \times Q \to \bar{R}$, taking advantage of the equalities

$$\varphi_N(t, x) \triangleq \min_{u \in Q} f^0(t, x, u), (t, x) \in W_N,$$

$$\varphi_N(t, x) \triangleq \infty, \ (t, x) \in Y_N \setminus W_N,$$

$$\chi_k(t, x, u) \triangleq \varphi_k(\tau(t), f(t, x, u))\, \tau'(t) + f^0(t, x, u).$$

$$\varphi_i(t, x) \triangleq \min_{u \in Q} \chi_{i+1}(t, x, u), (t, x) \in Y_i, \ i = \overline{0, N-1}.$$

It follows from 7° and Lemma 5 that φ_N is l.s.c. on W_N, and as the set W_N is closed, also on the whole domain of definition Y_N. Hence, by 6°–8° and Lemma 4, the function χ_N is also l.s.c. on the whole domain of definition, and, by Lemma 5, the function φ_{N-1} is also l.s.c.

In a similar way we may prove that the functions χ_{N-1}, φ_{N-2} are l.s.c. Repeating the process we may conclude that each of the functions χ_k, $k = \overline{1, N}$ and φ_i, $i = \overline{0, N}$ is l.s.c.

It is also easy to see that functions φ_i are finite on W_i. Moreover, they are bounded on any bounded subset of W_i.

Define the function $\varphi : [t_0, \tau_1] \times X \to \bar{R}$ by the equalities

$$\varphi(t, x) \triangleq \varphi_i(t, x), (t, x) \in [t^i, t^{i+1}) \times X, \ i = \overline{0, N-1},$$

$$\varphi(t, x) \triangleq \varphi_N(t, x), (t, x) \in Y_N.$$

It is important that the function φ is bounded on any bounded subset of W and turns into $+\infty$ outside W.

The function φ built above evidently satisfies the conditions 1), 2) of Theorem 3.

We show that φ is an originative function. For this purpose we build the control u^0, satisfying the conditions 3)–5) of Theorem 3.

Form the functions $u_i^0 : Y_i \to Q$, $i = 1, \bar{N}$, for which the following inclusions hold:

$$u_N^0(t, x) \in \operatorname{Arg} \min_{u \in Q} f^0(t, x, u), (t, x) \in Y_N,$$

$$u_0^i(t, x) \in \operatorname{Arg} \min_{u \in Q} \chi_i(t, x, u), (t, x) \in Y_i, \ i = \overline{0, N-1}.$$

The function f^0 is l.s.c. and finite (condition 7°). Each of the functions χ_i, $i=\overline{1,N}$, is also l.s.c., finite on the closed subset $W_i \times Q$ of its domain of definition and takes the value $+\infty$ outside this subset. With this in mind and making use of the theorem on measurable choice (Theorem 6), choose all the functions u_i^0, $i=\overline{0,N}$, in such a way that they are Borel-measurable. In this case the control $u^0 \in \mathscr{U}$ defined by the equalities

$$u^0(t,x) \triangleq u_i^0(t,x),\ (t,x) \in [t^i, t^{i+1}) \times X,\ i=\overline{0,N-1},$$
$$u^0(t,x) \triangleq u_N^0(t,x),\ (t,x) \in Y_N,$$

is also a Borel-measurable function.

The function u^0 thus constructed evidently satisfies the conditions 3), 4) of Theorem 3.

Now let us show that the condition 5) of Theorem 3 for the control u^0 constructed holds too.

To do this, it is sufficient to prove that for the initial element, satisfying the conditions 10°—12°, there exists a solution $x_\alpha(t)$ of the equation (25) with the initial condition (24), where $u_\alpha(t)$ is defined by the equality (27), satisfying the phase constraints (30) and it is such that the function

$$t \to \xi_\alpha(t) \triangleq f^0(t, x_\alpha(t), u_\alpha(t)) : [t_\alpha, \tau_1] \to R,$$

is summable.

