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COMPUTATION OF THE TIME-OPTIMAL CONTROL FOR SOME LINEAR SYSTEMS SUBJECT TO DISTURBANCES

by

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The time-optimal control computation presented in [4] for the linear time-invariant undisturbed systems is extended to the case, where also the effects of deterministic external disturbances are considered. It was shown that, if they do not depend directly on state variables and some additional assumptions concerning the reachability of the final state are valid, an effective numerical procedure can be applied in order to find the switching instants of the bang-bang control.

1. Introduction

In [4] the computational method of determining the time-optimal control for some linear time-invariant undisturbed systems was presented. The procedure applied there was based on the known general solution of the state equation.With the fundamental matrix obtained like in [3] the set of switching times, corresponding to the minimum of final error's norm can be found by an effective computation procedure.

This paper deals with the time-optimal control problem for the linear time-invariant system subject to deterministic external disturbances, described by the state equation

$$x(t)=Ax(t)+Bu(t)+Gz(t) \tag{1}$$

where $x(\cdot),u(\cdot),z(\cdot)$ are state, control and disturbance vectors of dimensions $n \times 1$, $r \times 1$ and $p \times 1$ respectively, A,B,G are constant matrices of appropriate dimensions, t is time. We accept the following assumptions (like in [4])

- the state vector is unconstrained,
- the control belongs to the closed, bounded admissible set

$$u(t)\in U_{ad} \tag{2}$$

with all components of $u(t)$ being functions of bounded variation on any bounded interval of time

$$|u_k(t)|\le U_{k\ max},\quad k=1,2,\dots,r \tag{3}$$

- the state matrix A is simple and all its eigenvalues s_l, $l=1,\dots,n$ are real negative

$$Re\ s_l<0 \tag{4}$$

$$Im\ s_l=0$$

- the system satisfies the normality condition of the time-optimal control

$$\det\,[b_k \mid Ab_k \mid A^2b_k \mid \ldots \mid A^{n-1}b_k] \neq 0; \qquad k=1,\ldots,r \tag{5}$$

where b_k is the k-th column of the matrix B.

Additionally we assume that the disturbance vector belongs to the compact set Z in R^p

$$z(t) \in Z \tag{6}$$

and all components $z_h(t)$, $h=1,\ldots,p$ of $z(t)$ are functions of bounded variation on any bounded interval of time.

The existence of the time optimal problem solution with the performance index

$$I = \int_{t_0}^{t} dt \rightarrow min \tag{7}$$

is for the considered system subject to disturbances related to the reachability ([1],[2],,[6]) of the final state.

We denote analogously like in [1] by $F(t,x_0,t_0,U_{ad},Z)$ the reachable set in R^n which consists of all the states x_f to which the system subject to the disturbance $z(.)\in Z$ can be transferred from x_0 at t_0 in the finite time $t \geq t_0$ by the admissible control $u(\cdot)\in U_{ad}$:

$$F(t,x_0,U_{ad},Z)=\{x_f:\ \exists u_{(t_0,t]} \in U_{ad} \text{ such that}$$

$$\Psi(t,u_{(t_0,t]},\ z_{(t_0,t]},x_0) = x_f\} \tag{8}$$

$$\text{for } z_{(t_0,t]} \in Z$$

where $\Psi(t,u_{(t_0,t]},z_{(t_0,t]},x_0)$ is the solution of the state equation with the initial condition x_0 at t_0.

By $F_k(t,x_0,t_0,U_{ad},Z)$ we denote the closed subset of states in R^n which our system subject to the disturbance $z(.)\in Z$ attains from x_0 at t_0 in the finite time $t \geq t_0$ by application of the single control component $u_k(t)$, $k=1,\dots,r$

$$F_k(t,x_0,t_0,U_{ad},Z) = \{x_f\colon \exists u_{k(t_0,t]} \in U_{k\ max} \text{ such that}$$

$$\Psi(t,u_{k(t_0,t]},z_{(t_0,t]},x_0) = x_f\} \qquad (9)$$

$$\text{for } u_1 = u_2 = \dots u_{k-1} = u_{k+1} = \dots u_r = 0, z_{(t_0,t]} \in Z.$$

The intersection F_f of all subsets F_k, $k=1,\dots,r$

$$F_f = F_1 \cap F_2 \cap \dots \cap F_r \qquad (10)$$

is the set of final states, which for $z(.)\in Z$ is reachable by application of any single control component $u_k(.)$, $k=1,2,\dots,r$ satisfying the condition (3).

