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New results on Optimal Pole Assignment

by

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The Optimal Pole Assignment Problem is concerned with finding the pole assignment feedback control which also minimises a certain quadratic performance measure. In this paper, a new theorem which lays down the necessary and sufficient conditions for optimality of the given closed loop system has been established. Using the theorem, the optimal pole regions for single and two pole assignments have been delineated. The design freedom thus generated has been utilised in choosing a desired set of poles from corresponding optimal regions. A recursive procedure for optimal pole assignment has been presented with a numerical example.

1. Introduction

It is well known that for a multi-input system there are many control laws which achieve the same closed loop pole configuration. It indicates thereby that apart from pole assignment, a state feedback could satisfy additional performance requirements such as minimization of a quadratic performance index. Such an Optimal Pole Assignment (OPA) amalgamates the advantages of improved transient response of pole assignment and the feedback properties of linear quadratic design. Recently, Juang and Lee (1984) have enunciated a theorem which gave necessary conditions for optimal pole assignment. This has been disproved by Amin and Hassan (1985). Amin (1985) has extended the mirror image property (Molinari, 1977) for OPA. But the application of this method is limited to shifting of the real parts of open loop poles only. In this paper a new theorem for OPA has been established. Using the theorem, a number of results on optimal pole locations have been obtained for 2-pole assignment problem which led to improved regions for OPA. A recursive procedure for optimal pole assignment using real Schur form (RSF) has been presented.

2. A new theorem on OPA

Let us consider the reduced order controllable system

$$\dot{X}_{kL} = A_{kL}\, X_{kL} + B_{kL}\, U. \tag{1}$$

With control law

$$U = -K_{kL}\, X_{kL}, \tag{2}$$

the closed loop system becomes,

$$\dot{X}_{kL} = \bar{A}_{kL}\, X_{kL} = (A_{kL} - B_{kL}\, K_{kL})\, X_{kL}. \tag{3}$$

In this system X_{kL} and U are $p_k \times l$ and $m \times l$ state and input vector respectively. The remaining matrices are of compatible dimensions. In addition B_{kL} is of full rank.

Let the linear quadratic cost function be

$$J = \int_0^\infty (X_{kL}^T\, Q_{kL}\, X_{kL} + U^T R U)\, dt, \tag{4}$$

where, $Q_{kL} = Q_{kL}^T \geqq 0$ (positive semidefinite) and $R = R^T > 0$ (positive definite). For this cost function to be minimum, the controller for the system is given by

$$K_{kL} = R_k^{-1}\, B_{kL}^T\, P_{kL}, \tag{5}$$

where, $P_{kL} = P_{kL}^T > 0$ (p.d.) is the solution of the algebraic Riccati equation. There is no loss of generality in assuming $R = I_m$ (Martin, 1973). Hence

$$K_{kL} = B_{kL}^T\, P_{kL}. \tag{6}$$

THEOREM. *For the controllable system with $p_k \leqq m$, the poles of the closed loop system can be assigned such that $P_{kL} = P_{kL}^T > 0$ and $Q_{kL} = Q_{kL}^T \geqq$ iff*

(1) $(B_{kL}\, B_{kL}^T)^{-1}\, (A_{kL} - \bar{A}_{kL})$ *is positive definite, and*

(2) $-(B_{kL}\, B_{kL}^T)^{-1}\, (A_{kL} - \bar{A}_{kL})\, \bar{A}_{kL} - A_{kL}^T\, (B_{kL}\, B_{kL}^T)^{-1}\, (A_{kL} - \bar{A}_{kL})$ *is positive semidefinite.*

Proof. The closed loop system matrix is given by

$$\bar{A}_{kL} = A_{kL} - B_{kL}\, K_{kL}.$$

Substituting for K_{kL} from (6) we get,

$$\bar{A}_{kL} = A_{kL} - B_{kL}\, B_{kL}^T\, P_{kL},$$

and

$$P_{kL} = (B_{kL}\, B_{kL}^T)^{-1}\,(A_{kL} - \bar{A}_{kL}). \tag{7}$$

Hence, $P_{kL} > 0$ iff

$$(B_{kL}\, B_{kL}^T)^{-1}\,(A_{kL} - \bar{A}_{kL}) > 0.$$

Further, consider

$$-(B_{kL}\, B_{kL}^T)^{-1}\,(A_{kL} - \bar{A}_{kL})\,\bar{A}_{kL} - A_{kL}^T\,(B_{kL}\, B_{kL}^T)^{-1}\,(A_{kL} - \bar{A}_{kL}).$$

Substituting from (7), this becomes

$$\begin{aligned}(-P_{kL}\,\bar{A}_{kL} - A_{kL}^T\,P_{kL}) &= P_{kL}\,A_{kL} - P_{kL}\,\bar{A}_{kL} - A_{kL}^T\,P_{kL} - P_{kL}\,A_{kL} = \\ &= P_{kL}\,(A_{kL} - \bar{A}_{kL}) - A_{kL}^T\,P_{kL} - P_{kL}\,A_{kL} = \\ &= P_{kL}\,(B_{kL}\,B_{kL}^T)\,(B_{kL}\,B_{kL}^T)^{-1}\,(A_{kL} - \bar{A}_{kL}) - A_{kL}^T\,P_{kL} - P_{kL}\,A_{kL} = \\ &= P_{kL}\,B_{kL}\,B_{kL}^T\,P_{kL} - A_{kL}^T\,P_{kL} - P_{kL}\,A_{kL} = Q_{kL},\end{aligned}$$

which is the matrix Riccati equation in P_{kL}. (8)

