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Optimality Conditions in Cooperative Differential Games

by

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Necessary conditions of the Pontryagin maximum principle type are proved for cooperative many players differential games with state constraints by reducing the games to vector optimization problems. This principle is also proved to be sufficient for optimality under additional assumptions.

1. Introduction

In the game theory in general, and in differential games in particular, there are many mode of playing. For the games in which each Player j has his cost $Jj(u_1,...,u_m)$ (dependent also on the controls of the other players), there are four mode of play. In the first one, each player assumes that all the other players are collectively playing against him and he must seek a **minimax solution** (see, e.g., [7]). In the second mode of play, the **Nash equilibrium solutions** are accepted (see, e.g., [6], [9]). This type of solution is secure against any attempt by one player to unilaterally alter his control. The third mode is the **cooperative game**, where all m players agree to cooperate exclusively. For this case a commonly accepted solution concept is the Pareto optimality [3], [6]. The final mode consists of the situation when only s players, $1 \leqslant s \leqslant m$, form a coalition and they assume that the **coalitive Pareto optimality** is the solution concept [1], [8].

In the present paper we consider cooperative many players differential games. The main difference between our consideration and the known results on cooperative differential games is the appearance of state constraints. In Section 2 we derive necessary conditions in the Pontryagin maximum principle form by using our preceding result on vector optimization [4]. Section 3 addresses sufficient conditions. Here, using a scalarization result, we prove that Pontryagin maximum principle provides also sufficient conditions under additional assumptions.

2. Necessary conditions

The cooperative differential game we consider is

$$\dot{x}(t) = \varphi(t, x(t), u_1(t), \dots, u_m(t)), \tag{1}$$

$$u_j(t) \in U_j \subset \mathbf{R}^{r_j}, \ j = 1, \dots, m, \tag{2}$$

$$h_o(x(t_o)) = 0, \ h_1(x(t_1)) = 0, \tag{3}$$

$$g_i(t, x(t)), \leqslant 0, \ i = 1, \dots, k, \tag{4}$$

$$\xi_j(t_1)) + \int_{t_o}^{t_1} f_j(t, x(t), u_1(t), \dots, u_m(t))\, dt \to \inf, \ j = 1, \dots, m, \tag{5}$$

where t_0 and t_1 are fixed; $\varphi : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{r}{}_1 \dots \times \mathbf{R}^{r_m} + \mathbf{R}^n$; $h_0 : \mathbf{R}^n \to \mathbf{R}^{s_0}$; $h_1 : \mathbf{R}^n \to \mathbf{R}^{s_1}$; $g_i : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$; $\xi_j : \mathbf{R}^n \to \mathbf{R}^{q_j}$, $f_j : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{r_1} \times \dots \times \mathbf{R}^{r_m} \to \mathbf{R}^{q_j}$, $\mathbf{R}^{q_j}$ being ordered by a closed convex cone K_j, $j = 1,\dots m$; (1) and (2) are satisfied almost everywhere (a.e.) on $[t_0,t_1]$, admissible controls are $u_j(.) \in L_\infty^{r_j}[t_0,t_1]$ which satisfy (2) a.e. on $[t_0,t_1]$. The set of all admissible controls $u_j(.)$ is denoted by U_j.

To describe the game (1) - (5) as a vector optimization problem we adopt the following notations:

$$G_i(x(.)) = \sup_{t \in [t_0,t_1]} g_i(t, x(t)),$$

$Y = \mathbf{R}^{q_1} \times \dots \times \mathbf{R}^{q_m}$, $q_1 + \dots + q_m = q$, $r_1 + \dots + r_m = r$, $f = (f_1,\dots,f_m)$, $u = (u_1,\dots,u_m)$,... (the same for ξ, g, G, U, U, K). Next we introduce a mapping $P : C^n[t_0, t_1] \times L_\infty^r[t_0, t_1] \to C^n[t_0, t_1] \times \mathbf{R}^{s_0} \times \mathbf{R}^{s_1}$ by

$$P(x(.), u(.)) = (y(.), b_0, b_1)$$

with

$$y(t) = x(t) - x(t_0) - \int_{t_0}^{t} \varphi(r, x(r), u(r))\, dr,$$

$$b_0 = h_0(x(t_0)), \ b_1 = h_1(x(t_1)),$$

and a mapping $F : C^n[t_0, t_1] \times L_\infty^r[t_0, t_1] \to \mathbf{R}^q$ by

$$F(x(.),u(.)) = \xi(x(t_1)) + \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt.$$

Then the game is reduced to the vector optimization problem

$$F(x(.), u(.)) \to \inf, \tag{6}$$

$$G(x(.)) \leqslant 0, \tag{7}$$

$$P(x(.), u(.)) = 0, \tag{8}$$

$$u(.) \in U. \tag{9}$$

Recall that a point $x^o \in S \subset X$, X being a topological space, is called a **local Pareto minimum** of a mapping $f\colon X \to Y$ on S, Y being a normed space ordered by a convex cone C, if there is a neighborhood N of x such that there is no $x \in S \cap N$ with $f(x^o) - f(x) \in C \setminus (-C)$. If there is no $x \in S \cap N$ with $f(x^o) - f(x) \in ri\, C$ (the relative interior), then x^o is called **local Slater minimum** of f on S.

Now we call controls $u_1^o(.), \dots, u_m^o(.)$, with the resulting state $x^o(.)$, **local-global weakly optimal** for the differential game (1) – (5) if $(x^o(.), u^o(.))$ is a local Slater minimum of the problem (6) – (9), considering $x(.)$ elements of $C^n\,[t_o, t_1]$ and $u(.)$ elements of U equipped with the trivial topology (containing only Φ and U).

If int $K \neq \Phi$ then $\boldsymbol{R}^q$ (ordered by K as above-defined) is a vector lattice. For $y \in \boldsymbol{R}^q$ let $|y|$ denote the Euclidean norm and $|y|^{\text{ord}}$ stand for the absolute value $\sup\{y, -y\}$ of y. Then we have

REMARK 2.1. For each $\delta > 0$, there exists $\gamma > 0$ such that $y \in \boldsymbol{R}^q$ and $|y| \leqslant \gamma$ imply $|y|^{\text{ord}} < \delta e$, e being a given order unit of $\boldsymbol{R}^q$.

Let, further, $\chi_n(.)$ denote the characteristic function of a set $M \subset [t_o, t_1]$ and, for a vector-valued function $y(.)$,

$$Y(t) = \int_{t_o}^{t} y(r)\, dr, \quad Y_M(t) = \int_{t_o}^{t} \chi_M(r)\, y(r)\, dr.$$

The following technical results will be used.

LEMMA 2.2 [2,p.245]. *Let $y_l(.)\colon [t_o, t_1] \to \boldsymbol{R}^{n_l}, l = 1, \dots, s$, be measurable bounded vector-valued functions. Then for every $\delta > 0$, there exist one-parameter families $M_s(\alpha), \dots, M_s(\alpha)$ of measurable subsets of $[t_o, t_1]$, $0 \leqslant \alpha \leqslant s^{-1}$, such that*

$$\text{mes } M_l(\alpha) = \alpha(t_1 - t_o) \text{ for } 1 = 1, \dots s,\ 0 \leqslant \alpha \leqslant s^{-1}; \tag{10}$$

$$M_l(\alpha') \subset M_l(\alpha) \text{ and } M_l(\alpha) \cap M_k(\alpha') = \Phi$$

$$\text{if} \quad 0 \leqslant \alpha' \leqslant \alpha \leqslant s^{-1} \quad \text{and } l \neq k; \tag{11}$$

$$|Y_{lM_l(\alpha)}(t) - Y_{lM_l(\alpha')}(t) - (\alpha - \alpha')\, Y_l(t)| \leqslant \delta |\alpha - \alpha'|$$

for all $t \in [t_o, t_1]$, $l = 1, \dots s$, $0 \leqslant \alpha$, and $\alpha' \leqslant s^{-1}$. Here „mes" means the Lebesgue measure.