Thus, let $w_\alpha \triangleq (t_\alpha, \alpha(\cdot))$ be an arbitrary initial element satisfying the conditions 10°—12°. As has already been mentioned, the structure of the set W is such that the condition 12° implies (38). Since $W \subset V$, for the initial function $\alpha(\cdot)$ the phase constraints (30) hold. Applying a step method (from left to right), it is easy to see that for the bounded initial function $\alpha(\cdot)$ (condition 11°) the trajectory $x_\alpha(t)$ is defined on the whole interval $[t_\alpha, \tau_1]$ and it is bounded. Besides, since the initial function $\alpha(\cdot)$ is Borel--measurable, the whole trajectory $x_\alpha(t)$, $t \in [t_\alpha, \tau_1]$, is also Borel-measurable. Next, the function f defining the equation (1) and thus forming the trajectory $x_\alpha(\cdot)$ maps the point $(t,x) \in W$, $t \in [t_\alpha, t_1]$, of the graph of the trajectory

$$\pi_\alpha \triangleq \{(t,x) \in [t_\alpha, \tau_1] \times X : x = x_\alpha(t)\},$$

into the point

$$y(t,x) \triangleq \big(\tau(t), f(t, x, u^0(t,x))\big) \in \pi_\alpha.$$

The point $y(t,x) \in W$, because otherwise, $\varphi(y(t,x)) = \infty$.

But this is impossible because of the structure of the function u^0 and the set W. Hence, the control u^0 does not drive the trajectory from the set W, nor from the set V.

It is clear that the trajectory $x_\alpha(t)$ is defined on the whole interval $[t_\alpha, \tau_1]$ and, as was just ascertained,

$$(t, x_\alpha(t)) \in V, \; t \in [t_\alpha, \tau_1]. \tag{39}$$

The inclusion (39) implies that the phase constraints (30) hold true.

Now we prove the measurability and boundedness of the function ξ_α. The functions $f^0(t, x, u)$ and $u^0(t, x)$ are Borelian and, therefore, the function $(t, x) \to f^0(t, x, u^0(t, x))$ is Borelian too. Moreover, since the trajectory $x_\alpha(\cdot)$ is also Borel-measurable, the function

$$t \to \xi_\alpha(t) = f^0\big(t, x_\alpha(t), u^0(t, x_\alpha(t))\big),$$

is Borel-measurable as well. Besides, owing to the boundedness of the function $x_\alpha(\cdot)$ the function $\xi_\alpha(\cdot)$ is bounded. The summability of the function ξ_α follows from its measurability and boundedness.

Thus, the condition 5) of Theorem 3 holds true and, so, all the conditions 1)–5) of this theorem are satisfied.

To complete the proof it is sufficient to check the validity of the inclusion $\varphi \in \Phi$, i.e. the condition 2° for all $(x(\cdot), u(\cdot)) \in D(w_\alpha)$, where w_α satisfied 10°–12°.

Using a step method (from left to right) it is easy to demonstrate that all the trajectories $x(\cdot)$, corresponding to the processes $(x(\cdot), u(\cdot) \in D(w_\alpha)$, are jointly bounded. At the same time all those trajectories lay in the set W (otherwise they violate the phase constraints). Since the function φ is bounded on any bounded subset of W, there exists a constant $c^* \in R$ such that $|\varphi(t, x(t))| \leqslant c^*$, $t \in [t_\alpha, \tau_1]$, for all $(x(\cdot), u(\cdot)) \in D(w_\alpha)$. But this means that the condition 2° holds, i.e. the inclusion $\varphi \in \Phi$ is valid. ∎

Proof of Theorem 5. The sufficiency of the conditions of the theorem follows directly from Theorem 2.

Let us prove the necessity. For this we have to apply the Theorem 4, already proved, to the one-element set $B_0 = \{(t_0, \alpha(\cdot))\}$. The restriction $\tilde{\varphi}$ of the originative function $\varphi : [t_0, \tau_1] \times X \to \bar{R}$, whose existence is guaranteed by Theorem 4, on $[\tau_0, \tau_1] \times X$, and the process $v_\alpha \triangleq (x_\alpha(\cdot), u_\alpha(\cdot))$, defined by the conditions (24), (25), (27), $t_\alpha = t_0$, satisfy all the conditions of Theorem 2. The proof of this fact repeats the proof of assertion a) of Theorem 3.

But the conditions of Theorem 2 are satisfied not only by the pair $(\tilde{\varphi}, v_\alpha)$ but also by a pair $(\tilde{\varphi}, \bar{v})$, where $\bar{v} \triangleq (\bar{x}(\cdot), \bar{u}(\cdot))$ is any other optimal process. This is the fact which guarantees the necessity of the conditions of Theorem 5 for optimality of the process $\bar{v}$. ∎

REMARK 5. It is easy to notice that the proof of Theorem 4 contains a constructive way of building the originative function φ in the most general case. This method can be used for solving particular problems.

REMARK 6. One and the same originative function φ built while proving Theorem 4 is acceptable for different sets B_0. It is only needed that their elements satisfy the conditions 10°–12°. Therefore this function φ is also an originative one for a maximal set B_0 which includes all initial elements w_α satisfying the conditions 10°–12°.