In order to extend the numerical procedure presented in [4] for $z(.)\equiv 0$ to the case of the system subject to disturbances, we assume that the final state x_f which must be attained in the minimal time belongs to the set F_f from (10)

$$x_f \in F_f \qquad (11)$$

2. Formulation of the result

We shall prove the

THEOREM. *In the case where the disturbance $z(.)$ does not depend on state $x(.)$ and the conditions (2), (3), (4), (5) are satisfied, the time optimal control of the system (1) is of the bang-bang type and the number of switching intervals is at most n.*

P r o o f. The Hamiltonian corresponding to the performance index (7) is given by

$$H(x(t),\lambda(t),u(t)z(t)) = 1+\lambda^T(t)Ax(t) + \lambda^T(t)Bu(t) \qquad (12)$$

$$+ \lambda^T(t)Gz(t)$$

where $\lambda(t)$ is the costate vector of dimension $n \times 1$.

We denote by $x^*(t), \lambda^*(t)$ the state and costate vectors corresponding to the time-optimal control $u^*(t)$, and from the necessary condition

$$H(x^*(t), \lambda^*(t), u^*(t), z(t)) \leq H(x^*(t), \lambda^*(t), u(t), z(t)) \qquad (13)$$

we obtain

$$u^*(t) = - U \; sign \; \{\lambda^{*^T}(t)B\} \qquad (14)$$

where

$$U = [U_{1 \; \max} \; U_{2 \; \max} \; \cdots \; U_{r \; \max}]^T. \qquad (15)$$

Hence $u^*(t)$ is of the bang-bang type and its components are given by

$$u_k^*(t) = U_{k \; max} \; \mathrm{sign}(\lambda^{*^T}(t)b_k, \qquad (16)$$

We observe that the canonical costate equation does not depend directly on the disturbance, hence

$$\dot{\lambda}^*(t) = - \frac{\partial H}{\partial x} = - A^T\lambda^*(t) \qquad (17)$$

and we conclude (Ref. [5]) that in the case where all the eigenvalues of the matrix A are real the number of switching intervals for each $u_k(t)$, $k=1,2,\ldots,r$ is at most n. That completes the proof. ■

The above result enables us to extend the procedure of time-optimal control computation (presented in [4]) on the system described by the equation (1). The general solution is given by

$$x(t)=\phi(t-t_0)x(t_0)+\int_{t_0}^{t}\phi(t-\tau)Bu(\tau)\,d\tau + \int_{t_0}^{t}\phi(t-\tau)Gz(\tau)\,d\tau. \tag{18}$$

The fundamental matrix $\phi(t)=e^{At}$ in the case where the state matrix A is simple can be obtained directly by computation like in [3].

First we shall consider the single-input system replacing the control vector by the scalar $u(t)$ satisfying the condition $|u(t)|\le U_{max}$. For the first computation we assume that the number of switching intervals is equal to n and choose the switching instants $t_I, t_{II},\dots,t_n$ and the sign σ of $u(t)$ in the first interval t_0, t_I. Next for the control

$$u(t)=\begin{cases} 0 & t<t_0 \\ \sigma\, U_{max} & t\in[t_0,t_I) \\ -\sigma\, U_{max} & t\in[t_I,t_{II}) \\ \vdots & \\ (-1)^{n-1}\delta\, U_{max} & t\in[t_{n-1},t_n) \\ 0 & t\ge t_n \end{cases} \tag{19}$$

we compute the state vector at switching instants

$$\left.\begin{aligned} x(t_I) &= \phi(\Delta t_I)x(t_0)+D(\Delta t_I)\delta\, U_{max}+\varphi(t_I)\\ x(t_{II}) &= \phi(\Delta t_{II})x(t_I)+D(\Delta t_{II})(-1)\sigma U_{max}+\varphi(t_{II})\\ &\vdots\\ x(t_n) &= \phi(\Delta t_n)x(t_{n-1})+D(\Delta t_n)(-1)^{n-1}\sigma\, U_{max}+\varphi(t_n) \end{aligned}\right\} \tag{20}$$

where

$$\left.\begin{aligned} \Delta t_{I} &= t_{I}-t_{0} \\ \Delta t_{II} &= t_{II}-t_{I} \\ &\vdots \\ \Delta t_{n} &= t_{n}-t_{n-1} \end{aligned}\right\} \tag{21}$$

By $D(\Delta t)$ we denote the $n\times r$ matrix

$$(\Delta t) = \int_{0}^{\Delta t} \phi(\vartheta)B(\vartheta)\,d\vartheta \tag{22}$$

whose elements are given in [4]

$$d_{iq}(\Delta t)=\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{f_{ijl}}{s_l}\left(e^{s_l\Delta t}-1\right)b_{jq} \tag{23}$$

$$i=1,2,\dots,n \qquad q=1,2,\dots,r$$

where

$$f_{ijl}=\frac{p_{il}\,\mathrm{cof}\,p_{jl}}{\det P} \tag{24}$$

and P is a nonsingular modal matrix (whose columns are eigenvectors of A).