Recall, further, that a mapping $f: X \to Y$, X being now a Hausdorff locally convex space, is said to be **locally convex** at $\bar{x}$ if its directional derivative $f'(\bar{x};x)$ exists for all $x \to X$ and is convex in the sense that

$$f'(\bar{x};\alpha x^1 + (1-\alpha)x^2) \leqslant \alpha f'(\bar{x};x^1) + (1-\alpha)f'(\bar{x};x^2)$$

for all $x^1, x^2 \in X$, $\alpha \in [0,1]$. f is said to be **uniformly differentiable in the direction** $\boldsymbol{x}$ at $\bar{x}$ if for every neighborhood of zero $V \subset Y$ there corresponds a neighborhood N of x and $\gamma_o > 0$ such that

$$\gamma^{-1}(f(\bar{x}+\gamma z) - f(\bar{x})) - f'(\bar{x};x) \in V$$

whenever $z \in N$ and $\gamma \in (0, \gamma_o)$. If f is locally convex and uniformly differentiable in all directions at $\bar{x}$ we say that f is **regularly locally convex** at $\bar{x}$.

LEMMA 2.3. *Let U_1 be a neighborhood of some $x^o(.) \in C^n[t_o, t_1]$ and $V = \{x(t) \in \boldsymbol{R}^n \,/\, t \in [t_o, t_1],\ x(.) \in U_1\}$. Let $g: [t_o, t_1] \times \boldsymbol{R}^n \to \boldsymbol{R}$ be continuous in t for each $x \in V$, and in x on V for each $t \in [t_o, t_1]$. Let, furthermore, g satisfy the following strong uniform differentiability in all directions at the point $x^o(t)$, $t \in [t_o, t_1]$: for every $\Sigma > 0$ and $z(.) \in C^n[t_o, t_1]$, there exist $\gamma_o > 0$ and a neighborhood U_2 of $z(.)$ such that*

$$\left| \frac{g(t, x^o(t) + \gamma y(t)) - g(t, x^o(t))}{\gamma} - g'_x(t, x^o(t); z(t)) \right| < \varepsilon \qquad (12)$$

for all t $\in [t_o, t_1]$, *$y(.) \in U_2$, and $\gamma \in (0, \gamma_o)$.*

Then the functional $G(x(.)) = \sup_{t\in[t_0 t_1]} g(t, x(t))$ is uniformly differentiable in every direction $z(.)$ at $x^o(.)$ and

$$G'(x^o(.);z(.)) = \max_{t\in T_0} g'_x(t, x^o(t);z(t)),$$

where $T_o = \{t \in [t_o, t_1] \,/\, G(x^o(t)) = g(t, x^o(t))\}$.

COROLLARY 2.4. *Let $g: \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}$ be jointly continuous and continiuously differentiable in $x \in \boldsymbol{R}^n$. Then $G(x(.))$ is regularly locally convex on $C^n[t_o, t_1]$ and $\partial G(\bar{x}(.))$ contains those $x^* \in C^n[t_o, t_1]^*$* defined by

$$< x^*, x(.) > = \int_{t_o}^{t} < g'_x(t,\bar{x}(t)), x(t) > d\mu,$$

where μ is a regular measure supported on $T = \{t \in [t_o, t_1] \,/\, g(t, \bar{x}(t)) = G(\bar{x}(.))\}$ with total variation 1.

For proofs of Lemma 2.3 and Corollary 2.4 see the appendix.

For general vector optimization problems of the form (6) – (9), where $F: X \times U \to Y$, $G: X \times U \to Z$, and $P: X \times U \to W$ are general mappings; X and W are Banach spaces; U is a set; Y and Z are normed spaces ordered by convex cones K and M, respectively, the following multiplier rule was proved.

THEOREM 2.5. [4]. *Assumptions*

(i) *int* $K \neq \emptyset$ *and int* $M \neq \emptyset$;

(ii) *for each* $u \in U$, $P(.,u)$ *is continuously differentiable at* x^o;

(iii) $F(.,u)$ *and* $G(.,u)$ *are continuous in a neighborhood* V *of* x^o *and regularly locally convex at* x^o *for each* $u \in U$;

(iv) *for every finite set of points* $u^1, \ldots, u^S$ *of* U *and every* $\delta > 0$, *there are a neighborhood* $V' \subset V$ *of* x^o, *an* $\varepsilon > 0$, *a mapping* $v: V' \times \varepsilon \Sigma^S \to U$ *and points* $e \in K$, $g \in M$ *such that, for all* $x, x' \in V$ *and* $a, a' \in \varepsilon \Sigma^S$,

$$v(x, 0) = u^o \text{ for some } u^o \in U,$$

$$\|P(x, v(x, a)) - P(x', v(x', a')) - P'_x(x^o, u^o)(x - x')$$

$$- \textstyle\sum_{j=1}^{s} (\alpha_j - \alpha'_j) P(x^o, u^j) \| \leqslant \delta(\|x - x'\| + \sum_{j=1}^{s} |\alpha_j - \alpha'_j|),$$

$$F(x, v(x, a)) - F(x, u^o) - \textstyle\sum_{j=1}^{s} \alpha_j (F(x, u^j) - F(x, u^o))$$

$$\leqslant \delta(\| x - x^o \| + \textstyle\sum_{j=1}^{s} \alpha_j) e$$

$$G(x, v(x, a) - G(x, u^o) - \textstyle\sum_{j=1}^{s} \alpha_j (G(x, u^j) - G(x, u^o))$$

$$\leqslant \delta(\| x - x^o \| - \textstyle\sum_{j=1}^{s} \alpha_j) g,$$

where $\sum^s = \{a = (\alpha_1, \ldots, \alpha_s) \in \boldsymbol{R}^s / \alpha_j \geqslant 0, \sum_{j=1}^{s} \alpha_j \leqslant 1\}$;

(v) $P'_x(x^o, u^o)\, X$ *has finite codimension.*

Then, if (x^o, u^o) *is a local Slater minimum there exist* $\lambda_o \in K^*$, $\mu_o \in M^*$ *and* $v_0 \in W^*$, *not all zero, such that*

$$\langle \lambda_o, F'_x(x^o, u^o; x) \rangle + \langle \mu_o, G'_x(x^o, u^o; x) \rangle + \langle v_o, P'_x(x^o, u^o) x \rangle \geqslant 0$$

for all $x \in X$,

$$\mathscr{L}(x^o, u^o, \lambda_o, \mu_o, v_o) = \min_{u \in U} \mathscr{L}(x^o, u, \lambda_o, \mu_o, v_o),$$

$$\langle \mu_o, G(x^o, u^o) \rangle = 0,$$

where L is the Lagrangian

$\mathscr{L} = \langle \lambda, F(x, u) \rangle + \langle \mu, G(x, u) \rangle + \langle v, P(x, u) \rangle.$