Thus, $Q_{kL} \geqq 0$ iff

$$-(B_{kL}\, B_{kL}^T)^{-1}\,(A_{kL} - \bar{A}_{kL})\,\bar{A}_{kL} - A_{kL}^T\,(B_{kL}\, B_{kL}^T)^{-1}\,(A_{kL} - \bar{A}_{kL}) \geqq 0.$$

■

3. Recursive Optimal Pole Assignment

Consider the controllable system

$$\dot{X} = AX + BU. \tag{9}$$

With control law

$$U = -KX, \tag{10}$$

the closed loop system is

$$\dot{X} = \bar{A}X = (A - BK)\,X, \tag{11}$$

wherein the dimensions of state and input vector are $n \times 1$ and $m \times 1$ respectively. The other system matrices are of compatible dimensions.

A maximum of $p_k\,(\leqq m)$ poles can be optimally assigned using our theorem stated in Section 2. In practice, $n > m$ and a recursive process for OPA becomes necessary for complete pole assignment. Let $p_k\,(\leqq m)$ poles be assigned at kth recursion and let all the poles be assigned in say, q recursions. The real Schur form (RSF) has been used since it has a number of computational advantages such as speed and stability. At the kth recursion the system matrices in RSF are

$$A_k = \begin{bmatrix} A_{kH} & a_k \\ 0 & A_{kL} \end{bmatrix}; \quad B_k = \begin{bmatrix} B_{kH} \\ B_{kL} \end{bmatrix}. \tag{12}$$

The dimensions of A_{kL} and B_{kL} are respectively $p_k \times p_k$ and $p_k \times m$ and $p_k \leqq m$ (Juang and Lee, 1984).

Let the controller be given by

$$K_k = [0 \;\; K_{kL}), \tag{13}$$

where K_{kL} is $m \times p_k$, and the closed loop system matrix is given by

$$\bar{A}_k = A_k - B_k K_k = \begin{bmatrix} A_{kH} & a_k - B_{kH} K_{kL} \\ 0 & A_{kL} - B_{kL} K_{kL} \end{bmatrix}. \tag{14}$$

Thus the p_k ($\leqq m$) poles of A_{kL} can be assigned optimally using our theorem with the eigenvalues of A_{kH} undisturbed. At each recursion P_k, K_k and Q_k can be calculated. Finally from these values, overall P, K and Q can be calculated as (Solheim, 1972)

$$P = \sum_{k=1}^{q} \prod_{i=0}^{k} U_i P_k \prod_{i=k}^{0} U_i^T, \tag{15}$$

$$K = \sum_{k=1}^{q} K_k \prod_{i=k}^{0} U_i^T, \quad \text{and} \tag{16}$$

$$Q = \sum_{k=1}^{q} \prod_{i=0}^{k} U_i Q_k \prod_{i=k}^{0} U_i^T, \tag{17}$$

where U_k is the kth unitary similarity transformation matrix which transforms the system matrices of $(k-1)$th recursion to RSF.

4. Optimal pole regions for two pole assignment

The transient response of higher order systems is mostly decided by a dominant pole pair. Hence, by a two pole assignment, it is possible to improve the response. With this in view, the OPA for a second order subsystem has been studied in detail in this section. The optimal pole regions for various cases have been delineated.

Let the open loop system matrices be

$$A_{kL} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_2 & \alpha_2 \end{bmatrix} \quad \text{and} \quad B_{kL} = \begin{bmatrix} b_{k1} \\ b_{k2} \end{bmatrix},$$

where the dim $[b_{ki}] \geqq 2$, $i = 1, 2$ and B_{kL} is of full rank.

$$B_{kL}\, B_{kL}^T = \begin{bmatrix} b_{k1} \\ b_{k2} \end{bmatrix} [b_{k1}^T \;\; b_{k2}^T] = \begin{bmatrix} b_{k1}\, b_{k1}^T & b_{k1}\, b_{k2}^T \\ b_{k2}\, b_{k1}^T & b_{k2}\, b_{k2}^T \end{bmatrix}.$$

Substituting $u = b_{k1}\, b_{k1}^T$, $v = b_{k1}\, b_{k2}^T = b_{k2}\, b_{k1}^T$, $w = b_{k2}\, b_{k2}^T$ and inverting we get

$$(B_{kL}\, B_{kL}^T)^{-1} = \Delta^{-1} \begin{bmatrix} w & -v \\ -v & u \end{bmatrix}, \tag{18}$$

where $\Delta = (uw - v^2) > 0$

The inner products u and w are positive constants while v can be positive, zero or negative.

