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Rational, not strictly monotonous logic operators

by

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A lot of papers deal with the fuzzy logic operators. Such operators play very important role in the application to expert systems e.g. Zadeh [18] and multi-critiria decision making e.g. Kacprzyk & Iwanski [10]. In this paper we present a new class of operators related to Hamacher's one: rational (polynomial by polynomial) and not strictly monotonous.

KEY WORDS: t-norms, Archimedean, rational, de Morgan class, classification property.

1. Introduction,.

In many real situations it is not possible to formulate precisly and deterministicaly the problem. In such cases fuzzy logic becomes very helpful and fruitful. Many authors have dealt with different operators of fuzzy logic Albert [2], Dombi [3,4], Drewniak [6], Dubois & Prade [7], Hamacher [9], Sugeno [14], Trillas and Valverde [15], Yager [17] and others. They assume that operators are associative, monotonous, continuous and that they fulfil the permanence principle i.e.

$$c(0,0) = c(0,1) = c(1,0) = 0, \quad c(1,1) = 1$$
$$d(1,1) = d(0,1) = d(1,0) = 1, \quad d(0,0) = 0$$

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where c and d are conjunctive and disjunctive operators, respectively. The represantation theorem for such class of operators was examined only under the assumption of strong monotonicity. Under such assumption Hamacher found the rational solution:

$$d(x,y) = \frac{x + y + (\gamma - 2)xy}{1 + (\gamma - 1)xy} \qquad t\text{—conorm}$$

$$c(x,y) = \frac{xy}{\gamma + (1 - \gamma)\ (x + y - xy)} \qquad t\text{—norm}$$

for all $\gamma > 0$.

The parameter γ ranges from 0 to infinite. For $\gamma = 1$ we get probabilistic model and if γ tends to infinite we get the drastic one. Hamacher gave the condition when such operators build de Morgan class. Recently some publications appeared Dombi [5], Weber [16] considering properties of logic operators assuming only monotonicity instead of strong monotonicity. It is a more general case and from that point of view it is interesting to study. In the following paper we present a rational family of monotonous operators, the homogeneous rational logical system and also we show when they build de Morgan class and when they fulfil classification property.

2. Basic results.

We begin by recalling the definition of t-norm and t-conorm, Schweizer-Sklar [12] and the Representation Theorem.

DEFINITION 1. The operator $c(x, y)$ and the operator $d(x, y)$ mapping $[0, 1] \times [0, 1] \rightarrow [0, 1]$ are called t-norm and t-conorm, respectively if they are:

(i) associative

(ii) monotonous

(iii) continuous

(iv) and they satisfy boundery contitions

$$c(0,0) = c(0,1) = c(1,0) = 0, \quad c(1,1) = 1$$
$$d(1,1) = d(0,1) = d(1,0) = 1, \quad d(0,0) = 0.$$

Some authors leave out the condition (iii). For more details see Dubois and Prade [8].

DEFINITION 2. The conjunctive operator $c(x, y)$ (disjunctive operator $d(x, y)$) is called Archimedean if $c(x, x) < x$ $(d(x, x) > x)$ for $x \in (0, 1)$.

DEFINITION 3. The pseudo-inverse of a strictly monotonously decreasing function f: $[a, b] \to [f(a), f(b)]$ is

$$f^{(-1)}(x) = \begin{cases} b & \text{if } \quad x \leqslant f(b) \\ f^{(-1)}(x) & \text{if } f(b) \leqslant x \leqslant f(a) \\ a & \text{if } f(a) \leqslant x \end{cases}$$

The pseudo-inverse of a strictly monotonously increasing function is defined in a similar way.

Representation Theorem (Ling [11]).
For an Archimedean t-norm and t-conorm there always exist strictly monotonous function f_1 and f_2, respectively, for which

