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Stochastic price-quantity optimal control problem with constraints

by

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This paper considers a finite-horizon quantity and price control problem, where the dynamics of the price is described by bilinear stochastic process, controlled by production rate. The production rate and the price are restricted to be nonnegative. The closed-form solution for optimal feedback policy is given.

1. Introduction

Consider a free market system with many producers and buyers of a homogeneous nonstorable good. Assume that the price of the good is stochastically perturbed by supply and demand fluctuations. If the rate of the price change $\frac{p_{t+\Delta t}-p_t}{p_t}$ is a result of a big number of stochastically independent effects such that is highly probable that each effect is sufficiently small, then by applying the central limit theorem we obtain the asymptotically Gaussian distribution of the price change rate. If we assume further, that this rate is an autonomous stochastic process, then its Gaussian distribution has a mean $\alpha\Delta t$ and a variance $\sigma^2\Delta t$, and in the limit we can write

$$\frac{dp_t}{p_t}=\alpha\, dt+\sigma\, dw_t, \tag{1.1}$$

where α, σ are some constants and w_t, $t \geq 0$, is a Wiener process, (see [1], where the process given by (1.1) was used for price fluctuation model).

Suppose now that hitherto existing free market system changed radically owing to appearance of a group of a few producers, which have a common production-price policy, and possess jointly such a degree of monopolistic power, that they can influence the price of the good, by varying their own supply on the market. Subsequently, this group will be called "the price leader".

We suppose also that any quantity of the good could be sold if the price would be low enough. As examples of such a situation we can mention oil and/or generaly energy markets.

Taking the above-mentioned assumptions and (1.1) into consideration we assume that the evolution of the price is described by the following SDE

$$dp_t = p_t(\alpha\, dt + \sigma\, dw_t) - \beta u_t\, dt, \qquad p_0 = p > 0, \tag{1.2}$$

where $\beta > 0$ and α, σ, w_t as in (1.1) and u_t denotes production rate of the price leader. The first term on the RHS of (1.2) represents an influence on the price p_t exerted jointly by all non-monopolistic producers and the second term represents an influence of the price leader.

The price-quantity differential equation (1.2) can be formulated in many different ways but we chose one of the simplests.

We shall list now some properties of the processes (1.1), (1.2) which turn out to be important in the sequel.

(a) These processes are easy to work with, as we can effectively compute their transition density, their moments and even obtain their solution as functions of a Wiener process.

(b) It is possible to obtain explicit formulae for the maximum likelihood estimators $\alpha_t(\hat{p}), \sigma_t(\hat{p}), \beta_t(\hat{p})$ of the coefficients α, σ, β as functionals of the trajectory $\{p_s; 0 \leq s \leq t\}$, (see [4]). Thus by observing continuously the real market price we can estimate the values of α, σ, β making our model more realistic.

(c) The process given by (1.1) satisfies the implication $p_t > 0$ for $t \geq 0$ if $p_0 > 0$. There are of course several kinds of processes which satisfy the conditions given in (a) and (c), for instance the Bessel process being a solution of the SDE

$$dp_t = \frac{\alpha}{p_t} dt + \sigma\, dw_t, \qquad p_0 > 0, \qquad \sigma > 0, \qquad \alpha > 0. \tag{1.3}$$

This process has a reflecting barrier in $p = 0$ so we have also $p_t \geq 0$ if $p_0 \geq 0$, but obtaining the maximum likelihood estimators for α and σ

in this case is a much more difficult task than for (1.2). The reason is that the Bessel process is not absolutely continuous with respect to the Wiener process.

Looking at the SDE (1.2) we can state that the implication (c) is not true, so nonnegativity of the price p cannot be guaranteed.

To overcome this inconvenience we impose explicitly the restriction, that the price must be nonnegative i.e. $p_t \geqslant 0$ for $t \geqslant 0$.

2. Statement of the optimal regulator problem

The problem of optimal regulator is formulated from standpoint of the price leader, who seeks such a production-price policy which would maximize his expected profits over time,

$$\sup_{0 \leqslant u_t} E \int_0^T [p_t u_t - g(u_t)] \, dt, \tag{2.1}$$

where p_t is a solution of the controlled SDE

$$dp_t = p_t (\alpha \, dt + \sigma \, dw_t) - \beta u_t \, dt, \tag{2.2}$$

with initial condition

$$p_0 = p > 0, \tag{2.3}$$

satisfying the restriction

$$p_t \geqslant 0 \quad \text{for} \quad t \in [0, T]. \tag{2.4}$$

We shall seek the solution of our problem in the set U of the admisible feedback controls $u_t = u(t, p_t)$ defined in the following way

(I) $u(\cdot, \cdot)$ is a Borel measurable function on $[0, T] \times \mathbf{R}$

(II) $|u(t, p_1) - u(t, p_2)| \leqslant M \, |p_1 - p_2|$ on $[0, T] \times \mathbf{R}$

(III) $|u(t, p)| \leqslant M(1 + |p|)$ on $[0, T] \times \mathbf{R}$

(IV) $u_t = u(t, p_t) \geqslant 0$ for $t \in [0, T]$.

