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A MARKOV-CHAIN APPROACH TO LONG-RUN INVESTMENT MODELS[1]

by

ALESSANDRA D'AMICO FINARDI

Dipartimento di Matematica,
Statistica, Informatica
e Applicazioni
Universita' di Bergamo,
Italy

A long-run period approach is considered for aggregate investments in a stochastic framework, in which Markov process models are introduced to characterize best the long-run investment evolution in a Samuelson-like approach of "overlapping generations". Then the work examines the characteristics of the transition matrix T and its consequences on its states to describe the economic-financial properties of relative investments. A "steady-state" hypothesis is then introduced to limit the analysis on the one side and to emphasize the financial implications on the other side, underlying the "inertia" effect on investments in the long-run horizon.

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Introduction

In a market economy in which investments (real or financial ones) are performed, we want to consider an aggregate investor in the long run period. Moreover, the considered time horizon is extensible to the infinite one, admitting a free and unconditioned allocation of goods over the period without predetermined time and quantity limits.

Long-run investments are theoretically performed on rational criteria basis (indifference, equivalence, etc.) and are expected-utility maximizing: these conditions are of great importance in a mathematical model, whose main aim is the real value maintenance and growth, and whose variables are the growth level and rate.

If the growth rate is known, then our problem is deterministic and the solution is trivial, according to the growth model introduced.

On the contrary, the problem can be stochastic if the "growth rate" is a random variable.

In the more recent financial-economic literature analysis of stochastic processes has had many succesful applications.

In this paper we consider a long-run model well fitted for an investment analysis through the Markov stochastic process. In particular, we examine the role and significance of the transition matrix in a regular Markov chain to explain the dynamic evolution of the long-run aggregate investment and the economic financial implication both from the theoretic and the applied point of view.

A long-run investment model

Looking at the investment as an aggregate, it is admittable to consider a long-run period: this aggregate will be a well-balanced composition and a well-differentiale one, in order to obtain very low risk and erraticity values in the considered investment.

A dynamic analysis of these values can employ the theory of fluctuations, in the special case of homogenous fluctuations with regularity characteristics (cyclicity, trend etc.) for the particular aggregate composition.

Only in case of erratic fluctuations, in fact, it should be necessary to use some chaotic analysis models, while simpler models can be useful here.

A sufficient long-run model has been introduced by Samuelson (1958) as the "overlapping generations" model: it considers, in the original approach, each generation living in two periods, young and old, and for each period both the young and the old generation exist.

In this scheme we can introduce two kinds of goods: perishable, for consumption, and durable, for investments, that can be transmitted among generations and individuals in the infinite horizon. We do not consider particular cases of particular goods as, for example, money, because they complicate too much the model in a first approach. In each period investments and consumer goods are exchanged in order to reach a temporal periodic equilibrium and then the economic system reaches a periodic equilibrium position.

There is a price vector for perishable and durable goods and their values are relative or can be normalized, being percentages in order that $\sum_{j=1}^{n} p_j=1$.

In this model each operator is represented by a couple (i,b) in which:

$i \in I$ finite operator set $i=1,2,\ldots,n$

$b=1,2$ is the period of the operator's life

Each operator has an investment sum of property E_b $(b=1,2)$ that is a compact subset of strictly positive vectors.

Another hypothesis is that each operator has the same preference order and the same expectation function. Following the Grandmont-Hildebrand model, from the dynamic point of view, we want to study the economic system and its stability characteristics through the study of the process $\{z^k\}$ and the Markov process properties.

Suppose that agents own random investment amounts: then at time k, for $k=1,2,\ldots$ these investments will be described by random variables defined in a probability space $(\Omega, \mathcal{F}, \nu)$.