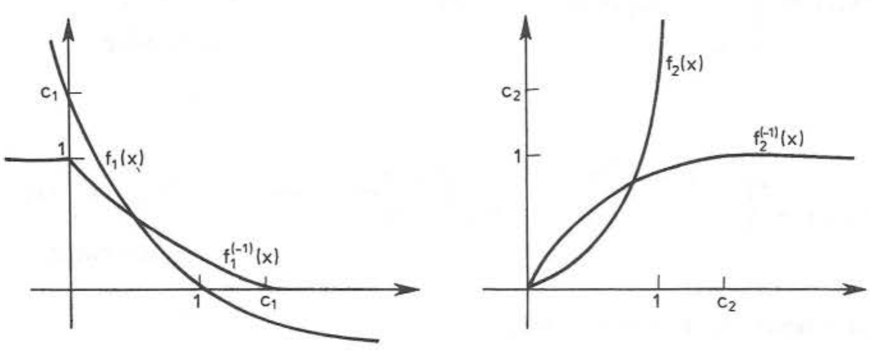
$$f_1(1) = 0, \quad f_2(0) = 0$$
$$f_1(0) = c_1, \quad f_2(1) = c_2$$

where c_1 and c_2 can be either $+\infty$ or $-\infty$ and they are such that

$$c(x,y) = f_1^{(-1)}(f_1(x) + f_1(y)). \; d(x,y) = f_2^{(-1)}(f_2(x) + f_2(y)).$$

where, apart from a multiplication factor α, f_1 and f_2 are uniquely defined.
COROLLARY. *Such operators are commutative.*

Aczel [1] proved the above theorem for the case when $c_1 = c_2$ are either $+\infty$ or $-\infty$ that is when a generator function and its invers are strictly monotonous. Ling showed that also for $-\infty < c_1, \quad c_2 < +\infty$ this theorem is true. The former case is called strictly monotonous and the latter not strictly monotonous.



Rys. 1

More comments and the extension of Representation Theorem can be found in Schweizer and Sklar [13]. Dickson 1916, Alt 1940 and Kuwagaki 1951 in Aczel [1] showed that the function $F(x,y) = f^{(-1)}(f(x)+f(y))$ has a rational form if and only if a generator function has a form

$$f(x) = \frac{Ax+B}{Cx+D} \qquad \text{or} \qquad f(x) = \frac{Ae^{Kx}+B}{Ce^{Kx}+D}$$

The application of these two function leads to the same result so let us take under consideration only the first one. The generator functions are unique with respect to a factor $\alpha \neq 0$ i.e. $\alpha f(x)$ is a generator for the same operator as $f(x)$, so we can assume that $c_1 = c_2 = 1$. Hence we will deal with the following generator functions for conjunction and for disjunction respectively

$$f_1(x) = f_2^{(-1)}(x) = \frac{v_1(1-x)}{v_1(1-x)+(1-v_1)x}$$

$$f_2^{(-1)}(x) = \frac{v_2 x}{v_2 x+(1-v_2)(1-x)} \qquad t_2(x) = \frac{(1-v_2)x}{(1-v_2)x+v_2(1-x)}$$

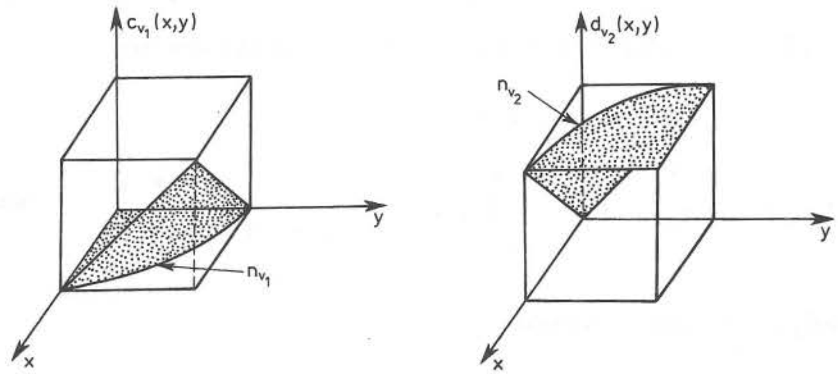
These generator functions together with rational negation Dombi [3] yield a rational logical system:

$$n_{v_3}(x) = \frac{v_3^2(1-x)}{v_3^2(1-x)\ (1-v_3)^2 x}$$

$$c(x,y) = \begin{cases} \dfrac{v_1^2(1-x)\ (1-y)-(1-v_1)^2\, xy}{(1-2v_1)^2\ (1-x)-(1-v_1)^2} & \text{if } y > n_{v_1}(x) \\ 0 & \text{otherwise} \end{cases}$$

$$d(x,y) = \begin{cases} \dfrac{v_2^2(1-x)\ (1-y)-v_2(2-3v_2)\,xy-v_2^2}{(1-2v_2)^2\, xy-v_2^2} & \text{if } y < n_{v_2}(x) \\ 1 & \text{otherwise} \end{cases}$$

where parameters v_1, v_2 and $v_3 \in [0,1]$.