Let the closed loop system be

$$\bar{A}_{kL} = \begin{bmatrix} \bar{\alpha}_1 & \bar{\beta}_1 \\ \bar{\beta}_2 & \bar{\alpha}_2 \end{bmatrix},$$

From equation (7), P_{kL} is given by

$$P_{kL} = \Delta^{-1} \begin{bmatrix} w & -v \\ -v & u \end{bmatrix} \begin{bmatrix} \alpha_1 - \bar{\alpha}_1 & \beta_1 - \bar{\beta}_1 \\ \beta_2 - \bar{\beta}_2 & \alpha_2 - \bar{\alpha}_2 \end{bmatrix} =$$

$$= \Delta^{-1} \begin{bmatrix} w(\alpha_1 - \bar{\alpha}_1) - v(\beta_2 - \bar{\beta}_2) & w(\beta_1 - \bar{\beta}_1) - v(\alpha_2 - \bar{\alpha}_2) \\ -v(\alpha_1 - \bar{\alpha}_1) + u(\beta_2 - \bar{\beta}_2) & -v(\beta_1 - \bar{\beta}_1) + u(\alpha_2 - \bar{\alpha}_2) \end{bmatrix}.$$

For P_{kL} to be symmetric

$$-v(\alpha_1 - \bar{\alpha}_1) + u(\beta_2 - \bar{\beta}_2) = w(\beta_1 - \bar{\beta}_1) - v(\alpha_2 - \bar{\alpha}_2). \tag{19}$$

Rearranging the terms we get,

$$(w\bar{\beta}_1 - u\bar{\beta}_2) = v\{(\alpha_1 - \bar{\alpha}_1) - (\alpha_2 - \bar{\alpha}_2)\} + w\beta_1 - u\beta_2, \tag{20}$$

and

$$\beta_1 - \bar{\beta}_1 = -\frac{v}{w}(\alpha_1 - \bar{\alpha}_1) + \frac{u}{w}(\beta_2 - \bar{\beta}_2) + \frac{v}{w}(\alpha_2 - \bar{\alpha}_2). \tag{21}$$

Thus the elements of P_{kL} are given by

$$P_{11} = w(\alpha_1 - \bar{\alpha}_1) - v(\beta_2 - \bar{\beta}_2), \tag{22}$$

$$p_{12} = -v(\alpha_1 - \bar{\alpha}_1) + u(\beta_2 - \bar{\beta}_2), \tag{23}$$

and

$$p_{22} = -v(\beta_1 - \bar{\beta}_1) + u(\alpha_2 - \bar{\alpha}_2).$$

Substituting from (21) for $(\beta_1 - \bar{\beta}_1)$ and simplifying

$$p_{22} = -\frac{v}{w}\, p_{12} + \frac{\Delta}{w}(\alpha_2 - \bar{\alpha}_2). \tag{24}$$

For P_{kL} to be positive definite the leading principal minors should be positive.

Thus

$$p_{11} = \{w(\alpha_1 - \bar{\alpha}_1) - v(\beta_2 - \bar{\beta}_2)\} > 0,$$

which implies

$$\left(\bar{\alpha}_1 - \frac{v}{w}\bar{\beta}_2\right) < \left(\alpha_1 - \frac{v}{w}\beta_2\right). \tag{25}$$

Further

$$(p_{11}p_{22} - p_{12}^2) > 0.$$

Substituting from (22), (23), (24) and simplifying

$$\left(\bar{\alpha}_2 - \frac{p_{12}}{p_{11}}\bar{\beta}_2\right) < \left(\alpha_2 - \frac{p_{12}}{p_{22}}\beta_2\right). \tag{26}$$

Furthermore Q_{kL} is given by

$$Q_{kL} = -\Delta^{-1}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix} \bar{\alpha}_1 & \bar{\beta}_1 \\ \bar{\beta}_2 & \bar{\alpha}_2\end{bmatrix} - \begin{bmatrix} \alpha_1 & \beta_2 \\ \beta_1 & \alpha_2\end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22}\end{bmatrix}\Delta^{-1}.$$

Substituting from (21) and (24) for $\bar{\beta}_1$ and p_{22} respectively,

$$Q_{kL} = -\Delta^{-1}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & -\frac{v}{w}p_{12} + \frac{\Delta}{w}(\alpha_2 - \bar{\alpha}_2)\end{bmatrix} \times \begin{bmatrix} \bar{\alpha}_1 & \bar{\beta}_1 - \frac{p_{12}}{w} - \frac{v}{w}(\alpha_2 - \bar{\alpha}_2) \\ \bar{\alpha}_1 & \bar{\alpha}_2 \end{bmatrix}$$

$$-\Delta^{-1}\begin{bmatrix} \alpha_1 & \beta_2 \\ \beta_1 & \alpha_2\end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & -\frac{v}{w}p_{12} + \frac{\Delta}{w}(\alpha_2 - \bar{\alpha}_2)\end{bmatrix} =$$

$$= -\Delta^{-1}\begin{bmatrix} q_{11} & d - \frac{\Delta}{w}\bar{\alpha}_2\bar{\beta}_2 \\ d - \frac{\Delta}{w}\bar{\alpha}_2\bar{\beta}_2 & e + \frac{\Delta}{w}(\alpha_2^2 - \bar{\alpha}_2^2)\end{bmatrix} \tag{27}$$

where

$$q_{11} = p_{11}(\alpha_1 + \bar{\alpha}_1) + p_{12}(\beta_2 + \bar{\beta}_2), \tag{28}$$

$$d = p_{11}\beta_1 + p_{12}(\bar{\alpha}_1 + \alpha_2) - \frac{v}{w}p_{12}\bar{\beta}_2 + \frac{\Delta}{w}\alpha_2\bar{\beta}_2, \tag{29}$$