Now we are able to prove the Pontryagin maximum principle for the game (1)—(5). We call

$$H(t, x, u, p, \lambda) = \langle p, \varphi(t, x, u) \rangle - \langle \lambda, f(t, x, u) \rangle$$

the **Pontryagin function** and

$$\mathscr{H}(t, x, p, \lambda) = \sup_{u \in U} H(t, x, u, p, \lambda)$$

the **Hamiltonian.**

THEOREM 2.6. *Assume that int $K \neq \emptyset$ and that $\varphi, h_o, h_1, g, \xi, f$ are jointly continuous and continuously differentiable with respect to x. Let $u_1^0(.), \dots, u_m^o(.)$ be local-global weakly optimal controls, with the resulting state $x^o(.)^m$. Then there exist $\lambda \in K^*$, $l_o \in \neq \boldsymbol{R}^{S_o}$, $l_l \in \neq \boldsymbol{R}^{S_1}$,* a mapping $p(.) : [t_o, t_l] \to \boldsymbol{R}^n$, *and nonnegative regular measures μ_i, $i = 1, \dots$,*k, *on $[t_o, t_1]$, supported on the sets $T_i = \{ t \in [t_o, t_l] \mid g_i(t, x^o(t)) = 0 \}$, respectively, not all zero and such that*
$(a)\, p(.)$ is a solution of the integral equation

$$p(t) = -\xi'^*(x^o(t_1))\lambda - h_1'^*(x^o(t_1))\, l_1 + \int_t^{t_1} H_x'(r, x^o(r), u_1^o(r), \dots, u_m^o(r),$$

$$p(r), \lambda)\, dr - \sum_{i=1}^{k} \int_t^{t_1} g_{ix}'(r, x^o(r))\, d\mu; \tag{13}$$

with the initial condition

$$p(t_o) = h_o'^*(x^o(t_o))\, l_o; \tag{14}$$

(b) *the equality*

$$H(t, x^o(t), u_1^o(t), \dots, u_m^o(t), p(t), \lambda) = H(t, x^o(t), p(t), \lambda)$$

holds a.e. on $[t_o, t_l]$.

Proof. Assumptions (i)—(iii), and (v) of Theorem 2.5 are trivially satisfied by Corollary 2.4. We consider Assumption (iv). Let $u^l(.), \dots, u^s(.) \in U$ and $\delta > 0$ be given. We shall choose $\varepsilon > 0$, a neighbourhood V of $x^o(.)$ and a mapping $v : V \times \varepsilon\Sigma^s \to U$ of the form $v(.) = v(a)(.)$ for all $a \in \varepsilon\Sigma^s$ such that

$$v(0)(t) = \bar{u}(t) \tag{15}$$

a.e. on $[t_o, t_l]$, and the inequalities in Assumption (iv) hold. First, consider the inequality for $P(x(.), u(.))$. Since h_o and h_l are strongly differentiable, the inequality for $h_i's$ part is trivially satisfied. It remains to show, for all $x(.)$, $x'(.) \in V$ and $a, a' \in \varepsilon\Sigma^s$, that

$$\max_{t\in[t_0,t_1]} \left| \int_{t_o}^{t} (\varphi(r,x(r),v(a)(r)) - \varphi(r,x'(r),v(a')(r)) - \varphi'_x(r, x^o)(r),u^o(r)).\right.$$

$$\left.(x(r) - x'(r)) - \sum_{j=1}^{s}(\alpha_j - \alpha'_j)(\varphi(r, x^o(r), u^j(r)) - \varphi(r, x^o(r), u^o(r)))\,dr\right|$$

$$\leqslant \delta(\|x(.) - x'(.)\|_c + \sum_{j=1}^{s}|\alpha_j - \alpha'_j|). \tag{16}$$

The inequality for G is of course satisfied. To prove the one for F it suffices to show that

$$F(x(.), v(a)(.) - F(x(.), u^o(.)) - \sum_{j=1}^{s}\alpha_j(F(x(.), u^j(.)$$

$$- F(x(.), u^o(.))) \leqslant \delta e \sum_{j=1}^{s}\alpha_j, \tag{17}$$

where e is an order unit of the vector lattice Y.

We proceed to prove (16) and (17). Consider $(n+q)$-dimensional vectors $y^j(t) = (\varphi(t, x^o(t), u^j(t)) - \varphi(t, x^o(t), u^o(t)), f(t, x^o(t), u^j(t)) - f(t, x^o(t), u^o(t)))$ for $t \in [t_o, t_l]$. By Remark 2.1, for the given δ, there is $\gamma > 0$ such that $f \in Y$ and $|f| \leqslant \gamma$ imply

$$|f|^{ord} \leqslant \inf\left\{\frac{\delta e}{2}, \frac{\delta e}{8(t_1 - t_o)}\right\}. \tag{18}$$

Next, since $u^o(.), u^1(.), \ldots, u^s(.)$ are bounded, their values are contained in a compact set $U_l \subset |R^r$. Hence, by the continuity and the strong differentiability with respect to x of φ and f, there is $\delta > 0$ such that for all $u \in U_l$, t, x, and x' satisfying $|x - x^o(t)| < \delta$, $|x' - x^o(t)| < \delta$, we have

$$|\varphi(t, x, u) - \varphi(t, x^o(t), u)| \leqslant \frac{\delta}{2(t_1 - t_o)}, \tag{19}$$

$$|\varphi(t, x, u) - \varphi(t, x', u) - \varphi'_x(t, x^o(t), u)(x - x')| \leqslant \frac{\delta|x - x'|}{2(t_1 - t_o)}, \tag{20}$$

$$|f(t, x, u) - f(t, x^o(t), u)| \leqslant \gamma. \tag{21}$$

We set $V = \{x(.) \in C^n[t_o, t_1] / \ \|x(.) - x^o(.)\| < \delta\}$ and choose $\varepsilon \in \left(0, \frac{1}{s}\right]$ such that

$$\varepsilon (t_1 - t_o) \max_{t \in [t_o, t_1]} |\varphi'_x(t, x^o(t), u)| \leqslant \delta / 4, \tag{22}$$

where $|\varphi'_x|$ is the norm of the linear mapping $\varphi'_x : \boldsymbol{R}^n \to \boldsymbol{R}^n$. We now construct the desired mapping v. For the mentioned δ, γ and $y^j(.)$ we have, in accordance with Lemma 2.2, one parameter families $\{M_j(\alpha)\}$, $0 \leqslant \alpha \leqslant {}^1/_s$, $j = 1, \dots s$, of measurable subsets of $[t_o, t_1]$ such that (10), (11), and

$$|Y_{jMj(\alpha)}(t) - Y_{jMj(\alpha)}(t) - (\alpha - \alpha') Y_j(t)| \leqslant \min \left\{\gamma, \frac{\delta}{2}\right\} |\alpha - \alpha'| \tag{23}$$

hold for all $t \in [t_o, t_1]$, $j = 1, \dots, s$, $0 \leqslant \alpha$, and $\alpha' \leqslant s^{-1}$. We set $Q^s = \{a = (\alpha_1, \dots, \alpha_s) / 0 \leqslant \alpha_j \leqslant {}^1/_s\}$ (then $\varepsilon \Sigma^s \subset Q^s$) and for every $a \in Q^s$ we define

$$v(a)(t) = u^o(t) + \sum_{j=1}^{s} \chi_{Mj(\alpha j)}(t)(u^j(t) - u^o(t)).$$

Then (15) is satisfied. Since $M_j(\alpha_j)$, $j = 1, \dots, s$, are disjoint, we have

$$g(t, v(a))(t) = g(t, u^o(t)) + \sum_{j=1}^{s} \chi_{Mj(\alpha j)}(t)(g(t, u^j(t)) - g(t, u^o(t))) \tag{24}$$

for any vector-valued function g on $[t_o, t_1] \times \boldsymbol{R}^r$.