REMARK 7. Let the set B_0, whose elements satisfy the conditions 10°–12°, be such that the constant c in the condition 11° may be chosen the same for all its elements. Then, with the assumptions of Theorem 4 true, there exists an originative function bounded on any bounded subset from $[t_0, \tau_1] \times X$.

This result is obtained in the following way. Under the assumption made with respect to the set B_0, the trajectories of the system are uniformly bounded, i.e. there exists a compact $W_c \subset [\tau_0, \tau_1] \times X$, such that for all $w_\alpha \in B_0$ and for any process $(x(\cdot), u(\cdot)) \in D(w_\alpha)$ the inclusion $(t, x(t)) \in W_c$, $t \in [\tau_0, \tau_1]$, holds true.

Let φ be an originative function built while proving Theorem 4. Then the function $\varphi^0 : [t_0, \tau_1] \times X \to R$, defined by the equalities

$$\varphi^0(t, x) \triangleq \varphi(t, x), \ (t, x) \in W,$$

$$\varphi^0(t, x) \triangleq c^0 + 1, \ (t, x) \in ([t_0, \tau_1] \times X) \backslash W,$$

where

$$c^0 \triangleq \max_{(t, x) \in W \cap W_c, \, u \in Q} \left[\varphi(\tau(t), f(t, x, u)) \, \tau'(t) + f^0(t, x, u)\right],$$

is also an originative function, containing the required property of boundedness on the compact subsets of the area of definition.

5. Local optimality conditions

Theorem 5 contains necessary and sufficient conditions for global optimality of noninertial dynamic systems with delays. Along with them we may find local optimality conditions similar to Pontryagin maximum principle in classical optimal control problem or its analogue for discrete controllable systems [14].

We can obtain these local conditions as necessary conditions for existence of the function φ in Theorem 5, differentiable with respect to x. Note that the theorem 5 guarantees the existence of the function φ but not its differentiability, which appears here as an additional assumption. In this connection such a method of obtaining local optimality conditions is not rigorous. Yet it is possible to give a rigorous proof applying the technique well elaborated for discrete systems [15]. The restricted size of the paper does not allow to include this proof here.

Let us take for granted that the assumptions of Part 4 and these to follow are satisfied.

13°. The functions $f(t,\cdot,\cdot):X\times Q\to X$ for $t\dot{\in}[t_0,t_1]$ and $f^0(t,\cdot,\cdot):X\times\times Q\to\bar{R}$ for $t\dot{\in}[t_0,\tau_1]$ are continuously differentiable.

THEOREM 7. *Let the assumptions of Theorem 5 and 13° hold true. Let also the function φ, whose existence is guaranteed by Theorem 5, be such that the function $\varphi(t,\cdot):X\to\bar{R}$ for $t\dot{\in}[t_\varphi,t_1]$ is continuously differentiable.*

Then for the process $(\bar{x}(\cdot),\bar{u}(\cdot))\in D_0$ to be optimal it is necessary that the following condition

1) $f^0(t,\bar{x}(t),\bar{u}(t))=\min\limits_{u\in Q} f^0(t,\bar{x}(t),u),\ t\dot{\in}[t_1,\tau_1]$,

holds true and that a vector-function $t\to\psi(t):[t_0,\tau_1]\to X$ exists, satisfying the conditions

2) $\left[H_x\left(\psi(\tau(t)),t,\bar{x}(t),\bar{u}(t)\right)-\psi(t)\right]^T\delta x\geqslant 0$,
$\delta x\in K_x(\bar{x}(t)),\ t\dot{\in}[\tau_0,t_1]$;

3) $H_u\left(\psi(\tau(t)),t,\bar{x}(t),\bar{u}(t)\right)^T\delta u\geqslant 0$,
$\delta u\in K_u(\bar{u}(t)),\ t\dot{\in}[t_0,t_1]$;

4) $[-\psi(t)+f_x^0(t,\bar{x}(t),\bar{u}(t))]^T\delta x\geqslant 0$,
$\delta x\in K_x(\bar{x}(t)),\ t\dot{\in}[t_1,\tau_1]$.