$$e = \frac{p_{12}}{w}(2w\beta_1 - p_{12} - 2v\alpha_2). \tag{30}$$

For Q_{kL} to be positive semidefinite, by considering the leading principal minors, we can show that

$$\left(\bar{\alpha}_1 - \frac{v}{w}\bar{\beta}_2\right)^2 + \frac{\Delta}{w^2}\bar{\beta}_2^2 \geqq \left(\alpha_1 - \frac{v}{w}\beta_2\right)^2 + \frac{\Delta}{w^2}\beta_2^2, \tag{31}$$

and

$$\left(\bar{\alpha}_2 - \frac{d\bar{\beta}_2}{q_{11} + \frac{\Delta}{w}\bar{\beta}_2^2}\right)^2 \geqq \frac{q_{11}}{q_{11} + \frac{\Delta}{w}\bar{\beta}_2^2} \times \left[\alpha_2^2 + \frac{w\Delta^{-1}(q_{11}e - d^2) + e\bar{\beta}_2^2}{q_{11} + \frac{\Delta}{w}\bar{\beta}_2^2}\right]. \tag{32}$$

4.1. To shift a complex conjugate pair of poles ($\alpha \pm j\beta$) to a pair of real poles

The system matrices are given by

$$A_{kL} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_2 & \alpha_2 \end{bmatrix},$$

where

$$\alpha = \frac{\alpha_1 + \alpha_2}{2} \quad \text{and}$$

$$\beta^2 = -\left\{\left(\frac{\alpha_1 - \alpha_2}{2}\right)^2 + \beta_1\beta_2\right\} > 0$$

$$\bar{A}_{kL} = \begin{bmatrix} \bar{\alpha}_1 & \bar{\beta}_1 \\ 0 & \bar{\alpha}_2 \end{bmatrix}.$$

Substituting $\bar{\beta}_2 = 0$ in equations (25) and (31)

$$\bar{\alpha}_1 < \left(\alpha_1 - \frac{v}{w}\beta_2\right), \tag{33}$$

$$\bar{\alpha}_1^2 \geqq \left(\alpha_1 - \frac{v}{w}\beta_2\right)^2 + \frac{\Delta}{w^2}\beta_2^2. \tag{34}$$

From these two equations, the optimal pole region for $\bar{\alpha}_1$ is

$$\bar{\alpha}_1 \leqq -\left|\left[\left(\alpha_1 - \frac{v}{w}\beta_2\right)^2 + \frac{\Delta}{w^2}\beta_2^2\right]^{\frac{1}{2}}\right|. \tag{35}$$

Further, from equations (26) and (32)

$$\bar{\alpha}_2 < \left(\alpha_2 - \frac{p_{12}}{p_{11}}\beta_2\right), \tag{36}$$

and

$$\bar{\alpha}_2^2 \geqq \left\{\alpha_2^2 + w\Delta^{-1}\left(e - \frac{d^2}{q_{11}}\right)\right\}. \tag{37}$$

Hence, optimal region for $\bar{\alpha}_2$ is

$$\bar{\alpha}_2 \leqq -\left|\left[\alpha_2^2 + w\Delta^{-1}\left(e - \frac{d^2}{q_{11}}\right)\right]^{\frac{1}{2}}\right|. \tag{38}$$

Thus for a given open loop system the optimal pole region (OPR) can be determined from equations (35) and (38) as illustrated in Figure 1.

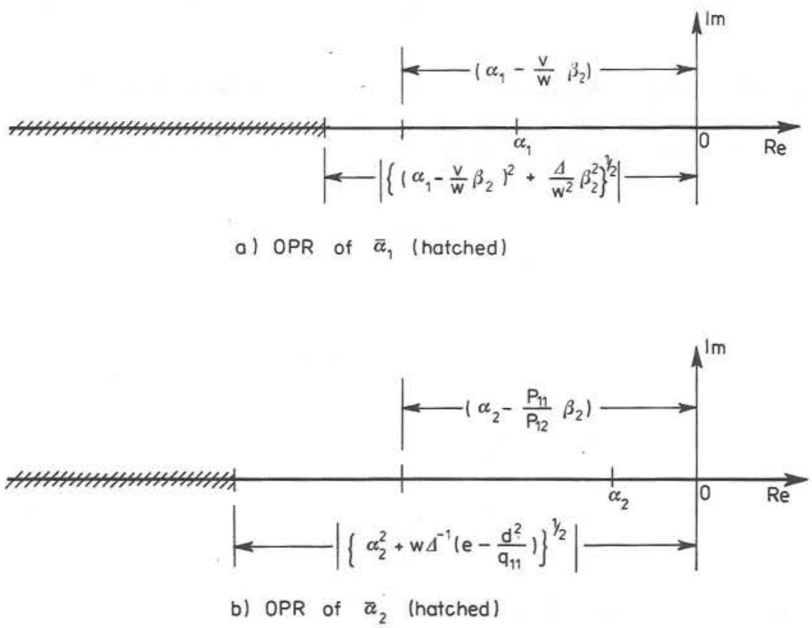

Fig. 1. OPR — complex to real pole shift

Continuing further $\bar{\beta}_1$ is obtained from equation (21) as

$$\bar{\beta}_1 = \beta_1 + \frac{v}{w}\{(\alpha_1 - \bar{\alpha}_1) - (\alpha_2 - \bar{\alpha}_2)\} - \frac{u}{w}\beta_2. \tag{39}$$