Now let $x(.), x'(.) \in V$ and $a, a' \in \varepsilon \Sigma^s \subset Q^s$. To prove (16) we estimate, for all $t \in [t_o, t_1]$,

$$|\int_{t_o}^{t} (\varphi(r, x(r), v(a)(r)) - \varphi(r, x'(r), v(a')(r)) - \varphi'_x(r, x^o(r), u^o(r))(x(r) - x'(r))$$

$$- \sum_{j=1}^{s} (\alpha_j - \alpha'_j)(\varphi(r, x^o(r), u^j(r)) - \varphi(r, x^o(r), u^o(r)))) dr|$$

$$\leqslant |\int_{t_o}^{t} (\varphi(r, x(r), v(a)(r)) - \varphi(r, x'(r), v(a)(r)) -$$

$$\varphi'_x(r, x^o(r), v(a)(r))(x(r) - x^o(r))) dr| + |\int_{t_o}^{t} (\varphi'_x(r, x^o(r), v(a)(r)) -$$

$$- \varphi'_x(r, x^o(r), u^o(r)))(x(r) - x'(r))) dr|$$

$$+ \left| \int_{t_o}^{t} (\varphi(r, x'(r), v(a)(r)) - \varphi(r, x'(r), v(a')(r)) \right.$$

$$\left. - \varphi(r, x^o(r), v(a')(r))) \, dr \right|$$

$$+ \left| \int_{t_o}^{t} (\varphi(r, x^o(r), v(a)(r)) - \varphi(r, x^o(r), v(a')(r)) \right.$$

$$\left. - \sum_{j=1}^{s} (\alpha_j - \alpha'_j)(\varphi(r, x^o(r), u^j(r)) - \varphi(r, x^o(r), u^o(r)))) \, dr \right|. \qquad (25)$$

Let us estimate each of the four terms in the right-hand side of (25). By (20) the first term does not exceed $\frac{\delta}{2} \| x(.) - x'(.) \|$.

By virtue of (27), the second term reads as

$$\left| \int_{t_o}^{t} (\sum_{j=1}^{s} x_{Mj(\alpha_j)}(r)(\varphi'_x(r, x^o(r), u^j(r)) - \varphi'_x(r, x^o(r), u^o(r)))(x(r) - x'(r)) \, dr \right|.$$

This, following (22), is dominated by

$$2 \| x(.) - x'(.) \| (\max_{\substack{t \in [t_o, t_1] \\ u \in U_1}} | \varphi'_x(t, x^o(t), u)|) \int_{t_o}^{t} (\sum_{j=1}^{s} x_{M_j(\alpha_j)}(t)) \, dt$$

$$= 2 \| x(.) - x'(.) \| (\max_{\substack{t \in [t_o, t_1] \\ u \in U_1}} | \varphi'_x(t, x^o(t), u)|)(\sum_{j=1}^{s} \alpha_j)(t_1 - t_0)$$

$$\leqslant \frac{\delta}{2} \| x(.) - x'(.) \|.$$

By (19) and (24) the third term is equal to

$$\left| \int_{t_o}^{t} \sum_{j=1}^{s} (\chi_{Mj(\alpha_j)}(r) - \chi_{Mj(\alpha'_j)}(r))(\varphi(r, x'(r), u^j(r)) - \varphi(r, x^o(r), u^j(r))) \, dr \right|$$

$$\leqslant \frac{\delta}{2(t_1 - t_0)} \sum_{j=1}^{s} \int_{t_o}^{t_1} |\chi_{Mj(\alpha_j)}(t) - \chi_{Mj(\alpha'_j)}(t)| dt = \frac{\delta}{2} \sum_{j=1}^{s} |\alpha_j - \alpha'_j|.$$

Finally, the forth term can be rewritten, by (24), as

$$\left| \sum_{j=1}^{s} \int_{t_o}^{t} ((\chi_{Mj(\alpha_j)}(r) - \chi_{Mj(\alpha'_j)}(r))(\varphi(r, x^o(r), u^j(r)) \right.$$

$$- \varphi(r, x^o(r), u^o(r)) - (\alpha_j - \alpha_j')(\varphi(r, x^o(r), u^j(r)) -$$

$$\varphi(r, x^o(r), u^o(r)))) \, dr| \quad . \tag{26}$$

The difference $\varphi(r, x^o(r), u^j(r)) - \varphi(r, x^o(r), u^o(r))$ constitute the first n components of the vector $y^j(r)$. Hence, by (23) and (26), it does not exceed

$$| \sum_{j=1}^{s} \int_{t_o}^{t} (\chi_{Mj(\alpha_j)}(t) - \chi_{Mj(\alpha_j')}(r)) y^j(r) - (\alpha_j - \alpha_j') y^j(r)) \, dr|$$

$$\leqslant \sum_{j=1}^{s} |Y_{jMj(\alpha_j)}(t) - Y_{jMj(\alpha_j')}(t) - (\alpha_j - \alpha_j') Y_j(t)| \leqslant \frac{\delta}{2} \sum_{j=1}^{s} |\alpha_j - \alpha_j'|.$$

Combining the estimations of the four terms we obtain (16). To verify (17), making use of (18), (23) and (24) we have

$$\int_{t_o}^{t_1} (f(t, x(t), v(a)(t) - f(t, x(t), u^o(t)))$$

$$- \sum_{j=1}^{s} \alpha_j (f(t, x(t), u^j(t)) - f(t, x(t), u^o(t))) \, dt$$

$$= \sum_{j=1}^{s} \int_{t_o}^{t_1} (\chi_{Mj(\alpha_j)} 1(f(t, x(t), u^j(t)) - f(t, x(t), u^o(t)))$$

$$- \alpha_j (ft, x(t), u^j(t)) - f(t, x(t), u^o(t)))) \, dt$$

$$\leqslant \sum_{j=1}^{s} |\int_{t_o}^{t_1} (\chi_{Mj(\alpha_j)}(t) (f(t, x^o(t), u^j(t)) - f(t, x^o(t), u^o(t)))$$

$$- \alpha_j (f(t, x^o(t), u^j(t)) - f(t, x^o(t), u^o(t)))) \, dt|^{ord}$$

$$+ \sum_{j=1}^{s} (\int_{t_o}^{t_1} \chi_{Mj(\alpha_j)}(t) (|f(t, x(t), u^j(t)) - f(t, x^o(t), u^j(t))|^{ord}$$

$$+ |f(t, x^o(t), u^o(t)) - f(t, x(t), u^o(t))|^{ord}) \, dt$$

$$+ \alpha^j \int_{t_o}^{t_1} |f(t, x^o(t), u^j(t)) - f(t, x(t), u^j(t))|^{ord}$$

$$+ |f(t, x(t), u^o(t)) - f(t, x^o(t), u^o(t))|^{ord}) \, dt)$$

$$\leqslant \frac{\delta}{2} e \sum_{j=1}^{s} \alpha_j 4 \sum_{j=1}^{s} \alpha_j \frac{\delta}{8} e = \delta e \sum_{j=1}^{s} \alpha_j,$$

i.e., (17) holds.