Rys. 2

The above system is a generalization of Hamachers's one. It seems very adequate for numerical calcutations because of its rationality.

2. Special and limes operators.

Let us remember two definitions of non-continuous operators, which play important role in the theory of logic operators.
The t-norm t_c and t-conorm t_d are called drastic if

$$t_c(x,y) = \begin{cases} x \text{ if } y = 1 \\ y \text{ if } x = 1 \\ 0 \text{ otherwise} \end{cases} \qquad t_d(x,y) = \begin{cases} x \text{ if } y = 0 \\ y \text{ if } x = 0 \\ 0 \text{ otherwise} \end{cases}$$

and the second definition.
The t-norm s_c and t-conorm s_d are called strong-drastic if

$$s_c(x,y) = \begin{cases} 1 \text{ if } x = 1 \text{ and } y = 1 \\ 0 \text{ otherwise} \end{cases} \qquad s_d(x,y) = \begin{cases} 0 \text{ if } x = 0 \text{ and } y = 0 \\ 1 \text{ otherwise} \end{cases}$$

For the following particular value of v_1 and v_2 we get special operators:

$$v_1 = 1/2 \qquad c(x,y) = \max(x + y - 1, 0) \qquad \text{(Lukasiewicz)}$$

$$v_1 = 0 \qquad c_H(x,y) = \begin{cases} \dfrac{xy}{x + y - xy} & \text{if } x \neq 0 \text{ or } y \neq 0 \\ 0 & \text{if } x \neq 0 \text{ and } y = 0 \end{cases} \qquad \text{(Hamacher)}$$

$$v_1 = 1 \qquad c(x,y) = s_c(x,y)$$

and respectively for disjunction operators

$v_2 = 1/2 \qquad d(x,\, y) = \min(x+y, 1) \qquad$ (Lukasiewicz)

$v_2 = 0 \qquad d(x,y) = s_d(x,y)$

$$v_2 = 1 \qquad d_H(x,y) = \begin{cases} \dfrac{x+y-2xy}{1-xy} & \text{if } x \neq 1 \text{ or } y \neq 1 \\ 1 & \text{if } x = 1 \text{ and } y = 1 \end{cases} \qquad \text{(Hamacher)}$$

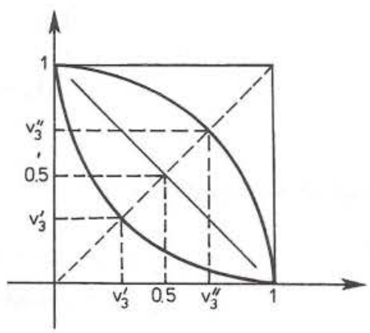
Similarly for negation operators:

$$v_3 = 1/2 \qquad n_{1/2}(x) = 1 - x$$

$$v_3 = 0 \qquad n_0(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

$$v_3 = 1 \qquad n_1(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$$

The following two Theorems are true in general.



Rys. 3

THEOREM 1. *Let $r > 0$ be a real number. If $f_1(x)$ ($f_2(x)$) is the generator function of a conjunctive (disjunctive) operator $c(x,y)$ ($d(x,y)$) then*

$$f_1^r(x) \qquad (f_2^r(x))$$

are also the generator function of some conjunction (disjunction) denoted by $c_r(x,y)$ ($d_r(x,y)$).

P r o o f. If $f_1(x)$ is the generator function of some conjunction, then it is easy to check that $f_1^r(x)$ satisfies the assumptions of generator function too. Thus

$$c_r(x,y) = f_1^{(-1)}((f_1^r(x) + f_1^r(y))^{1/r}).$$

THEOREM 2. *Let $c(x,y)$ and $d(x,y)$ be an Archimedean conjunction and disjunction, respectively. Then the following results are true:*

1. $\lim_{r\to\infty} c_r(x,y) = \min(x,y)$ $\qquad$ $\lim_{r\to\infty} d_r(x,y) = \max(x,y)$