Here $K_x(\bar{x}(t))\triangleq\{\delta x\in X:\bar{x}(t)+\varepsilon\,\delta x\in B(t),\ 0<\varepsilon<\varepsilon_1\}$, $K_u(\bar{u}(t))\triangleq\{\delta u\in U:\bar{u}(t)+\varepsilon\,\delta u\in Q,\ 0<\varepsilon<\varepsilon_1\}$ are the cones of permissible variations with respect to a phase vector and the control correspondingly [15]; $H(\psi,t,x,u)\triangleq\triangleq\psi^T f(t,x,u)\,\tau'(t)+f^0(t,x,u)$, $\psi\triangleq(\psi_1,\dots,\psi_n)^T\in R^n=X$, $f\triangleq(f_1,\dots,f_n)^T$, $f_x^0\triangleq(\partial f^0/\partial x_1,\dots,\partial f^0/\partial x_n)^T$, $H_x\triangleq(\partial H/\partial x_1,\dots,\partial H/\partial x_n)^T$, $H_n\triangleq(\partial H/\partial u_1,\dots,\partial H/\partial u_m)^T\in R^m=U$, $\delta x\triangleq(\delta x_1,\dots,\delta x_n)^T\in R^n$, $\delta u\triangleq(\delta u_1,\dots,\delta u_m)^T\in R^m$, $(\cdot)^T$ is the transposition operation.

Proof. Let $(\bar{x}(\cdot),\bar{u}(\cdot))$ be an optimal control process. According to Theorem 5 there exists a function $\varphi\in\Phi_0$, satisfying the conditions 1)–3) of Theorem 2. ■

The conditions 1)–4) of Theorem 7 are the first order necessary conditions [15] for extremal problems, contained in the conditions 1)–3) of Theorem 2. In writing them down it is necessary to take into account the notations $\psi(t)\triangleq\varphi_x(t,\bar{x}(t))$, $t\dot{\in}[t_0,\tau_1]$, $\varphi_x\triangleq(\partial\varphi/\partial x_1,\dots,\partial\varphi/\partial x_n)^T\in R^n=X$, and set $\psi(t)\triangleq 0$ for such t where the derivative φ_x is not defined.

In case when the restrictions on x and u are absent the following result is evident.

COROLLARY 2. *If. in addition to the conditions of Theorem 7, $B(t)=X$, $t\in[t_0,\tau_1]$, $Q=U$, then the conditions 2)–4) of the theorem take the form*

2′) $H_x\left(\psi(\tau(t)),t,\bar{x}(t),\bar{u}(t)\right)-\psi(t)=0,\ t\dot{\in}[\tau_0,t_1]$,

3′) $H_u\left(\psi(\tau(t)),t,\bar{x}(t),\bar{u}(t)\right)=0,\ t\dot{\in}[t_0,t_1]$,

4′) $\psi(t)=f_x^0(t,\bar{x}(t),\bar{u}(t)),\ t\dot{\in}[t_1,\tau_1]$.

6. Generalization of the optimality conditions to the case of several delays

There are some generalizations for a dynamic system model (1) to the case of several delays of the argument. They are of interest from the viewpoint of their applications. For simplicity, let us confine ourselves to a consideration of the case when two delays are available and, therefore, instead of the equation (1) we take the equation

$$x\left(\tau(t)\right)=f\left(t, x(t), u(t), x\left(\xi(t), u\left(\xi(t)\right)\right)\right),\ t\in[t_0, t_1], \tag{40}$$

where $\xi:[t_0, t_1]\to R$, the rest of the notation having the same meaning as before.

Introducing the notation $y(t)\triangleq x\left(\xi(t)\right)$, $v(t)\triangleq u\left(\xi(t)\right)$ we can rewrite (40) as a system of equations

$$\begin{aligned} x\left(\tau(\dot{t})\right)&=f\left(t, x(t), u(t), y(t), v(t)\right),\\ y(t)&=x\left(\xi(t)\right),\\ v(t)&=u\left(\xi(t)\right), \end{aligned} \tag{41}$$

or as a system

$$\begin{aligned} x\left(\tau(t)\right)&=f\left(t, x(t), u(t), y(t), v(t)\right),\\ y\left(\tau(t)\right)&=x\left(\xi\left(\tau(t)\right)\right),\\ v\left(\tau(t)\right)&=u\left(\xi\left(\tau(t)\right)\right). \end{aligned} \tag{42}$$

In the case when $\xi\left(\tau(t)\right)=t$ for $t\in[t_0, t_1]$, the model (42) of the dynamic system evidently has the form (1). Therefore, the problem reduces to that already considered. Such a reduction is possible in a somewhat general case when there exists a $\eta:[t_0, t_1]\to R$ and the integers $k, s>0$ are such that

$$\eta_{(k)}\left(\tau(t)\right)=t,\ \eta_{(s)}(t)=\xi(t),\ t\in[t_0, t_1].$$

Here $\eta_{(\tau)}(\cdot)$ is the τ-times superposition of the function η (e.g. $\eta_{(2)}(\cdot)=$ $=\eta\left(\eta(\cdot)\right)$). In particular, the reduction described is possible if

$$\tau(t)=t+k\Delta t,\ \xi=t-s\Delta t,\ \Delta t=\text{const}>0.$$

The given method for reducing the model (40) to the model (1) may demand addition of a considerable number of auxilliary equations. Their total number can be up to $2(k+s)-1$. Besides, the method is evidently far from being always applicable. That is why it is of interest to study more closely the model (40) or the model (41).