Operators, born at time k, for $k = 1,2,\ldots,$ have random investments described by an $(a_i^k)_{i\in I}$ family of random variables with values in $E=E_1xE_2$.
A market economy, therefore, is completely characterized by: the operator set I, a fixed amount of money M, common preferences and expectation functions, an initial state $z(0)$ and an investment process a_i^k :

$$\varepsilon = (I,M,\gamma,\psi,(a_i^k)_{i\in I},z(0))$$

Random investments of both young and old agents are described by a random variable e^k with values in E^n; then we want to consider the market paths as a stochastic process realization $\{z^k,\ k \geq 0\}$ defined in (Ω,F,ν) by:

$$z^0(\omega) = z(0)$$

$$z^k(\omega)=f(e^k(\omega),\quad z^{k-1}(\omega)) \quad \text{for any } k\geq 1,\omega\in\Omega$$

In order to obtain a stationary Markov process for (z^k) it is sufficient that:
-the process depend on the previous state at time $k-1$
-$\{e^k,k\geq 0\}$ is such that e^k with $k=0,1,2,..$ (a random variable sequence of investments) be a Markov process with stationary probability Q on E^n.
$Q(e,B)\in[0,1]$, for any $e\in E^n$ and $B\in\beta(E^n)$ is a value assigning the probability that an investment is B at time $k+1$ if it was e at time k and $P(z,B)$ is the probability that the market is B at time $k+1$ if at time k it was z.

This process, being a Markov chain in the finite and homogeneous case, is a system composed in the following way:
-a finite set of states $S=\{s_1,s_2,s_3,\ldots s_n\}$
-an nxn matrix $T=(t_{ij})$ with t_{ij} being the probability that the system passes directly from s_i to s_j
-a vector $\Pi^o= \Pi_1^o,\ldots\Pi_n^o$ with Π_i^o being the probability that the system is initially in the state s_i, $i=1,2,\ldots,n$.
Such conditions are specified in the notation $\mu(T,\Pi^o)$.
In particular there is:

a) $t_{ij} \geq 0 \qquad 1 \leq i, j \leq n$

b) $\Sigma_{j=1}^{n} t_{ij} = 1 \qquad 1 \leq i \leq n$

T is therefore a row stochastic matrix and

$$\Pi_i^k \geq 0$$

$$\Sigma_{i=1}^{n} \Pi_i^k = 1 \qquad \forall\ k$$

There is also:

$$\Pi^k = \Pi^{k-1} T = \Pi^o\ T^k$$

If $$\Pi^o = \Pi^k \qquad k = 1, 2, \ldots ,$$

this is defined as stationary distribution vector.
If the definition of "equivalence classes of a Markov chain" given by the communication relation on the state set is known, the character of a Markov chain is determined by the study of its transition matrix T with the goal of describing the investment evolution.

The transition matrix T and its characteristics

The stochastic matrix T is formed by the elements t_{ij}: these are estimates and probabilities with values in the interval [0,1] but it is necessary to obtain a transition matrix to be used in the model for any time k.
The parameter estimate of a Markov chain can be obtained through regressive techniques on time series of data: in this way we can obtain maximum likelihood estimates. In fact it is obvious that it should be very difficult - if not impossible - to obtain a perfect information about any transition, even if minimal, among different states.

For the different states the essential condition is:

$$s_i \cap s_j = \emptyset\ \forall\ i = j \qquad i, j = 1, 2, \ldots, n$$

Such an independence is fundamental, for example, in the investment case in order to assign univocally each kind of investment to its relative state. The k positive integer values define the time instants of the process evolution:

the time cadency can be chosen according to the performed analysis and the studied case characteristics. For example, we can observe that national aggregate investments in the researches performed by "Banca d'Italia" are long period data (months, quarters, etc.) - see for instance the application of Clements-Taylor to the Australian case with quarterly data.

Another problem is to define exactly the application field: in an aggregate case we can observe that in the same group there are investments which are very different one from another, mainly for the different market characteristics.

In the case of sector disaggregations there are much more details and homogeneity (for instance, the financial investment sector is divided into shares, bank deposits, bonds, government security, etc.). It should be simpler to consider infinite transition matrices in case of homogeneity that continue to be stochastic matrices. We consider, however, finite matrices in which different states are different investments and each state is an aggregation into the same economic sector.

The transition matrix T being available, its analysis is of great importance in terms of examining the state and evolution of the considered investments, with the constraints sub a) and b) as above in order to apply the Markov model properties.