This problem goes back to some earlier works, for example [2], [3] where some static versions of this problem were analyzed.

3. Solution of the problem

We shall give the solution of the problem (2.1)–(2.4) only for a quadratic production cost function.

THEOREM 3.1. *Let $g(u) = au^2$, $a > 0$. If the parameters α, β, σ, a, satisfy the inequality*

$$\left|1-\frac{a}{\beta}(\sigma^2+2\alpha)\right| \geqslant 2, \tag{3.1}$$

then there exists, in the set U of admissible controls, the optimal control

$$u^*(t, p) = (2a)^{-1}[1-2\beta A(t)]\, p, \tag{3.2}$$

where

$$A(t) = \frac{k}{\beta}\sqrt{k^2\beta^2-1}\ \mathrm{th}\left[\frac{\beta(t-T)}{a}\sqrt{k^2\beta^2-1}\right], \tag{3.3}$$

$$k = \frac{1}{2\beta} - \frac{a}{2\beta^2}(\sigma^2+2\alpha). \tag{3.4}$$

Proof. First we shall prove that $u^* \in U$. To do this it will be sufficient to show that the condition (IV) is satisfied. Note that (3.1) implies that $A(\cdot)$ is a real function satisfying $1-2\beta A(t) \geqslant 0$ for $t \in [0, T]$. On the other hand, after substituting u^* into (2.2) we get the bilinear equation so the condition $p_t > 0$, by property (c), is satisfied and consequently we have $u^* \geqslant 0$.

Now we shall show optimality of u^*. Let us ignore for a moment the conditions $u \geqslant 0$, $p \geqslant 0$. The solution of the problem (2.1)–(2.4) will be carried out via the development of the Bellman equation (see [5]). Let $V = V(t, p)$ denote the Bellman function for our control problem. Then it can be shown that $V(t, p)$ satisfies on $[0, T] \times \mathbf{R}$ the Bellman equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 p^2}{2}\frac{\partial^2 V}{\partial p^2} + \alpha p \frac{\partial V}{\partial p} + \sup_u \left\{pu - au^2 - \beta u \frac{\partial V}{\partial p}\right\} = 0, \tag{3.5}$$

with the boundary data

$$V(T, p) = 0. \tag{3.6}$$

From (3.5) we get $\bar{u} = (2a)^{-1}\left(p - \beta\frac{\partial V}{\partial p}\right)$ as a presumable optimal control. After substituting u into (3.5) we have

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 p^2}{2}\frac{\partial^2 V}{\partial p^2} + \alpha p \frac{\partial V}{\partial p} + (4a)^{-1}\left(p - \beta\frac{\partial V}{\partial p}\right)^2 = 0. \tag{3.7}$$

Simple calculations show that the function $A(t)\, p^2$ is the solution of (3.7) (3.6), so we have $\bar{u} = (2a)^{-1}[1-2\beta A(t)]\, p = u^*$. As $u^* \in U$, then u^* is the solution of our optimal regular problem.

REMARKS

1. In the present paper the closed-form solution was given for the problem of controlling a bilinear SDE, with the nonnegativity constrains put on the state and control variables.

   We use methods which follow the spirit of [1] although these do not apply directly since we assumed state-dependent optimization criterion.
2. If the parameters a, α, σ, β did not satisfy (3.1) then obtaining the closed-form solution could be much more difficult task in the case analyzed in [6] where the closed-form solution had not been given explicitly.
3. It seems that the results presented in this paper could serve as a point of departure for attacking a problem of optimal price-quantity policies in the situation of competition among few, namely the oligopolistic competition.

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Stochastyczne zagadnienie sterowania cenami i ilościami produktów z ograniczeniami

W artykule rozpatrzono zagadnienie sterowania cenami i ilościami produktów ze skończonym horyzontem czasowym, w którym dynamika cen opisana jest biliniowym procesem stochastycznym sterowanym intensywnością produkcji. Intensywności produkcji i ceny są nieujemne. Podano rozwiązanie w postaci reguł optymalnego sterowania w pętli sprzężenia zwrotnego.

Стохастическая задача управления ценами и количеством продуктов с ограничениями

В статье рассматривается задача управления ценами и количеством продуктов с конечным временным горизонтом, в которой динамика цен описывается билинейным стохастическим процессом, управляемым интенсивностью производства. Интенсивности производства и цены являются неотрицательными. Дается решение в виде правил оптимального управления в цепи обратной связи.