Following the representation (41) of the equation (40) consider a somewhat general model

$$x\left(\tau(t)\right)=f\left(t, x(t), z(t), u(t)\right),$$
$$z(t)=g\left(\xi(t), x\left(\xi(t)\right), u\left(\xi(t)\right)\right),\ t\in[t_0, t_1], \tag{43}$$

and study the problem of minimum of the functional

$$v\to J(v)\triangleq\int_{\xi_0}^{t_0} f_0^0\left(t, x(t), u(t)\right)dt+$$
$$+\int_{t_0}^{t_1} f^0\left(t, x(t), z(t), u(t)\right)dt+\int_{t_1}^{\tau_1} f_1^0\left(t, x(t), u(t)\right)dt: D_0\to\bar{R}. \tag{44}$$

Here $x\in X$, $u\in U$, $z\in Z$ where X, U, Z are metric spaces; $f:[t_0, t_1]\times\times X\times Z\times U\to X, g:[\xi_0,\xi_1]\times X\times U\to Z, f_0^0:[\xi_0,t_0]\times X\times U\to\bar{R}, f^0:[t_0,t_1]\times\times X\times Z\times U\to\bar{R}$, $f_1^0:[t_1, \tau_1]\times X\times U\to\bar{R}$; τ, $\xi:[t_0, t_1]\to R$; $\xi_0\triangleq\xi(t_0)\leqslant\leqslant t_0\leqslant\tau_0\triangleq\tau(t_0)<\xi_1\triangleq\xi(t_1)\leqslant t_1\leqslant\tau_1\triangleq\tau(t_1)$; the functions $\tau(\cdot)$, $\xi(\cdot)$ are absolutely continuous and non-decreasing. For all $t\in[\xi_0, \tau_1]$ the restriction $x(t)\in B(t)$ must hold, for all $t\in[t_0, t_1]$ the restriction $z(t)\in C(t)$ must hold and for all $t\in[\xi_0, \tau_1]$, $x\in B(t)$ the restriction $u(t)\in Q(t, x)$ must hold. The sets $B(t)\subset X$, $C(t)\subset Z$, $Q(t, x)\subset U$ are known beforehand. The initial condition is of the form: $x(t)=\alpha(t)$, $t\in[\xi_0, \tau_0)$. Here, as has been assumed earlier, suppose $[\xi_0, \xi_0)\triangleq\{\xi_0\}$. The function $\alpha:[\xi_0, \tau_0)\to X$ is defined and it satisfies the restriction: $\alpha(t)\in B(t)$ for $t\in[\xi_0, \tau_0)$.

The elements of the set D_0 here are the control processes $v\triangleq\triangleq\left(x(\cdot), z(\cdot), u(\cdot)\right)$, $x:[\xi_0, \tau_1]\to X$, $z:[t_0, t_1]\to Z$, $u:[\xi_0, \tau_1]\to U$ for which the functional (44) is defined and all the conditions enumerated hold true. Denote the lower bound of the functional (44) by d_0 as before.

In transferring the optimality conditions into the case considered in this section two main complications occur.

In formulating the optimality conditions, similar to those of Theorems 1, 2, a natural desire is to use the function $(t, x, z)\to\varphi^*(t, x, z)$ instead of the function φ. But it is impossible. The fundamental role in the proof of Theorems 1 and 2 was played by the equality

$$(\mathbf{P})\int_{\tau_0}^{t_1}\varphi\left(\tau(t), x\left(\tau(t)\right)\right)\tau'(t)\,dt=(\mathbf{P})\int_{t_0}^{\tau_1}\varphi\left(t, x(t)\right)dt,$$

(see the proof of Lemma 1), based on deformation of the interval of integration. In the case under consideration the functions $\tau(t)$ and $\xi(t)$ give rise to various deformations. Thereby, we are forced to confine ourselves to the functions φ^*, representable as: $\varphi^*(t, x, z)=\varphi(t, x)+\gamma(t, z)$. To be more precise, two functions $\varphi:[\tau_0, \tau_1]\times X\to\bar{R}$ and $\gamma:[\xi_0, \xi_1]\times\times Z\to\bar{R}$ are used.