Having all the assuptions of Theorem 2-5 satisfied we now apply this theorem. The Lagrangian has the form

$$\mathscr{L}(x(.),u(.),\lambda,v,l_0,l_1,\theta) = \langle \lambda, \xi(x(t_1)) + \int_{t_0}^{t_1} f(t,x(t),u(t))\,dt \rangle$$

$$+ \int_{t_0}^{t_1} \langle x(t) - x(t_0) - \int_{t_0}^{t} \varphi(r,x(r),u(r))dr\, dv \rangle + \langle l_0, h_0(x(t_0)) \rangle$$

$$+ \langle l_1, l_1(x(t_1)) \rangle + \langle \theta, G(x(.)) \rangle ,$$

where $\lambda \in \boldsymbol{R}^1$, v is a regular vector measure on $[t_0, t_1]$, $l_0 \in \boldsymbol{R}^{s_0}$, $l_1 \in \boldsymbol{R}^{s_1}$, and $\theta \in \boldsymbol{R}^k$. According to Theorem 2.5 there exist $\lambda \in K^*$, $\theta \in \boldsymbol{R}^k$, a vector measure v, $l_0 \in \boldsymbol{R}^{s_0}$, $l_0 \in \boldsymbol{R}^{s_1}$, not all zero and such that

$$0 \in \delta_{x(.)} L(x^o(.), u^o(.), \lambda, v, 1_o, 1_1, \theta), \tag{27}$$

$$\mathscr{L}(x^o(.),u^o(.),\lambda,v,l_o,l_1,\theta) = \min_{u(\cdot)\in U} \mathscr{L}(x^o(.),u(.),\lambda,v,l_o,l_1,\theta), \tag{28}$$

$$\langle \theta, \mathrm{G}(\mathrm{x}^o(.)) \rangle = 0. \tag{29}$$

First, consider (27). This means, for all $x(.) \in C^n[t_o, t_1]$, that

$$\mathscr{L}'_{x(.)}(x^o(.), u^o(.), \lambda, v, l_o, l_1, \theta) \geqslant 0.$$

Substituting the differentials of P and F, and the subdifferential of G (given by Corollary 2.4) into this inequality leads to

$$\langle \lambda, \xi'(x^o(t_1))\, x(t_1) \rangle + \langle \lambda, \int_{t_0}^{t_1} f'_x(t, x^o(t), u^o(t))\, x(t)\, dt \rangle$$

$$+ \int_{t_o}^{t_1} \langle x(t) - x(t_o) - \int_{t_0}^{t} \varphi'_x(r, x^o(r), u^o(r))\, x(r)\, dr,\, dv \rangle$$

$$+ \langle l_o, h'_o(x(t_o)) \rangle + \langle l_1, h'_1(x(t_1)) \rangle +$$

$$\sum_{i=1}^{k} \theta_i \int_{t_0}^{t_1} \langle g'_{ix}(t, x^o(t)), x(t) \rangle\, d\tilde{\mu}_i = 0, \tag{30}$$

where $\tilde{\mu}_i$, $i = 1,\ldots,k$, are nonnegative regular measure supported on the set $\tilde{T}_i = \{ t \in [t_o, t_1] / g_i(t, x^o(t)) = G_i(x^o(.)) \}$, respectively, and with total variation 1. Setting $\theta_i \tilde{\mu}_i = \mu_i$ and changing the order of integration in the third term (with the abbreviations $f'_x = f'_x(t, x^o(t), u^o(t))$, $h'_o = h'_o(x^o(t_o)), \ldots$) we obtain, for all $x \in C^n[t_o, t_1]$,

$$\int_{t_0}^{t_1} \langle (f_x'^* \lambda - \varphi'^*_x \int_t^{t_1} dv)\, dt + dv + \sum_{i=1}^{k} g'_{ix} d\mu_i,\, x(t) \rangle$$

$$+ \langle h_o'^* l_o - \int_{t_0}^{t_1} dv,\, x(t_o) \rangle + \langle \xi'^* \lambda + h_1'^* l_1,\, x(t_1) \rangle = 0.$$

Now, denoting $p(t) = \int_t^{t_1} dv$, with Riesz's representation theorem we arrive at (13) and (14).

Further, the relation (28) is equivalent to

$$\int_{t_0}^{t_1} \langle \lambda, f(t, x^o(t), u^o(t)) \rangle\, dt - \int_{t_0}^{t_1} \langle \int_{t_o}^{t} \varphi(r, x^o(r), u^o(r))\, dr,\, dv \rangle$$

$$= \min_{u(\cdot)\in U} \left\{ \int_{t_0}^{t_1} \langle \lambda, f(t, x^o(t), u(t)) \rangle\, dt - \int_{t_0}^{t_1} \langle \int_{t_o}^{t} \varphi(r, x^o(r), u(r))\, dr,\, dv \rangle \right\}.$$

Changing the order of integration in the second terms of each side we obtain Assertion (b).

Finally, by (29), if $G_i(x^o(.)) < 0$, then $\theta_i = 0$, and so $\mu_i = \theta_i\, \tilde{\mu}_i = 0$. Therefore, only those μ_i, which correspond to the indices i with $G_i(x^o(.)) = 0$, can be nonzero. Combining with the fact that $\tilde{\mu}_i$ are supported on the set $\tilde{T}_i$ we see that μ_i are supported on the set

$$T_i = \{ t \in [t_o, t_1] / g_i(t, x^o(t)) = 0 \}.$$

If $m = q = 1$, i.e., we deal with a scalar optimal control problem, and $\xi \equiv 0$, then Theorem 2.6 coincides with Theorem 1 of § 5.2 in [2].

Now we pass to another cooperative differential game, namely, to the one with a state constraint of the integral type

$$\int_{t_0}^{t_1} g(t, x(t), u_1(t), \ldots, u_m(t))\, dt \leqslant 0, \tag{4'}$$

where $g: \boldsymbol{R} \times \boldsymbol{R}^n \times \boldsymbol{R}^{r_1} \times \ldots \times \boldsymbol{R}^{r_m} \to \boldsymbol{R}^k$, $\boldsymbol{R}^k$ being ordered by a convex cone M, instead of (4).

For this game the Pontryagin function is

$$H(t, x, u, p, \lambda, \mu) = \langle p, \varphi(t, x, u) \rangle - \langle \lambda, f(t, x, u) \rangle - \langle \mu, g(t, x, u) \rangle.$$

THEOREM 2.7. *Assume that int $K \neq \emptyset$, int $M \neq \emptyset$, and $\varphi, h_o, h_1, g, \xi, f$ are jointly continuous and continuously differentiable with respect to x. Let $u_1^o(.), \ldots, u_m^o(.)$ be local-global weakly optimal controls for m players, with the resulting state $x^o(.)$.*

Then, there exist $\lambda \in K^*$, $l_o \in \boldsymbol{R}^{s_o}$, $l_1 \in \boldsymbol{R}^{s_1}$, *a mapping* $p(.):[t_o, t_1] \to \boldsymbol{R}^n$, and $\mu \in M^*$, *not all zero, such that*