By $\Phi(t_i)$, $i=1,2,\dots,n$ we denote the value at $t=t_i$ of the third term on the right hand side of (18)

$$\Phi(t_i)=\int_{t_{i-1}}^{t_i}\phi(t_i-\tau)Gz(\tau)\,d\tau. \tag{25}$$

For the known $z(t)$ we can compute $\Phi(t_i)$ by the iterative method from the formula

$$\Phi(t_\nu)=M(\Delta\tau)z(t_{\nu-1}) \tag{26}$$

at instants

$$t_1=t_o+\Delta\tau, \qquad t_2=t_1+\Delta\tau,\dots, \qquad t_\nu=t_{\nu-1}+\Delta\tau$$

(choosing the sufficiently small value of $\Delta\tau$).

The elements $m_{iq}(\Delta\tau)$ of the $n \times p$ matrix $M(\Delta\tau)$ are

$$m_{iq}(\Delta\tau)=\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{f_{ijl}}{s_l}(e^{s_l\Delta\tau}-1)g_{jq} \tag{27}$$

$i=1,2,\ldots,n,\ l=1,2,\ldots,n,\ j=1,2,\ldots,n,\ q=1,2,\ldots,p.$

Knowing the deterministic disturbance $z(t)$ for $t \geq t_0$ we find $\varphi(t)$ from (26) for a chosen time interval $[t_0,t_z]$ with $t_z>t_n$ and store the obtained results. That enables us to extend $\varphi(t_I)$, $\varphi(t_{II}),\ldots,\varphi(t_n)$ from computer's memory and to introduce them into (20). For the state $x(t_n)$ we find the norm $N(t_n)$ in R^n representing the distance between $x(t_n)$ and prespecified final state x_f

$$N(t_n)=\|x_f-x(t_n)\| = \sqrt{\sum_{i=1}^{n}[x_{fi}-x_i(t_n)]^2}. \tag{28}$$

If this norm is bigger than a positive value ε corresponding to the desired accuracy we repeat the computation applying the minimization of $N(t_n)$ as function of switching instants. We accept as optimal switching instants $t_I^*, t_{II}^*,\ldots,t_n^*$ those corresponding to

$$N(t_n^*) \leq \varepsilon. \tag{29}$$

In order to check the choice of σ we repeat the above procedure for the opposite value of σ. After the comparison of obtained results we fix σ definitely.

For a multiple-input system we find first the switching instants for all cases where only one of r control components is applied.We proceed like for the single-input system.The values of ε_k, $k=1,2,\ldots,r$ corresponding to the desired accuracy can be chosen bigger than ε imposed for the multiple-input system.

Next we consider the case where two control components $u_a(t)$, $u_b(t)$ are applied and compute the switching instants

$t_{Ia}, t_{IIa}, \ldots, t_{na}, t_{Ib}, t_{IIb}, \ldots, t_{nb}$ corresponding to the accuracy given by the value ε_{ab}. It can be bigger than ε (imposed for the multiple-input system with all $u_k(t)$, $k=1,2,\ldots,r$ being active) and smaller than $\varepsilon_a, \varepsilon_b$ in previous computation where only $u_a(t)$ or $u_b(t)$ were applied; $\varepsilon<\varepsilon_{ab}<\varepsilon_a=\varepsilon_b$.

In further procedures we increase consecutively the number of applied control components and finally obtain the switching instants in the case where all r of them are applied. In order to investigate the implication of the disturbance on the system, we can compute also the switching instants for $z(t)\equiv 0$.