Naturally, this results in more strict sufficient conditions of optimality. In particular, these optimality conditions, generally speaking, cannot be used for constructing a feedback control. They are intended mainly for solving a program control problem (with the initial condition fixed).

The second complication is the following. The use of the lower Perron integral in proving the theorems 1—3 enabled us to minimize the assumptions on the problem. It became possible because the inequalities, arisen when using the lower Perron integral instead of the Lebesgue integral, enabled us to estimate the lower bound d_0. However, some of the inequalities have a sign different from that which would have been desirable. This circumstance necessitates contraction of the class of problems in order to guarantee the possibility of using the Lebesgue integral.

Some of the restrictions on the problem have already been observed (X, Z, U are metric spaces; τ, ξ are absolutely continuous). Now let us formulate some additional assumptions.

14°. The admissible control processes $(x(\cdot), z(\cdot), u(\cdot))$ (elements of the set D_0) are such that:

a) the functions $x(\cdot), z(\cdot)$ are Borel-measurable and bounded

b) the function $t \to f_0^0(t, x(t), u(t)): [\xi_0, t_0] \to \bar{R}$, $t \to f^0(t, x(t), z(t), u(t)): : [t_0, t_1] \to \bar{R}$, $t \to f_1^0(t, x(t), u(t)): [t_1, \tau_1] \to R$ is (Lebesgue) summable.

The condition 14° holds true trivially, provided that the following asumptions 15°–17° are valid, while for any element $(x(\cdot), z(\cdot), u(\cdot)) \in D_0$ the assumption 18° is valid.

15°. $\tau(t) > t > \xi(t)$ for all $t \in [t_0, t_1]$.

16°. The functions f, g, f_0^0, f^0, f_1^0 are Borel-measurable and bounded on any subset of domain of definition.

17°. The initial function $\alpha(\cdot)$ is Borel-measurable and bounded.

18°. The function $u(\cdot)$ is Borel-measurable and bounded.

Note that for the control $u(\cdot): [\xi_0, \tau_1] \to U$ fixed and the initial function $\alpha(\cdot)$ defined the conditions 15°–18° guarantee the existence and uniqueness of the functions $x(\cdot): [\xi_0, \tau_1] \to X$, $z(\cdot): [t_0, t_1] \to Z$, Borel-measurable, bounded and satisfying (43). This fact, as well as the sufficiency of 15°–18° for 14°, can easily be proved using a step method and taking into account the superpositional measurability of Borelian functions.

To formulate the optimality conditions introduce a class Φ^0 of functions $\varphi: [\tau_0, \tau_1] \times X \to \bar{R}$ and a class Γ^0 of functions $\gamma: [\xi_0, \xi_1] \times Z \to \bar{R}$, satisfying the condition:

19°. Functions $\varphi \in \Phi^0$ and $\gamma \in \Gamma^0$ are Borel-measurable and bounded on any bounded subset of the domain of definition.

By means of the functions φ and γ construct the equations similar to (5)—(7):

$$G_0^0(t, w) \triangleq -\gamma\left(t, g\left(t, \alpha(t), w\right)\right) + f_0^0\left(t, \alpha(t), w\right), \ t \in [\xi_0, t_0];$$

$$G_0(t, z, w) \triangleq \varphi\left(\tau(t), f\left(t, \alpha(t), z, w\right)\right) \tau'(t) + J\left(\xi(t), z\right) \xi'(t) - \\ - J\left(t, g\left(t, \alpha(t), w\right)\right) + f^0\left(t, \alpha(t), z, w\right), \ t \in [t_0, \tau_0);$$

$$S(t, x, z, u) \triangleq \varphi\left(\tau(t), f(t, x, z, u)\right) \tau'(t) + J\left(\xi(t), z\right) \xi'(t) -$$

$$-\gamma\left(t, g\left(t, x, u\right)\right)-\varphi\left(t, x\right)+f^{0}\left(t, x, z, u\right),\ t\in[\tau_0, \xi_1];$$

$$G_1\left(t, x, z, u\right) \triangleq \varphi\left(\tau\left(t\right), f\left(t, x, z, u\right)\right)\tau'\left(t\right)+J\left(\xi\left(t\right), z\right)\xi'\left(t\right)-$$
$$-\varphi\left(t, x\right),\ t\in[\xi_1, t_1];$$

$$G_1^1\left(t, x, u\right) \triangleq -\varphi\left(t, x\right)+f_1^0\left(t, x, u\right),\ t\in[t_1, \tau_1];$$

where

$$x\in B\left(t\right),\ z\in C\left(t\right),\ w\in Q\left(t, \alpha\left(t\right)\right),$$
$$u\in Q\left(t, x\right),\ \xi'\left(t\right)=d\xi/dt.$$