(a) $p(.)$ *is a solution of the integral equation*

$$p(t) = -\xi'^*(x^o(t_1))\lambda - h_1'^*(x^o(t_1))\, l_1$$

$$+\int_t^{t_1} H_x'(r, x^o(r), u_1^o(r),\ldots,u_m^o(r), p(r), \lambda, \mu)\, dr,$$

with the initial condition $p(t_o) = h_o'^*(x^o(t_o))\, l_o$;

(b) *the equality*

$$H(t, x^o(t), u_1^o(t),\ldots,u_m^o(t), p(t), \lambda, \mu) = H(t, x^o(t), p(t), \lambda, \mu)$$

holds for almost all $t \in [t_o, t_1]$;

(c) $\int_{t_0}^{t_1} \langle \mu, g(t, x^o(t), u^o(t)) \rangle dt = 0.$

P r o o f. The proof parallels the one of Theorem 2.6 and we present only what is different. $G(x(.), u(.)) = \int_{t_0}^{t_1} g(t, x(t), u(t))\, dt$ is continuously differentiable with respect to $x(.)$ and we have

$$G'_{x(.)}(x^o(.), u^o(.))\, x(.) = \int_{t_0}^{t_1} g_x'(t, x^o(t), u^o(t))\, x(t)\, dt.$$

To apply Theorem 2.5, only Assumption (iv) needs to be dicussed. For the present game, besides (14) — (16) we have also to prove the following inequality for G

$$G(x(.), v(a)(.)) - G(x(.), u^o(.))$$

$$-\sum_{j=1}^{s}\alpha_j(G(x(.), u^j(.)) - G(x(.), u^o(.))) \leqslant \delta g \sum_{j=1}^{s}\alpha_j, \qquad (16')$$

where g is an order unit of the vector lattice $\boldsymbol{R}^k$.

The vector y^j now must be $(n+q+k)$-dimensional vector

$$y^j(t) = (\varphi(t, x^o(t), u^j(t)) - \varphi(t, x^o(t), u^o(t)), f(t, x^o(t), u^j(t))$$

$$-f(t, x^o(t), u^o(t)), g(t, x^o(t), u^j(t)) - g(t, x^o(t), u^o(t))).$$

For the given δ, using Remark 2.1 we choose $\gamma > 0$ such that if $f \in Y, |f| \leqslant \gamma$ and $h \in |R^k, |h| \leqslant \gamma$, we have

$$|f|^{ord} \leqslant inf\left\{\frac{\delta e}{2}, \frac{\delta e}{8(t_1 - t_o)}\right\},$$
$$|h|^{ord} \leqslant inf\left\{\frac{\delta g}{2}, \frac{\delta g}{8(t_1 - t_o)}\right\}. \qquad (18')$$

Next, we use (18′), (23) and (24) to get (17′) in the same way as to get (17). Making use of Theorem 2.5 we obtain also (27) — (29). In this case (27) implies the following, instead of (30),

$$\langle \lambda, \xi' x(t_1) + \int_{t_0}^{t_1} f'_x x(t)\,dt \rangle + \int_{t_0}^{t_1} \langle x(t) - x(t_o) - \int_{t_o}^{t} \varphi'_x x(r)\,dr, dv \rangle$$
$$+ \langle l_o, h'_o x(t_o) \rangle + \langle l_1, h'_o x(t_1) \rangle + \langle \mu, \int_{t_0}^{t_1} g'_x x(t)\,dt \rangle = 0.$$

Changing the order of integration in the third term and applying Riesz's representation theorem we arrive at Assertion (a). Assertion (b) is obtained by the same way as for Theorem 2.6 and (c) is nothing else than (29). ■

3. Sufficient conditions

In this section we shall demonstrate, for cooperative differential games, the common assertion that under additional assumptions multiplier rules are also sufficient conditions. Namely, with the aid of some scalarization results we shall show that under additional assumptions the Pontryagin maximum principle stated in Section 2 is also a sufficient condition even for global optimal controls and global weakly optimal controls. However, we are able to consider only problems with fixed left end-point, i.e., with $h_o(x(t_o) \equiv x(t_o) - x^o$ for some fixed $x^o \in \boldsymbol{R}^n$.

We recall at first the needed notions and scalarization results. For a cone C in a linear space X, the sets

$$q.\ \text{int}\ C' = \{x' \in X' / \langle x', x \rangle > 0 \text{ for all } x \in C \setminus \{0\}\},$$
$$q.\ int\ C^* = \{x^* \in X^* / \langle x^*, x \rangle > 0 \text{ for all } x \in C \setminus \{0\}\}$$

are called (algebraic and topological, respectively), **quasiinteriors** of C' and C^*, respectively.

A functional f defined on a subset S of an ordered linear space X with the ordering cone C is called:

+ **increasing** (on S) if $x^1 \leqslant x^2$ implies $f(x^1) \leqslant f(x^2)$;

+ **strongly increasing** if $x^1 \leqslant x^2$, $x^1 \neq x^2$, implies $f(x^1) < f(x^2)$;

+ **strictly increasing**, in the case $\mathrm{cor}C \neq \emptyset$, if $x^2 - x^1 \in \mathrm{cor}C$ implies $f(x^1) < f(x^2)$,

where x^1 and x^2 are points in S.

It is easy to see that each $x' \in C'$ (and each $x^* \in C^*$) is increasing on X, that each $x' \in q\text{-int}C'$ (and each $x^* \in q\text{-int}C^*$) is strongly increasing on X, and that, in the case $\mathrm{cor}C \neq \emptyset$, each $x' \in C' \setminus \{0\}$ (and each $x^* \in C^* \setminus \{0\}$) is strictly increasing on X.

The following scalarization result (see, e.g., [3]) will be used to derive sufficient conditions.

LEMMA 3.1. *Let X be an ordered linear space. Let $S \subset X$ and $\bar{x} \in S$ be given. Then*

(a) *if the ordering cone C is pointed and there is a strongly increasing functional f on S with*

$$f(\bar{x}) \leqslant f(x) \text{ for all } x \in S, \tag{31}$$

then $\bar{x}$ is a Pareto minimum of S.

(b) *if $\mathrm{cor}C \neq \emptyset$ and there is a strictly increasing functional f on S with* (31), *then $\bar{x}$ is a weak minimum of S.*

Recall further that a Fréchet differentiable functional $f: X \to \boldsymbol{R}$, X being normed space, is said to be **quasiconvex** at $\bar{x}$ if, for each $x \in X$, $f(x) \leqslant f(\bar{x})$ implies $f'(\bar{x})(x - \bar{x}) \leqslant 0$.

We now consider the game (1)–(5) with fixed left end-point, i.e., with $h_o(x(t_o)) \equiv x(t_o) - x^o$ for some fixed $x^o \in \boldsymbol{R}^n$.