4.2. To shift real poles (α_1, α_2) to $(\bar{\alpha}_1, \bar{\alpha}_2)$

Substituting $\beta_2 = 0$ in constraints (33), (34), (36) and (37) we get

$$\bar{\alpha}_1 < \alpha_1, \tag{40}$$

$$\bar{\alpha}_1^2 \geqq \alpha_1^2, \tag{41}$$

$$\bar{\alpha}_2 < \alpha_2, \tag{42}$$

$$\bar{\alpha}_2^2 \geq \alpha_2^2 + \Delta^{-1}\,\frac{(\alpha_1-\bar{\alpha}_1)^2}{\bar{\alpha}_1^2-\alpha_1^2}(w\beta_1+v\alpha_1-v\alpha_2)^2.$$

Figure 2 illustrates the OPR delineated using these equations. The $\bar{\beta}_1$ is given by

$$\bar{\beta}_1 = \beta_1 + \frac{v}{w}\,\{(\alpha_1-\bar{\alpha}_1)-(\alpha_2-\bar{\alpha}_2)\}. \tag{44}$$

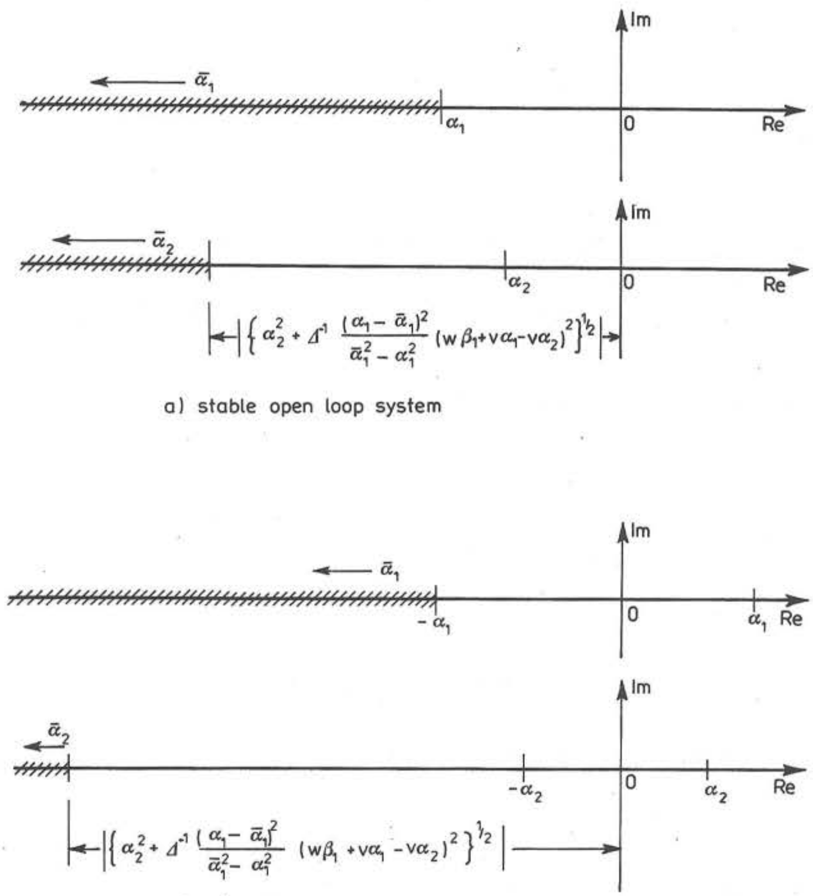

Fig. 2. OPR real to real pole shift

4.3. To shift a complex conjugate pair of poles $(\alpha \pm j\beta)$ to a complex conjugate pair $(\bar{\alpha} \pm j\bar{\beta})$

The open loop system matrix is

$$A_{kL} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_2 & \alpha_2 \end{bmatrix},$$

where

$$\alpha = \frac{\alpha_1 + \alpha_2}{2} \quad \text{and} \quad \beta^2 = -\left\{ \left(\frac{\alpha_1 - \alpha_2}{2} \right)^2 + \beta_1 \, \beta_2 \right\} > 0.$$

Let the closed loop matrix be

$$\bar{A}_{kL} = \begin{bmatrix} \bar{\alpha} & \bar{\beta}_1 \\ \bar{\beta}_2 & \bar{\alpha} \end{bmatrix},$$

where

$$\bar{\alpha} = \bar{\alpha}_1 = \bar{\alpha}_2 \quad \text{and} \quad \bar{\beta}^2 = -\bar{\beta}_1 \, \bar{\beta}_2. \tag{45}$$

We have from equation (20)

$$w\bar{\beta}_1 - u\bar{\beta}_2 = v\,(\alpha_1 - \alpha_2) + w\beta_1 - u\beta_2.$$

Substituting for $\bar{\beta}_1$ from (45) and rearranging the terms we get

$$\bar{\beta}_2^2 + \frac{1}{u}\,\{v\,(\alpha_1 - \alpha_2) + w\beta_1 - u\beta_2\}\,\bar{\beta}_2 + \frac{w}{u}\,\bar{\beta}^2 = 0.$$