THEOREM 8. *Let the assumption* 14° *hold true and the functions* $\varphi\in\Phi^0$, $\gamma\in\Gamma^0$ *and* $t\to\mu\left(t\right):[\xi_0, \tau_1]\to\bar{R}$ *be such that*:

1) *The function* μ *is summable*;
2) $G_0^0\left(t, u\right)\geqslant\mu\left(t\right),\ t\dot{\in}[\xi_0, t_0],\ u\in Q\left(t, \alpha\left(t\right)\right)$;
3) $G_0\left(t, z, u\right)\geqslant\mu\left(t\right),\ t\dot{\in}[t_0, \tau_0),\ z\in C\left(t\right),\ u\in Q\left(t, \alpha\left(t\right)\right)$;
4) $S\left(t, x, z, u\right)\geqslant\mu\left(t\right),\ t\dot{\in}[\tau_0, \xi_1),\ z\in C\left(t\right),\ x\in B\left(t\right),\ u\in Q\left(t, x\right)$;
5) $G_1\left(t, x, z, u\right)\geqslant\mu\left(t\right),\ t\dot{\in}[\xi_1, t_1],\ x\in B\left(t\right),\ z\in C\left(t\right),\ u\in Q\left(r, x\right)$;
6) $G_1^1\left(t, x, u\right)\geqslant\mu\left(t\right),\ t\dot{\in}[t_1, \tau_1],\ x\in B\left(t\right),\ u\in Q\left(t, x\right)$.

Then:

a) For all $v\triangleq\left(x\left(\cdot\right), z\left(\cdot\right), u\left(\cdot\right)\right)\in D_0$ the following inequality holds true:

$$J\left(v\right)\geqslant l\left(\varphi, \gamma\right)\triangleq\int_{\xi_0}^{\tau_1}\mu\left(t\right)dt.$$

b) If there exists a sequence of control processes $\{v_s\}\subset D_0$ such that

$$\lim_{s\to\infty} J\left(v_s\right)=l\left(\varphi, \gamma\right), \tag{45}$$

then $l\left(\varphi, \gamma\right)$ is the lower bound of the functional J on D_0 $(l=d_0)$ while the sequence $\{v_s\}$ is a minimizing sequence.

Besides, any other minimizing sequence satisfies the condition (45).

THEOREM 9. *Let the assumption* 14° *hold true and let the functions* $\varphi\in\Phi^0$, $\gamma\in\Gamma^0$ *and the control process* $\bar{v}\triangleq\left(\bar{x}\left(\cdot\right), \bar{z}\left(\cdot\right), \bar{u}\left(\cdot\right)\right)\in D_0$ *satisfy the conditions*:

1) $G_0^0\left(t, \bar{u}\left(t\right)\right)=\min\limits_{u\in Q\left(t, \alpha\left(t\right)\right)} G_0^0\left(t, u\right),\ t\dot{\in}[\xi_0, t_0]$;
2) $G_0\left(t, \bar{z}\left(t\right), \bar{u}\left(t\right)\right)=\min\limits_{z\in C\left(t\right),\, u\in Q\left(t, \alpha\left(t\right)\right)} G_0\left(t, z, u\right),\ t\dot{\in}[t_0, \tau_0)$;
3) $S\left(t, \bar{x}\left(t\right), \bar{z}\left(t\right), \bar{u}\left(t\right)\right)=\min\limits_{x\in B\left(t\right),\, z\in C\left(t\right),\, u\in Q\left(t, x\right)} S\left(t, x, z, u\right),\ t\dot{\in}[\tau_0, \xi_1]$;
4) $G_1\left(t, \bar{x}\left(t\right), \bar{z}\left(t\right), \bar{u}\left(t\right)\right)=\min\limits_{x\in B\left(t\right),\, z\in C\left(t\right),\, u\in Q\left(t, x\right)} G_1\left(t, x, z, u\right),\ t\dot{\in}[\xi_1, t_1]$;
5) $G_1^1\left(t, \bar{x}\left(t\right), \bar{u}\left(t\right)\right)=\min\limits_{x\in B\left(t\right),\, u\in Q\left(t, x\right)} G_1^1\left(t, x, u\right),\ t\dot{\in}[t_1, \tau_1]$.

