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LAGRANGE RELAXATION AND DANTZIG-WOLFE DECOMPOSITION

by

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The purpose of this paper is to show the deep connections between the Lagrange relaxation techniques and the well known Dantzig-Wolfe decomposition methods.

KEY WORDS: Lagrange relaxation, Linear programming, Decomposition.

1. Introduction

In this paper we are dealing with a large scale linear programming problem having a block-diagonal structure with coupling constraints. Such problems (*P*) are of the form

$$\min Z = cx = \sum_{k=1}^{K} c_k x_k$$

$$\sum_{k=1}^{K} A_k x_k = b \quad (m \text{ constraints})$$

$$D_k x_k = d_k \quad , \qquad k=1,\dots,K$$

$$x_k \geq 0$$

(P) can also be written

$$\min Z = cx = \sum_{k=1}^{K} c_k x_k$$

$$\sum_{k=1}^{K} A_k x_k = b$$

$$x_k \in X_k \ , \quad k=1,\dots,K$$

where $X_k=\{x_k \mid D_k x_k=d_k,\ x_k\geq 0 \mid\}$, or in more compact form

$$\min Z = cx$$

$$Ax = b$$

$$x \in X$$

where $X = \{x \mid x=(x_1,\dots,x_K),\ \ x_1\in X_1,\ x_2\in X_2,\dots,x_K\in X_K\}$.

For the sake of simplicity, we assume that the convex polytopes X_k are bounded and not empty. Extensions to more complicated situations are straightforward.

The purpose of this paper is to show that the classical Dantzig-Wolfe decomposition method as for instance presented in Lasdon [1], is just a Lagrange relaxation approach.

In section 2 we decompose the problem (P) applying a Lagrange relaxation method, thus obtaining K subproblems (linear programming problems) and an unconstrained maximization problem of a concave (dual) function. This last problem can be solved using a subgradient method, or, as shown in section 3, using linear programming. The application of generalized linear programming to the dual of the last mentioned linear programming problem gives the theoretical basis for the development of the Dantzig-Wolfe decomposition algorithm.

We assume that the reader has a previous knowledge of Lagrange relaxation, generalized linear programming and the Dantzig-Wolfe method. The main contribution of this paper is to show the deep connections between these methods.

2. Lagrange relaxation

Let us associate with each coupling constraint "i" of (P) a dual variable $\Pi_i \in R$. We denote by $\Pi \in R^m$ the row-vector of dual variables. The Lagrange function will be

$$L(x,\Pi) = cx - \Pi(Ax-b) = \Pi b + \sum_{k=1}^{K} (c_k - \Pi A_k) x_k$$

and the dual function will be

$$L(\Pi) = \min_{x \in X} [L(x,\Pi)] = \min_{x \in X} [(c-\Pi A)x] + \Pi b =$$

$$= \Pi b + \sum_{k=1}^{K} \min_{x_k \in X_k} [(c_k - \Pi A_k) x_k]$$

Then, the computation of the dual function decomposes into the solution of K subproblems (S_k) as

$$\text{Min} \quad \bar{c}_k \, x_k$$

subject to

$$D_k \, x_k = d_k$$

$$x_k \geq 0$$

where $\bar{c}_k$ are the reduced costs ($\bar{c}_k = c_k - \Pi A_k$). These subproblems are linear programming problems of very much smaller size than the original problem, then for given Π the computation of the dual function value is reduced to the solution of K independent linear programming problems.

The dual problem (D) of (P) is the following unconstrained maximization problem

$$\max_{\Pi \in R^m} L(\Pi)$$

We know from duality theory (see Geoffrion [2] or [3]) the following properties of (D) ($L(\Pi)$ is a concave function).