THEOREM 3.2. *Let the ordering cone K of the objective space Y be pointed. Let $u_1^o(.), \ldots, u_m^o(.)$ be admissible controls and $x^o(.)$ be the resulting state. Let the following differentiability conditions be satisfied: h_1 and ξ are Fréchet differentiable at $x^o(t_1)$; $f(t,.,.)$, $\varphi(t,.,.)$ and $g_i(t,.)$ have partial derivatives at $(x^o(t), u^o(t))$ and at $x^o(t)$, respectively, for all $t \in [t_o, t_1]$. Moreover, assume that there exist $\lambda \in q.\mathrm{int}\, K^*$, $l_1 \in \boldsymbol{R}^{s_1}$, a mapping $p(.): [t_o, t_1] \to \boldsymbol{R}^n$, and nonnegative regular measures μ_i, $i = 1, \ldots, k$, on $[t_o, t_1]$ such that*

(a) *$p(.)$ is a solution of the integral equation*

$$p(t) = -\xi'^*(x^o(t_1))\lambda - h_1'^*(x^o(t_1))l_1$$

$$+ \int_t^{t_1} H_x'(r, x^o(r), u_1^o(r), \ldots, u_m^o(r), p(r), \lambda)\, dr -$$

$$\sum_{j=1}^{k}\int_{t}^{t_1} g'_{ix}(r, x^o(r))\, d\mu_i;$$

(b) *the equality*

$$H(t, x^o(t), u_1^o(t), \ldots, u_m^o(t), p(t), \lambda) = H(t, x^o(t), p(t), \lambda)$$

holds for almost all $t \in [t_o, t_1]$;

(c) *the following convexity assumptions are satisfied:*

$U = U_1 \times \ldots \times U_m$ *is convex;* $\langle \lambda, \xi(.)\rangle$ *is convex at* $x^o(t_1)$; $\langle l_1, h_1(.)\rangle$ *is quasiconvex at* $x^o(t_1)$; $\langle \lambda, f(t,.,.)\rangle$ *and* $\langle p(t) + \sum_{j=1}^{k}\int_{t}^{t_1} g'_{ix}\, d\mu_i$, $\varphi(t,.,.)\rangle$ *are convex and concave, respectively, at* $(x^o(t), u^o(t))$ *for almost all* $t \in [t_o, t_1]$.

(d) *for almost all* $t \in [t_o, t_1]$ *we have*

$$\varphi_x'^{*}(t, x^o(t), u^o(t))\sum_{j=1}^{k}\int_{t}^{t_1} g'_{ix}(t, x^o(t))\, d\mu_i = 0$$

and

$$\langle \sum_{j=1}^{k}\int_{t}^{t_1} g'_{ix}(t,x^o(t))\, d\mu_i, \sum_{j=1}^{k}\varphi'_{u_j}(t,x^o(t),u^o(t))(u_j(t) - u_j(t))\rangle \leqslant 0$$

for all admissible controls $u_j(.)$, $j = 1, \ldots, m$.
Then $u_1^o(.), \ldots, u_m^o(.)$ *are global optimal controls.*

P r o o f. Let $u_1(.), \ldots, u_m(.)$ be arbitrary admissible controls with the corresponding state $x(.)$. By (a), the quasiconvexity of $\langle l_1, h_1(.)\rangle$ and the convexity of $\langle \lambda, \xi(.)\rangle$ we obtain

$$-\langle (p(t) + \sum_{j=1}^{k}\int_{t}^{t_1} g'_{ix}(r, x^o(r))\, d\mu_i)_{t=t_1}, x(t_1) - x^o(t_1)\rangle$$

$$= \langle \xi'^{*}\lambda + h_1'^{*} l_1, x(t_1) - x^o(t_1)\rangle \leqslant \langle \lambda, \xi(x(t_1)) - \xi(x^o(t_1))\rangle.$$

By (b) $H'_u(t, x^o(t), u^o(t), p(t), \lambda, \mu)$ is a support functional of U at the point $u^o(t)$ for almost all $t \in [t_o, t_1]$. That means, for each admissible $u_j(.)$, $j = 1, \ldots, m$, and for almost all $t \in [t_o, t_1]$,

$$\langle f_{u_j}'^{*}(t, x^o(t), u^o(t))\,\lambda - \varphi_{u_j}'^{*}(t, x^o(t), u^o(t))\, p(t), u_j(t) - u_j(t)\rangle \geqslant 0.$$

This together with (a) and (d) implies

$$\langle \lambda, f(t, x(t), u(t)) - f(t, x^o(t), u^o(t)) \rangle$$

$$- \frac{d}{dt} \langle p(t) + \sum_{j=1}^{k} \int_t^{t_1} g'_{ix} d\mu_i, x(t) - x^o(t) \rangle$$

$$\geqslant \langle \lambda, f(t, x(t), u(t)) - f(t, x^o(t), u^o(t)) \rangle - \langle p(t) + \sum_{j=1}^{k} \int_t^{t_1} g'_{ix} d\mu_i,$$

$$\varphi(t, x(t), u(t)) - \varphi(t, x^o(t), u^o(t)) \rangle - \langle f'^*_x \lambda - \varphi'^*_x p(t), x(t) - x^o(t) \rangle$$

$$- \sum_{j=1}^{m} \langle f'^*_{u_j} \lambda - \varphi'^*_{u_j} p(t), u_j(t) - u_j^o(t) \rangle$$

$$+ \langle \sum_{j=1}^{k} \int_t^{t_1} g'_{ix} d\mu_i, \varphi'_x(x(t) - x^o(t)) + \sum_{j=1}^{m} \varphi'_{u_j}(u_j(t) - u_j^o(t)) \rangle.$$

By the convexity property of $\langle \lambda, f(t,.,.) \rangle$ and $\langle p(t) + \sum_{j=1}^{k} \int_t^{t_1} g'_{ix} d\mu_i,$ $\varphi(t,.,.) \rangle$ this quanity must be nonnegative.

Consequently, integrating leads to the inequality

$$\int_{t_0}^{t_1} \langle \lambda, f(t, x(t), u(t) - f(t, x^o(t), u^o(t)) \rangle dt$$

$$+ \langle \lambda, \xi(x(t_1)) - \xi(x^o(t_1)) \rangle \geqslant 0.$$

This together with Lemma 3.1 (a) completes the proof.

With Lemma 3.1 (b), instead of (a), used in the proof, we get in the same way following sufficient condition for weakly optimal controls.

THEOREM 3.3. *Let the cone K (not necessarily pointed) have cor $K = \emptyset$. If the set q-int K^* is replaced by $K^* \setminus \{0\}$, and the words „optimal controls'' are replaced by „weakly optimal controls'', then Theorem 3.2. is still in force.*

Note that Condition (d) is also of the maximum condition type. It is rather restrictive, especially its first part. However, we can intuitively imagine that this conditions is essential, because the maximum condition (b) does not involve the mappings g_i. The situation will be simpler when we deal with the game (1) - (3), (4), (5) as the following two theorems assert. Namely, no condition of the type (d) is needed. The proofs are quite similar, even with less complexity since the Pontryagin function H involves the mapping g, and therefore are omitted.