Proof. 1. Let $x \leqslant y$ then $\min(x,y) = x$. Since always holds $c(x,y) \leqslant \min(x,y)$ we have

$$\min(x,y) \geqslant \lim_{r\to\infty} c_r(x,y) \geqslant \lim_{r\to\infty} f_1^{(-1)}((2f_1^r(x))^{1/r}) =$$

$$\lim_{r\to\infty} f_1^{(-1)}(2^{1/r}(fx))) = x = \min(x,y).$$

2. If $y = 1$ then $c_r(x,1) = x$. Similarly, if $x = 1$ then $c(1,y) = y$. Let us assume that $x<1$ and $y<1$ and $x \leqslant y$. Since

$$0 \leqslant \lim_{r\to 0} c_r(x,y) = \lim_{r\to 0} f_1^{(-1)}((f_1^r(x) + f_1^r(y))^{1/r}) \leqslant$$

$$\lim_{r\to 0} f_1^{(-1)}(2^{1/r} + f_1(y)) = 0,$$

we have

$c_0(x,y) = t_c(x,y)$, $d_0(x,y) = t_0(x,y)$

$c_\infty(x,y) = \min(x,y)$, $d_\infty(x,y) = \max(x,y)$

3. The de Morgan and classification properties.

Dombi [5] proved the general theorem.

THEOREM 3. *Let $c(x,y)$ be a conjunction, $d(x,y)$ disjunction operator and $n(x)$ a negation. Then de Morgan identity holds if and only if*

$$f_2(x) = \frac{f_1(n(x))}{\alpha}$$

where α *is a certain constant.*

In our case of monotonous and rational operators it simplifies to the Theorem 4.

THEOREM 4. *The system of monotonous and rational operators builds the de Morgan class if and only if*

$v_2 = n_{v_3}(v_1).$

The second important property is classification property:

DEFINITION 4. Conjunctive (disjunctive) operator has the classification property if $c(x, n(x)) = 0 \quad (d(x, n(x)) = 1)$.

THEOREM 5. *If $c(x, y)$ and $d(x, y)$ are monotonous, rational operators then they have classification property if and only if*

$$v_1 = v_3 \quad \textit{and} \quad v_2 = v_3, \quad \text{respectively.}$$

THEOREM 6. *The class of monotonous, rational logic operators has the de Morgan and classification property if and only if $v_1 = v_2 = v_3$. For $v_1 = v_2 = v_3 = 1/2$ we get the Lukasiewicz model.*

Proof. The result comes immediately from following property of negation $n_{v_3}(x)$:

$$n_{v_3}(x) = x \quad \text{iff} \quad x = v_3.$$

As simple conclusions from above Theorems we get the following ■

Corollaries:

COROLLARY 1. *Let $n_{v_3}(x)$ be the classical negation $n_{1/2}(x) = 1 - x$ i.e. $v = 1/2$. Then each pair of conjunction and disjunction such that $v_2 = 1 - v_1$ has the de Morgan property, but only exactly one for $v_1 = v_2 = 1/2$ has the classification property.*

COROLLARY 2. *The Hamacher's operators and strong drastic operators have de Morgan property for every negation $n_{v_3}(x)$, $v_3 \in [0, 1]$, but have no classification property for every negation $n_{v_3}(x)$, $v_3 \in [0, 1]$.*

The class under consideration is not distributive with respect to each other, because distributivity yields idempotency:

$$x = d(x, 0) = d(x, c(0, 0)) = c(d(x, 0), d(x, 0)) = c(x, x).$$

4. The homogeneous rational logical system.

We present below the rational logical system, which can be used fruitfully in multi-citeria decision making systems and in experts systems. In such systems a very important role is played by implication $x \rightarrow y = i(x, y)$, which in our system has the form:

$$i(x,y) = \begin{cases} \dfrac{v_4^2 xy + 2v_4(1-v_4)y(1-x) + (1-v_4)^2\,(1-x)(1-y)}{v_4^2 x + v_4(2-3v_4)y(1-x) + (1-v_4)^2\,(1-x)(1-y)} & \text{if } n_{v_4}(y) \geqslant n_{v_4}(x) \\ 1 & \text{otherwise} \end{cases}$$

where $v_4 \in [0.1]$.