If x is feasible in (P) and Π is feasible in (D) then

$$L(\Pi) \leq cx$$

(x^*, Π^*) is a saddle point for $L(x,\Pi)$ if and only if x^* is feasible in (P), Π^* is feasible in (D), and

$$L(\Pi^*) = cx^*$$

If $\overline{x}_k$ denotes an optimal solution to (S_k) for given Π, then the vector

$$\lambda = b - \sum_{k=1}^{K} A_k\overline{x}_k = b - A\overline{x}$$

is a subgradient of $L(\Pi)$ at the point Π.

Thus the solution of the dual problem reduces to the search for the maximum of a concave function, for which a subgradient is known at every point, therefore a subgradient method can be applied, see [4].

3. Solving the dual as a linear programming problem

Since the optimum of (S_k) is always reached at a vertex of X_k, the dual function can be expressed as

$$L(\Pi) = \Pi b + \sum_{k=1}^{K} \min_{y_k \in Y_k} [(c_k - \Pi A_k) y_k]$$

where the finite set Y_k contains the vertices of X_k, $k=1,\ldots,K$, or in more compact form

$$L(\Pi) = \Pi b + \min_{y \in Y} [(c-\Pi A) y]$$

where Y contains the vertices of X, with elements

$$Y = \{ y^1, y^2, \ldots, y^p \}$$

Obviously, the dual problem (D) is equivalent to the following linear programming problem

$$L(\Pi^*) = \max v$$
$$v \leq \Pi b + (c-\Pi A) y^t, \qquad t=1,2,\ldots,p$$

which can also be written as

$$L(\Pi^*) = \max v$$
$$v - \Pi(b-Ay^t) \leq cy^t, \qquad t=1,2,\ldots,p$$

The dual problems reads

$$\min \sum_{t=1}^{p} (cy^t)\, u_t$$

$$- \sum_{t=1}^{p} (b-Ay^t)\, u_t = 0$$

$$\sum_{t=1}^{p} u_t = 1$$

$$u_t \geq 0\,, \qquad \forall\, t$$

This linear programming problem contains $(m+1)$ constraints and an enormous number of variables. To be able to apply generalized linear programming, we must know whether there is an efficient method to find the variable u_s such that

$$cy^s + \Pi(b-Ay^s) - v = \min_t \{cy^t + \Pi(b-Ay^t) - v\}$$

That is we are looking for the column s such that

$$cy^s + \Pi(b-Ay^s) = \min_t \{cy^t + \Pi(b-Ay^t)\}$$

This is equivalent to looking for

$$\text{Min } \{\Pi b + (c-\Pi A)x\}$$

which is equivalent to evaluating the dual function $L(\Pi)$ at the point Π.

That is our procedure for column generation and it is nothing else but the method to compute $L(\Pi)$ consisting of solving, for given value of Π, the K subproblems s_k. The solutions thus obtained give an element y^s of Y. Now the well-known Dantzig-Wolfe decomposition algorithm can be developed as for instance in Lasdon [1].

4. Concluding remarks

In this paper we decompose a large scale linear programming problem having a block-diagonal structure with coupling constraints using a Lagrange relaxation method. We show that two types of algorithms can be developed, the first one is based on a subgradient approach while the second one is the well-known Dantzig-Wolfe decomposition algorithm. The first approach is of particular interest to obtain quickly a good approximation to $L(\Pi^*)$ and good lower bounds on Z^*. This approach has been applied with success to many large scale problems, see for instance Held et al. [4]. The second approach is recommended when suitable procedures for column generation are available, for applications see for instance [1].

References

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RELAKSACJA LAGRANZIANA A DEKOMPOZYCJA DANTZIGA-WOLFE'A

Celem artykułu jest ukazanie głębokich powiązań pomiędzy technikami odwołującymi się do relaksacji Lagranziana a znanymi metodami dekompozycji Danziga-Wolfe'a.

РЕЛАКСАЦИЯ ЛАГРАНЖИАНА А ДЕКОМПОЗИЦИЯ ДАНЦИГА-ВОЛЬФА

Цель статьи состоит в указании глубоких связей между методами относящимися к релаксации Лагранжиана, а известными методами декомпозиции Данцига-Вольфа