THEOREM 3.4. *Consider the game (1) — (3), (4'), (5). Let the ordering cone K of Y be pointed. Let $u_1^o(.), \ldots, u_m^o(.)$ be admissible controls and $x^o(.)$ be the resulting state. Let the following differentiability assumptions be satisfied: h_1 and ξ are Fréchet differentiable at $x^o(t_1)$, $f(t,.,.)$, $\varphi(t,.,.)$ and $g(t,.,.)$ have partial derivatives at $(x^o(t), u^o(t))$ for all $t \in [t_o, t_1]$. Moreover, assume that there exist $\lambda \in$ q-int K^*, $l_1 \in \boldsymbol{R}^{s_1}$, a mapping $p(.): [t_o, t_1] \to \boldsymbol{R}^n$, and $\mu \in M^*$ such that*

(a) $p(.)$ *is a solution of the integral equation*

$$p(t) = -\xi'^{*}(x^o(t_1))\lambda - h_1'^{*}(x^o(t_1))l_1$$

$$+\int_t^{t_1} H_x(r, x^o(r), u_1^o(r),\ldots,u_m^o(r), p(r), \lambda, \mu)\,dr;$$

(b) *for each admissible* $u_j(.), j = 1,\ldots,m$, *we have, for almost all* $t \in [t_o, t_1]$,

$$\langle f_{u_j}'^{*}(t, x^o(t), u^o(t))\lambda + g_{u_j}'^{*}(t, x^o(t), u^o(t))\mu$$

$$-\varphi_{u_j}'^{*}(t, x^o(t), u^o(t))\,p(t), u_j(t) - u_j^o(t)\rangle \geqslant 0;$$

(c) $\int_{t_o}^{t_1} \langle \mu, g(t, x^o(t), u^o(t)\rangle dt = 0;$

(d) *the following convexity assumptions are satisfied: U is convex;* $\langle \lambda, \xi(.)\rangle$ *is convex at* $x^o(t_1)$, $\langle l_1, h_1(.)\rangle$ *is quasiconvex at* $x^o(t_1)$, $\langle \lambda, f(t,.,.)\rangle$ *and* $\langle \mu, g(t,.,.)\rangle$ *are convex at* $(x^o(t), u^o(t))$ *for almost all* $t \in [t_o, t_1]$, *and* $\langle p(t), \varphi(t,.,.)\rangle$ *is concave at* $(x^o(t), u^o(t))$ *for almost all* $t \in [t_o, t_1]$.

Then $u_1^o(.),\ldots,u_m^o(.)$ *are global optimal controls.*

THEOREM 3.5. *Let the cone K (not necessarily pointed) have corK* $\neq \emptyset$. *If the set q-int*K^* *is replaced by* $K^* \setminus \{0\}$, *then Theorem* 3.4 *remains true for weakly optimal controls, i.e., in this case* $u_1^o(.),\ldots,u_m^o(.)$ *are global weakly optimal controls.*

Theorems 3.2 — 3.5. include Theorems 10.7 and 10.8 of [3], as special cases when $g_i \equiv 0$ and $g \equiv 0$.

Finally, we note that in this paper we consider necessary conditions only for local-global solutions of differential games. In [5] we prove the Pontryagin maximum principle for local solutions but with more restrictve differentiability assumptions.

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APPENDIX

Proof of Lemma 2.3. Note first that $g(., x^o(.))$ is continuous on $[t_o, t_1]$. The sequence $n(g(t, x^o(t) + 1/n\, z(t)) - g(t, x^o(t)))$ of continuous functionals on $[t_o, t_1]$ uniformly converges to $g'_x(t, x^o(t); z(t))$. So, the functional $g'_x(., x^o(.); z(.))$ is continuous on $[t_o, t_1]$, and the maximum in the expression for G' is indeed attained, and T_o is a compact subset.

To prove the lemma we have to convince ourselves that for every $\varepsilon > 0$ and $z(.) \in C^n[t_o, t_1]$, there exist a neighborhood U of $z(.)$ and some $\gamma_1 > 0$ such that

$$\left| \frac{G(x^o(.) + \gamma y(.)) - G(x^o(.))}{\gamma} - \max_{t \in T_o} g'_x(t, x^o(t); z(t)) \right| < \varepsilon \tag{32}$$

for all $y(.) \in U$ and $\gamma \in (0, \gamma_1)$. We have, for $y(.) \in U_2$ and $\gamma \in (0, \gamma_o)$,

$$G(x^o(.) + \gamma\, y(.)) \geqslant \sup_{t \in T_o} g(t, x^o(t) + \gamma\, y(t))$$

Therefore, to get (32) it suffices to find $U \subset U_2$, $\gamma_1 \leqslant \gamma_o$ so that $y(.) \in U$ and $\gamma \in (o, \gamma_1)$ would imply

$$G(x^o(.) + \gamma\, y(.)) < G(x^o(.)) + \gamma \max_{t \in T_o} g'_x(t, x^o(t); z(t)) + \gamma \varepsilon. \tag{33}$$

We proceed to do this. Since $g'_x(., x^o(.); z(.))$ is continuous there is a neighborhood $W \subset [t_o, t_1]$ of T_o such that

$$\sup_{t \in W} g'_x(t\, x^o(t); z(t)) = \max_{t \in T_o} g'_x(t, x^o(t); z(t) + \frac{\varepsilon}{2} \tag{34}$$

On the other hand, we can show the existence of a neighborhood U_o of $x^o(.)$ such that, for all $x(.) \in U_o$,

$$G(x(.)) = \sup_{t \in [t_o, t_1]} g(t, x(t)) = \sup_{t \in W} g(t, x(t)).$$

Indeed, we have, for some $\alpha > 0$, $\sup_{t \in [t_o, t_1] \setminus W} g(t, x^o(t)) = G(x^o(.)) - \alpha$. Setting $z(t) \equiv 0$ in (12) we find a neighborhood U_o of $x^o(.)$, for all $x(.)$ of which we have $|g(t, x(t)) - g(t, x^o(t))| < \alpha/2$ for all $t \in [t_o, t_1]$. This is just a required neighborhood, since, for $x(.) \in U_o$ and $t \in [t_o, t_1] \setminus W$,

$$g(t, x(t) < g(t, x^o(t) + \frac{\alpha}{2} \leqslant G(x^o(.) - \frac{\alpha}{2} \leqslant G(x(.)).$$

Now we can check directly (33) for $\gamma_1 = \min \{\gamma_o, \gamma_2\}$, where γ_2 is a number such that $x^o(.) + \gamma_2 z(.) \in U_o(z(.)$ given in (32)), and $U = \gamma_2^{-1}(\gamma_2 U_1 \cap (U_o - x^o))$. By (12) and (34) we have, for $y(.) \in U$ and $\gamma \in (0, \gamma_1)$,

$$G(x^o(.) + \gamma\, y(.)) = \sup_{t \in W} \; g(t, x^o(t) + \gamma\, y(t))$$

$$< \sup_{t \in W} g(t, x^o(t) + \gamma \sup_{t \in W} g'_x(t, x^o(t); z(t)) + \frac{\gamma\varepsilon}{2}$$

$$\leqslant G(x^o(.)) + \gamma \max_{t \in T_o} \; g'_x(t, x^o(t); z(t)) + \gamma\varepsilon.$$

Proof of Corollary 2.4. By Lemma 2.3, G is uniformly differentiable in every directions at each point $x \in C^n[t_o, t_1]$. Since $g'_x(t, x^o(t); z(t)) = g'_x(t, x^o(t))\, z(t)$ the convexity of $G'(x^o(.); z(.))$ with respect to $z(.)$ follows immediately from its expression. Thus, G is regularly locally convex.

The second part of the corollary follows from Subsection 4.5.3 of [2].

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Warunki optymalności w kooperatywnych grach różniczkowych

W pracy udowodniono konieczność warunków typu zasady maksimum Pontriagina dla wieloosobowych kooperatywnych gier różniczkowych ze zmiennymi stanu przez sprowadzenie gry do zadania optymalizacji wektorowej. Udowodniono również, że te same warunki są wystarczającymi do optymalności przy pewnych dodatkowych założeniach.

Условия оптимальности в кооперативных дифференциальных играх

В работе доказана необходимость условий вида принципа максимума Понтрягина для кооперативных дифференциальных игр со многими игроками путем сведения игры к задаче векторной оптимизации. Доказано также, что после принятия некоторных дополнитеьных предпосылок, такие же условия являются достаточными для оптимальности.