The roots of this quadratic equation are

$$\bar{\beta}_2 = b \pm c, \tag{46}$$

where

$$b = -\frac{1}{2u}\,\{v\,(\alpha_1 - \alpha_2) + w\beta_1 - u\beta_2\},$$

and

$$c = \left(b^2 - \frac{w}{u}\,\bar{\beta}^2 \right)^{\frac{1}{2}}. \tag{48}$$

Since c^2 should be a positive constant

$$\left(b^2 - \frac{w}{u}\,\bar{\beta}^2 \right) > 0,$$

which implies

$$\bar{\beta}_{\max} = \left| \left(\frac{u}{w} \right)^{\frac{1}{2}} b \right|. \tag{49}$$

This places an upper limit on imaginary part of the pole being assigned and $\bar{\beta}_2$ is obtained from (46) by choosing the value of $\bar{\beta}$ within this limit. From constraints (25) and (31)

$$\bar{\alpha} < \left\{ \alpha_1 - \frac{v}{w} (\beta_2 - \bar{\beta}_2) \right\}, \tag{50}$$

$$\left\{ \left(\bar{\alpha} - \frac{u}{w} \bar{\beta}_2 \right)^2 + \frac{\Delta}{w^2} \bar{\beta}_2^2 \right\} \geqq \left\{ \left(\alpha_1 - \frac{v}{w} \beta_2 \right)^2 + \frac{\Delta}{w^2} \beta_2^2, \tag{51}$$

$\bar{\alpha}$ is chosen to satisfy the above constraints along with the constraints (26) and (32). OPR of the closed loop complex poles $\bar{\alpha} \pm j\bar{\beta}$ is graphically illustrated in Figure 3.

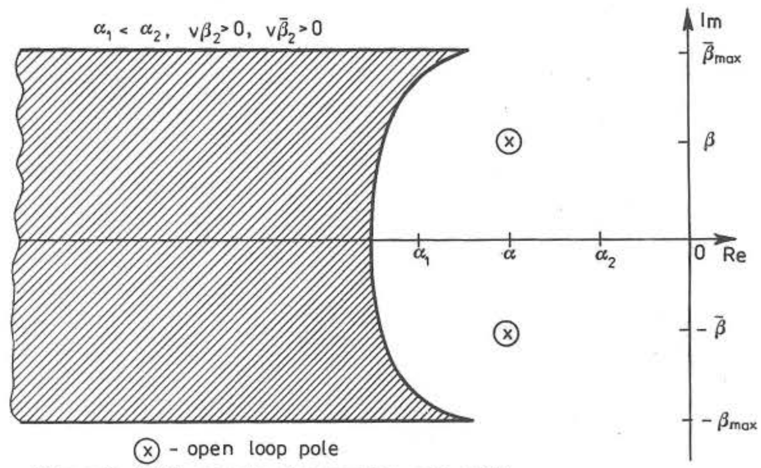

Figure 3 OPR – complex to complex pole shift.

4.4. To shift real poles (α_1, α_2) to a pair of complex conjugate poles $(\bar{\alpha} \pm j\bar{\beta})$

The constraint equations for this case are obtained by putting $\beta_2 = 0$ in the equations of section 4.3 as follows:

The system matrices are

$$A_{kL} = \begin{bmatrix} \alpha_1 & \beta_1 \\ 0 & \alpha_2 \end{bmatrix} \quad \text{and} \quad \bar{A}_{kL} = \begin{bmatrix} \bar{\alpha} & \bar{\beta}_1 \\ \bar{\beta}_2 & \bar{\alpha} \end{bmatrix},$$

$$\bar{\beta}^2 = -\bar{\beta}_1 \bar{\beta}_2, \tag{45}$$

$$\bar{\beta}_2 = b \pm c, \tag{46}$$

where

$$b = -\frac{1}{2u} \{ v (\alpha_1 - \alpha_2) + w\beta_1 \}, \tag{52}$$

and

$$c = \left(b^2 - \frac{w}{u} \bar{\beta}^2 \right)^{\frac{1}{2}}. \tag{48}$$

The maximum value $\bar{\beta}$ can have is

$$\bar{\beta}_{\max} = \left(\frac{u}{w}\right)^{\frac{1}{2}} b. \tag{49}$$

From constraints (50) and (51)

$$\left(\bar{\alpha} - \frac{v}{w}\bar{\beta}_2\right) < \alpha_1, \tag{53}$$

$$\left(\bar{\alpha} - \frac{v}{w}\bar{\beta}_2\right)^2 + \frac{\Delta}{w^2}\bar{\beta}_2^2 \geqq \alpha_1^2.$$

The OPR for $\bar{\alpha}$ has been delineated in Figure 4 using constraints (53) and (54).