For special v_4 we get

$$v_4 = 1/2 \qquad i(x,y) = \begin{cases} 1 - x + y & \text{if } y \leqslant x \\ 1 & \text{otherwise} \end{cases} \qquad \text{(Lukasiewicz)}$$

$$v_4 = 1 \qquad i(x,y) = \begin{cases} \dfrac{xy}{x - y + xy} & \text{if } x \neq 0 \text{ or } y \neq 0 \\ 1 & \text{if } x = 0 \text{ and } y = 0 \end{cases} \qquad \text{(Hamacher)}$$

$$v_4 = 0 \qquad i(x,y) = \begin{cases} 0 \text{ if } x = 1 \text{ and } y = 0 \\ 1 \text{ otherwise} \end{cases} \qquad \text{(drastic)}$$

Below we present the homogeneous class of logic operators.
$v = v_1 = v_2 = v_3 = v_4$, which has the de Morgan and classification property.
For simplicity we use the following notation:

$$[x] = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqslant x \leqslant 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Not strictly monotonous case	Linear case
$\left[\dfrac{v^2(1-x)\,(1-y) - (1-v)^2\,xy}{(1-2v)^2\,(1-x)\,(1-y) - (1-v)^2}\right]$.	$\max(x + y - 1{,}0)$

$$\left[\frac{v^2(1-x)\,(1-y)-v\,(2-3v)\,xy-v^2}{(1-2v)\,{}^2_x\,y-{}_{,}v^2}\right]. \qquad \max(x+y,1)$$

$$\frac{v^2\,(1-x)}{v^2\,(1-x)+(1-v)^2x}. \qquad 1-x$$

$$\left[\frac{v^2xy+2v(1-v)\,y(1-x)+(1-v)^2\,(1-x)\,(1-y)}{v^2x+v\,(2-3v)\,y\,(1-x)+(1-v)^2\,(1-x)\,(1-y)}\right].$$

$$\min(y-x+1,1)$$

Example 1.
The following two systems are boundary ones:

if $v =$ then $c_H(x,y)$, $s_d(x,y)$, $n_o(x)$
if $v = 1$ then $s_c(x,y)$, $d_H(x,y)$, $n_1(x)$
if $v =$ is nearer 1/2 we get more iteresting results:
if $v = 1/2$ then Lukasiewicz (linear) system
if $v = 1/4$ then

$$n_{1/4}(x) = \frac{1-x}{1+8x}$$

$$d_{1/4}(x,y) = \begin{cases} \dfrac{x+y+8xy-1}{4\,(x+y-xy)+5} & \text{if } y \geqslant \dfrac{1-x}{1+8x} \\ 0 & \text{otherwise} \end{cases}$$

$$c_{1/4}(x,y) = \begin{cases} \dfrac{x+y+4xy}{1-4xy} & \text{if } y \leqslant \dfrac{1-x}{1+8x} \\ 1 & \text{otherwise} \end{cases}$$

$$i_{1/4}(x,y) = \begin{cases} \dfrac{12-12x-6y+13xy}{12-11x-7y+7xy} & \text{if } y \leqslant x \\ 1 & \text{otherwise} \end{cases}$$

5. Summary

In this paper a new class of logic operators was proposed. They are related to Hamacher's ones, because of their rationality (polynomial by polynomial), but their generators are not strictly monotonous function.

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Operatory logiczne o wymiernych i monotonicznych generatorach

Wiele prac zajmuje się własnościami różnych klas operatorów logicznych. Odgrywają one dużą rolę w zagadnieniach w których nie może być użyta logika klasyczna, a więc w sytuacjach, gdzie występuje niepełność informacji. Znajdują one zastosowania w systemach ekspertowych Zadeh [18], czy w systemach wspomagania decyzji Kacprzyk i Iwański [10]. W pracy przedstawiono i zbadano nową klasę operatorów, których generatory są funkcjami monotonicznymi i wymiernymi. Są one uogólnieniem operatorów Hamachera, których generatory są funkcjami ściśle monotonicznymi i wymiernymi.

Логические операторы с рациональными и монотонными генераторами

Во многих работах рассматриваются свойства разных классов логических операторов. Они играют существенную роль в тех задачах, в которых невозможно использовать классическую логику, а значит в задачах с неполнотой информации. Они находят применение в экспертных системах, Заде [I8], или в автоматизированных системах принятия решений — Кацпшик и Иваньски [I0]. В работе представлен и исследован новый класс операторов, генераторы которых являются монотонными и рациональными функциями. Они являются обобщением операторов Гамахера, генераторы которых являются строго монотонными и рациональными функциями.