It will be useful to note some definitions and theorems and classify some different kinds of Markov chains for the considered model. We can remember one very important state classification: "For a Markov chain in general a state s_i is: <u>temporary</u> if and only if the i-th column of an L matrix is completely composed of zeroes, that is $Le_i=0$, with e_i=i-th unit vector (matrix $L=I-AA^*$, with A^* being generalized inverse matrix of A, and $A=I-T$ being a singular matrix); <u>ergodic</u> <u>or</u> <u>persistent</u> if and only if $Le_i \neq 0$" (see the Theorem 4.14 in Berman-Plemmons, 1979). Moreover for a Markov chain classification according to the kind of transition matrix, an important theorem is: "If T is a transition matrix for a Markov chain, the chain is:

1) ergodic if and only if T is irreducible;

2) regular if and only if T is primitive;

3) periodic if and only if T is irreducible and primitive".

For proof see Berman-Plemmons, 1979 (3.9). Another useful theorem for the study of the market evolution path in the previous problem is the one that if $A^{*}=(a_{ij})$ is the inverse matrix and I is the set of indices corresponding to the transient states, then $\Sigma_{j\in I}a_{ij}^{*}$ is the expected value of the number of time moments in which the chain will be in the temporary state, being at the beginning in the state s_i (Theorem 4.25 in Berman-Plemmons, 1979). A passage from a temporary class to a persistent one can be called absorption and in particular a state can be called absorbing when it can be identified with the same class and is characterized by a value one on the principal diagonal, that is $t_{ii}=1$. In particular there is a useful property stating that a system can not consist of only temporary classes, but it must have at least an ergodic class. When the system reaches the ergodic class:

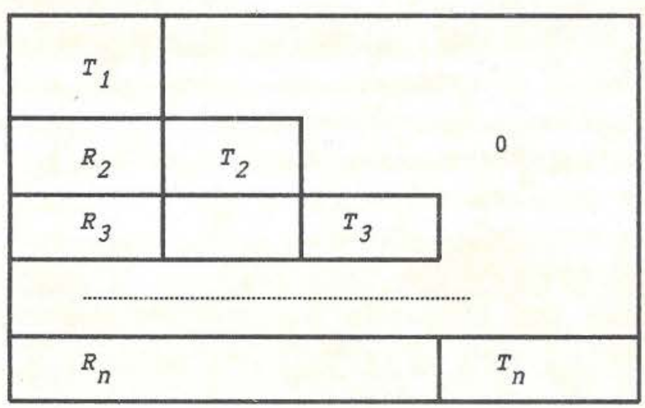
a) there is no possibility that the system leaves it;

b) the number of passages to reach it is finite.

In this way one can identify classes of equivalence and distinguish the states: the classes of temporary states that have to be left, and the classes of permanent states that cannot be left.

The canonic stochastic matrix is of the form:

$T =$



with:
- 0 null region with zero elements
- T_i square transition sub-matrices corresponding to the equivalence classes
- R_i submatrices containing null elements only in case of ergodicity of the correspondence class

An economic interpretation of the previous properties could be attempted: if the states represent different investments, then they can be divided into temporary and persistent ones. Temporary investments are performed in the short-run but it is useless to own them for a long period for a rational operator and then they tend to disappear or to be changed into persistent investments. A good example could be provided by high interest rate assets that, on the other side, do not prevent from an inflationary phenomenon: for some reasons there could be investors, who, after some periods, can prefer to change investment and choose to buy a real asset (for example a building). This one can be considered a persistent investment because it is not easy to buy and sell it at once (also if this asset can be substituted for by an other one of the same kind, and the investment can be considered the same). The ergodic property, therefore, seems to be a very strong condition but, truly, some investment classes seem to be characterized by a strong "inertia" effect and could approximate the persistency.