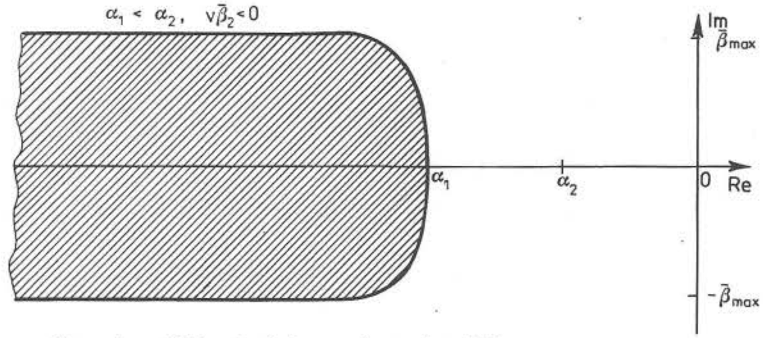

Figure 4 OPR – real to complex pole shift.

Example 1

Consider

$$A_{kL} = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix}; \quad B_{kL} = \begin{bmatrix} -0.707 & 0 \\ 0.707 & 1.414 \end{bmatrix}.$$

From (49) $\bar{\beta}_{\max} = 0.89$
Let $\bar{\beta} = 0.5$
With this value of $\bar{\beta}$, $\bar{\beta}_2 = 0.342$
From (50) and (51)

$$\bar{\alpha} \leqq -2.068.$$

The value $\bar{\alpha} = -3$ satisfies the necessary constraints and yields the following:

$$\bar{A}_{kL} = \begin{bmatrix} -3 & -0.731 \\ 0.342 & -3 \end{bmatrix},$$

$$P_{kL} = \begin{bmatrix} 2.329 & 0.329 \\ 0.329 & 0.866 \end{bmatrix}, \quad Q_{kL} = \begin{bmatrix} 11.538 & 3.35 \\ 3.35 & 4.03 \end{bmatrix},$$

and

$$K_{kL} = \begin{bmatrix} -1.414 & 0.379 \\ 0.465 & 1.224 \end{bmatrix}.$$

4.5. Remarks

1. It can be seen that from constraints (25), (26), (31) and (32) the optimal pole regions depend upon the elements of A_{kL} and B_{kL}. Further the choice of $\bar{\alpha}_1$ affects optimal region of α_2.
2. In case the diagonal elements of A_{kL} are positive, the diagonal elements of $\bar{A}_{kL}$ lie to left of the imaginary axis to ensure a positive semi-definite Q.
3. For single pole assignments the OPR lies to the left of open loop pole if stable, or to the left of mirror image of open loop pole if unstable.
4. If real poles are assigned one at a time instead of assigning them simultaneously, there is some gain in OPR at the cost of extra computation.
5. The theorem stated in section 2 is general and does not require A_{kL} to be in RSF. The optimal pole regions for such cases will be different.

5. Optimal Pole Assignment algorithm

A recursive pole assignment procedure based on section 4 is presented in this section.

(1) Transform A and B to RSF

$$A_0 = U_0^T A U_0 \quad \text{and} \quad B_0 = U_0^T B.$$

(2) Choose q the number of recursions necessary to carry out pole assignment and the order in which the poles are to be assigned. Set $k = 0$ and $\bar{A}_0 = A_0$.

(3) Set $k = k+1$.

(4) Obtain $A_k = U_k^T \bar{A}_{k-1} U_k$ and $B_k = U_k^T B_{k-1}$. If the desired closed loop poles are real go to step (5) otherwise go to step (8).

(5) Determine the optimal region for $\bar{\alpha}_1$ and hence choose $\bar{\alpha}_1$.

(6) Determine OPR for $\bar{\alpha}_2$ and choose $\bar{\alpha}_2$.

(7) Calculate $\bar{\beta}_1$. Go to step (11).

(8) Determine $\bar{\beta}_{\max}$, hence choose $\bar{\beta}$ and calculate $\bar{\beta}_2$.

(9) Choose $\bar{\alpha}$ to satisfy the necessary constraints.

(10) Calculate $\bar{\beta}_1$.

(11) Calculate P_{kL}, K_{kL} and Q_{kL}.

(12) Obtain $\bar{A}_k = A_k - B_k K_k$.

(13) If $k=q$ go to step (14) otherwise go to step (3).
(14) Calculate P, K and Q.

EXAMPLE 2

Consider

$$\dot{X}=\begin{bmatrix}-4 & 1 & 2\\ 0 & -2 & 0\\ 0 & 1 & -1\end{bmatrix}X+\begin{bmatrix}2 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}u.$$

By RSF transformation we get

$$A_0=\begin{bmatrix}-4 & -0.707 & 2.12\\ 0 & -2 & -1\\ 0 & 0 & -1\end{bmatrix},\quad B_0=\begin{bmatrix}2 & 0\\ -0.707 & 0\\ 0.707 & 1.414\end{bmatrix},$$

where

$$U_0=\begin{bmatrix}1 & 0 & 0\\ 0 & 0.707 & 0.707\\ 0 & -0.707 & 0.707\end{bmatrix}.$$

The eigenvalues of open loop system are -4, -2 and -1. The poles are assigned in two recursions ($q=2$). In first recursion $(-2, -1)$ are shifted to a complex conjugate pair. In second recursion (-4) is shifted.
For $k=1$

Since $U_1=I_3$ the identity matrix

$$A_{1L}=\begin{bmatrix}-2 & -1\\ 0 & -1\end{bmatrix} \quad \text{and} \quad B_{1L}=\begin{bmatrix}-0.707 & 0\\ 0.707 & 1.414\end{bmatrix}.$$