EXAMPLE. The system from the example in [4] where

$$A = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & -4 & 3 & 3 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 2 & 4 & 1 \\ 5 & 1 & 3 \end{bmatrix}$$

$$s_1 = -1, \quad s_2 = -2, \quad s_3 = -3, \quad s_4 = -4$$

$$U_{1\,max} = 1.5, \quad U_{2\,max} = 7, \quad U_{3\,max} = 8$$

$$\phi(t) = \begin{bmatrix} e^{-t} & 0 & 0 & 2e^{-t}-2e^{-2t} \\ 0 & e^{-4t} & 3e^{-3t}-3e^{-4t} & 1.5e^{-2t}-1.5e^{-4t} \\ 0 & 0 & e^{-3t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{bmatrix}$$

is subject to the disturbance

$$z(t) = \begin{bmatrix} z_{1\,max}(1-e^{\beta_1 t}) + z_{10}e^{\beta_1 t} \\ z_{2\,max}(1-e^{\beta_2 t}) + z_{20}e^{\beta_2 t} \end{bmatrix} \tag{30}$$

where $z_{1\,max} = 2.5$; $z_{2\,max} = 3.5$; $z_{10} = z_{20} = 0$

$\beta_1=-4.5;\quad \beta_2=-5.5.$

The matrix G in (1) is

$$G = \begin{bmatrix} 0 & 1.5 \\ 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix}$$

This system must be transferred in the shortest time from the initial state x_0

$$x_0 = [20 \quad -10 \quad 40 \quad -30]^T$$

to the final state $x_f=0$ with the accuracy corresponding to $\varepsilon=1\cdot 10^{-2}$. Tables I and II show the results computed by both: undisturbed (found in [4]),and subject to the disturbance - (30) - systems.

TABLE I

u_k	$z(t)$	t^*_{I}	t^*_{II}	t^*_{III}	t^*_{IV}
$u_1=\pm1.5$	0	0.5590975	1.1071200	1.3475340	1.3890230
$\sigma_1=1$	from (30)	0.8035849	1.4960260	1.9527250	1.9688400
$u_2=\pm7$	0	0.6151865	1.1260130	1.3168710	1.3890230
$\sigma_2=1$	from (30)	0.9961923	1.8065360	1.8505530	1.9688410
$u_3=\pm8$	0	0.7521554	1.110476	1.3513780	1.3890230
$\sigma_3=1$	from (30)	1.0820740	1.5749200	1.8961510	1.9688410

TABLE II

$z(t)$	$x_1^*(t_{IV}^*)$	$x_2^*(t_{IV}^*)$	$x_3^*(t_{IV}^*)$	$x_4^*(t_{IV}^*)$	$x_5^*(t_{IV}^*)$
0	-0.3997028	-0.6423354	-0.242306	-0.1702905	0.8124561
from (30)	-0.3912095	0.6266915	0.2216495	-0.6167312	0.9875592

all values $.10^{-2}$

3. Conclusive remarks

In the case where the deterministic disturbance does not depend on state variables and all the above accepted assumptions hold - the presented computational procedure enables us to find effectively the time optimal control of the bang-bang type. In order to choose appropriately the initial data for the described numerical procedure it can be useful to compute first the switching instants for the undisturbed system (with $z(\cdot)\equiv 0$).

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NUMERYCZNA METODA WYZNACZANIA STEROWANIA CZASOWO-OPTYMALNEGO DLA PEWNYCH UKŁADÓW LINIOWYCH PODDANYCH DZIAŁANIOM ZAKŁÓCEŃ

Podana w [4] numeryczna metoda wyznaczania sterowania czasowo-optymalnego dla inwariantnych w czasie i niezakłóconych układów liniowych została rozszerzona na przypadek, gdy układy te są poddane działaniom zdeterminowanych zakłóceń. Wykazano, że jeżeli zakłócenia te nie są funkcjami zmiennych stanu oraz są spełnione dodatkowe założenia dotyczące osiągalności stanu końcowego to można zastosować procedurę numeryczną umożliwiającą wyznaczenie chwil przełączeń przy sterowaniu typu bang-bang.

ЧИСЛЕННЫЙ МЕТОД ОПРЕДЕЛЕНИЯ ВРЕМЯ-ОПТИМАЛЬНОГО УПРАВЛЕНИЯ ДЛЯ НЕКОТОРЫХ ЛИНЕЙНЫХ СИСТЕМ ПОДВЕРГАЕМЫХ ВОЗДЕЙСТВИЮ ПОМЕХ

Приведенный в [4] численный метод определения время-оптимального управления для инвариантных по времени и без воздействия помех линейных систем был расширен для случая, когда эти системы подвержены воздействию определенных помех. Показано, что если эти помехи не являются функциями переменных состояния и удовлетворяются дополнительные предпосылки, касающиеся достигаемости конечного состояния, то можно применить численную процедуру позволяющую определять моменты переключений при управлении типа банг-банг.