Then the process $\bar{v}$ *is optimal and any other optimal control process satisfies the conditions* 1)–5).

The outline of the proof of Theorems 8, 9 is the same as in the Theorems 1, 2. The equality below plays the role of equality (15)

$$
\begin{aligned}
J\big(x(\cdot), z(\cdot), u(\cdot)\big) = & \int_{\xi_0}^{t_0} f_0^0\big(t, x(t), u(t)\big)\, dt + \\
& + \int_{t_0}^{t_1} f^0\big(t, x(t), z(t), u(t)\big)\, dt + \int_{t_1}^{\tau_1} f_1^0\big(t, x(t), u(t)\big)\, dt + \\
& + \int_{t_0}^{t_1} \varphi\big(\tau(t), x(\tau(t))\big)\, \tau'(t)\, dt - \int_{\tau_0}^{\tau_1} \varphi\big(t, x(t)\big)\, dt + \\
& + \int_{t_0}^{t_1} J\Big(\xi(t), g\big(\xi(t), x(\xi(t)), u(\xi(t))\big)\Big)\, \xi'(t)\, dt - \\
& - \int_{\xi_0}^{\xi_1} J\Big(t, g\big(t, x(t), u(t)\big)\Big)\, dt, \qquad (15)
\end{aligned}
$$

which, by (43) and the notations introduced, can be rewritten as follows:

$$
\begin{aligned}
J\big(x(\cdot), z(\cdot), u(\cdot)\big) = L\big(x(\cdot), z(\cdot), u(\cdot)\big) \triangleq & \\
\triangleq & \int_{\xi_0}^{t_0} G_0^0\big(t, u(t)\big)\, dt + \int_{t_0}^{\tau_0} G_0\big(t, z(t), u(t)\big)\, dt + \\
= & \int_{\tau_0}^{\xi_1} S\big(t, x(t), z(t), u(t)\big)\, dt + \int_{\xi_1}^{t_1} G_1\big(t, x(t), z(t), u(t)\big)\, dt + \\
& + \int_{t_1}^{\tau_1} G_1^1\big(t, x(t), u(t)\big)\, dt. \qquad (46)
\end{aligned}
$$

The role of equality (46) here is the same as that of (13), (14) in the proof of Theorems 1, 2.

In principle it is possible to formulate an analogue of Theorem 3. However, the functions $\varphi \in \Phi^0$ and $\gamma \in \Gamma^0$ have quite a few "degrees of freedom" to provide in the general case a solution of the optimization problem for an arbitrary initial element. As has been already mentioned, the optimality conditions obtained in this part of the paper are effective only for an optimal control problem in the program form (with a fixed initial condition).

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Warunki globalnej optymalności dla sterowanych systemów bezinercyjnych z opóźnieniami

W pracy rozpatrzono zadanie sterowania optymalnego bezinercyjnymi systemami dynamicznymi z opóźnieniami.

Przy braku sterowania systemy takie są specjalnym przypadkiem równań funkcjonalnych bez swobodnych zmiennych [1—3]. Z drugiej strony, są one blisko związane z systemami wieloetapowymi. W odróżnieniu jednak od nich zmienna niezależna zmienia się tu w sposób ciągły.

Wyprowadzono warunki dostateczne optymalności globalnej dla sterowania programowego i sterowania w zamkniętej pętli. W przypadku, gdy wektory stanu i sterowania są skończenie wymiarowe, przy słabych założeniach wykazano, że są one także warunkami wystarczającymi. Przebadano też lokalne warunki wystarczające pierwszego rzędu.

Omówiono trudności związane z przeniesieniem otrzymanych wyników na systemy ogólniejszego typu.

Условия глобальной оптимальности управляемых безинерционных систем с запаздыванием

Рассматривается задача оптимального управления безинерционными динамическими системами с запаздыванием.

При отсутствии управления такие системы являются частным случаем функциональных уравнений без свободных переменных [1—3]. С другой стороны, они тесно соприкасываются с многошаговыми системами, но в отличие от последних независимая переменная здесь изменяется непрерывно.

Получены достаточные условия глобальной оптимальности програмного управления и управления с обратной связью. В случае, когда фазовый вектор управления конечномерны, при весьма слабых предположениях установлена и необходимость этих условий. Исследуются необходимые условия локальной оптимальности первого порядка.

Обсуждаются трудности перенесения результатов на системы более общего вида.