This has been solved in example 1. The closed loop system is

$$\bar{A}_{1L}=\begin{bmatrix}-3 & -0.732\\ 0.341 & -3\end{bmatrix}.$$

The chosen closed loop optimal poles are $(-3\pm j0.5)$.

$$P_{1L}=\begin{bmatrix}2.329 & 0.329\\ 0.329 & 0.866\end{bmatrix},\quad Q_{1L}=\begin{bmatrix}11.533 & 3.35\\ 3.35 & 4.03\end{bmatrix},$$

and

$$K_{1L}=\begin{bmatrix}-1.414 & 0.379\\ 0.465 & 1.224\end{bmatrix}.$$

Therefore

$$\bar{A}_1 = \begin{bmatrix} -4 & 2.121 & 1.362 \\ 0 & -3 & -0.732 \\ 0 & 0.342 & -3 \end{bmatrix}.$$

For $k = 2$, by RSF transformation, A_2 and B_2 are given by

$$A_2 = \begin{bmatrix} -3.018 & 0.196 & 0.982 \\ -1.277 & -2.978 & 2.089 \\ 0 & 0 & -4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.296 & 0.762 \\ 1.765 & 0.3 \\ -0.45 & 1.152 \end{bmatrix},$$

where

$$U_2 = \begin{bmatrix} 0.166 & 0.922 & -0.350 \\ -0.825 & 0.324 & 0.462 \\ 0.539 & 0.212 & 0.815 \end{bmatrix}.$$

From A_2 and B_2 matrices we have,

$$A_{2L} = [-4], \quad B_{2L} = [-0.45 \;\; 1.152].$$

Since $\bar{\alpha}_3 < \alpha_3$
Chossing $\bar{A}_{2L} = [\bar{\alpha}_3] = [-5.53]$ we get

$$P_{2L} = [1], \quad Q_{2L} = [9.53] \quad \text{and} \quad K_{2L}\begin{bmatrix} -0.45 \\ 1.152 \end{bmatrix}.$$

Referred to original system coordinates

$$P = U_0\, U_1 \left[\begin{array}{c:cc} 0 & 0 & 0 \\ \hdashline 0 & & \\ 0 & & P_{1L} \end{array}\right] U_1^T\, U_0^T +$$

$$+U_0\, U_1\, U_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P_{2L} \end{bmatrix} U_2^T\, U_1^T\, U_0^T = \begin{bmatrix} 0.123 & -0.316 & -0.087 \\ -0.316 & 2.741 & -0.505 \\ -0.087 & -0.505 & 1.33 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.167 & -3.012 & -0.833 \\ -3.012 & 18.898 & -1.596 \\ -0.833 & -1.596 & 5.025 \end{bmatrix},$$

and

$$K = \begin{bmatrix} 0.159 & -1.137 & 1.156 \\ -0.403 & 2.236 & 0.825 \end{bmatrix}.$$

With this controller K, the closed system becomes

$$\bar{A} = \begin{bmatrix} -4.318 & 3.274 & -0.312 \\ 0.403 & -4.236 & -0.825 \\ 0.244 & -0.099 & -2.981 \end{bmatrix},$$

The evigenvalues of $\bar{A}$ are $(-3 \pm j0.5)$ and (-5.53) as assigned.

6. Conclusion

A new theorem on optimal pole assignment has been established for linear time-invariant systems. Using the theorem optimal pole regions for various two pole assignment problems have been delineated. By a recursive procedure, based on the theorem, all the poles are optimally assigned by considering one or two poles at each step. The various results have been illustrated with numerical examples. The theorem established is however applicable for p_k ($\leqq m$) pole assignment at each recursion. The delineation of OPR for such cases ($p > 2$) and other allied problems of OPA are being currently studied.

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Nowe wyniki w dziedzinie optymalnego przesuwania biegunów

Optymalne przesuwanie biegunów polega na wyborze takiego układu sterującego w zamkniętej pętli, który sprowadza układ zamknięty do zadanych biegunów a jednocześnie pozwala minimalizować pewien kwadratowy wskaźnik jakości sterowania. Poniższy artykuł zawiera nowe twierdzenie, które podaje warunki konieczne i wystarczające optymalności takiego układu zamkniętego. Na podstawie tego twierdzenia określono rejony położenia optymalnych biegunów w przypadku jednego i dwóch biegunów. Uzyskana w ten sposób dowolność jest wykorzystana do wyboru pożądanego zestawu biegunów z odpowiednich rejonów optymalnych. Podano rekurencyjną procedurę optymalnego przesuwania biegunów oraz ilustrujący jej działanie przykład liczbowy.

Новые результаты в области оптимального сдвига полюсов

Оптимальное перемещение полюсов состоит в выборе такой системы, управляющей в замкнутой цепи, которая сводит замкнутую систему к заданным полюсам, а одновременно позволяет минимизировать некоторый квадратный показатель качества управления. Данная статья содержит новую теорему, которая дает необходимые и достаточные условия оптимальности такой замкнутой системы. На основе этой теоремы определены области расположения оптимальных полюсов в случае одного и двух полюсов. Достигнутая таким образом произвольность используется для выбора желаемого состава полюсов из соответствующих оптимальных областей. Дается рекуррентная процедура оптимального сдвига полюсов и иллюстрирующий её действие численный пример.