The "steady state" hypothesis

An interesting hypothesis for a preliminary analysis is the "steady state" hypothesis: it has been shown by Theil that, with steady state hypothesis, the system is always converging to the same steady distribution $\Pi=[\Pi_i]$ for $k\rightarrow\infty$ indifferently from the initial distribution s_0.
That is:

$$\lim_{k\to\infty} s^{o'}T^k = \Pi'$$

and $$\Pi_i > 0 \quad \forall\ i$$

$$\Sigma_{i=1}^{n} \Pi_i = 1$$

The vector Π is steady because once the system has reached it, it stops in it. The system from now on develops in "statistic equilibrium" conditions: $\Pi' = \Pi' T$.
In terms of investments, this cannot eliminate value movements among investments but these investments are balancing each other in order to have a constant investment distribution.

A regular Markov chain corresponds to this situation: a Markov chain is regular when the stochastic matrix T^k for any k integer positive is formed only with positive elements. Then, also by the previously proven theorems one can say that the corresponding T matrix is primitive and irreducible and it is aperiodic because of the cyclicity index is d=1. The process convergence to the steady state model is an interesting property for this model also after an exogeneous growing in the invested amount. This growing quantity could influence only one state and then only one kind of investment. For this investment j there will be:

$$s_j^{o'} = (1/W_1)(W_o\Pi_1, \ldots W_o\Pi_j + W_1 - W_o, \ldots W_n\Pi_n)\ ,$$

and the system evolution will be $s_j^{k'} = s_j^{o'}T^k$. The relative error for each investment compared with the previous steady state can be computed:

$$\Delta\ s_j^{ik} = \frac{W_1 s_j^{ik} - W_o\Pi_i}{W_o\Pi_i} \qquad i, j=1,2,\ldots,n.$$

with the index i indicating the i-th element of s_j^k .
But the steady state is independent from the initial increase in the investment, and then $\lim_{k\to\infty} s_j^{ik} = \Pi_i$ and $\Delta\ s_j^{ik}$

converges to the value of the relative error for any i and j.Then the increase may be found in all the investments with steady state hypothesis and therefore the steady distribution is confirmed. Moreover, the adjustment can take place in short time and the adjustment speed depends on the investment class that has been directly increased and on the link with other classes in terms of transition probability and then of increased transmission capability.

An interesting extension of this consideration is the fact that for an exogeneous negative variation products the same opposite effect and then a decrease of invested wealth in a sector (that can take place also for a fiscal move of government or a change in expectations) in a steady state situation can influence all the sectors of global investments in the same class of the considered economy. Moreover, if in the steady state there is a situation of an "overlapping generations" economy, as above described, we hypothesize:

$\bar{r}$ =interest rate, constant for the economy of this model,

n =population growth rate.

If and only if $\bar{r}<n$ there is a steady state in the economy with rent creation; this can be consistent with the existence of a bubble (bubble creation), that is a good value increase (real or financial goods for investment, and transmissible among generations with an infinite horizon -durable goods-).This bubble, that can be valued, for example, comparing with the "fundamental value" or reconstruction cost, is caused by speculative market operations.

Therefore the existence of a speculative component is possible in the investment evolution in the long run and in a Markov process model.

Then, the possibility of inserting in a model some other elements is very interesting, like e.g. speculation and its effects, as is the possibility of changing some hypotheses, e.g. elimination of the steady state condition and insertion of a progress, or consideration of a cyclical trend or a jump growth). The long term investment problem surely would be enlarged and complicated into a more complete model.

Economic-financial implications of the suggested model characteristics

The definition of state, class of states and of the grouping criteria has already been considered in the suggested model and it has some important consequences under the economic-financial point of view.

First of all, it is clear that temporary state classes correspond to short run investment opportunities, which are temporary in their own nature. These investments could be represented, for example, by some government asset (Italian BOT, CCT, etc.) that often produce in the short run a major rent compared with other investment opportunities, but do not offer a defense against the depreciation of the investment. In the long run the value of invested amount tend to become null or negligible (consider, for example, old government assets which, due to the economic changes, wars and strong money depreciations have diminished their value completely).

Furthermore, in the short run there are no opportunities for bubbles and speculation phenomena to appear in order to increase the price of goods if this kind of assets exists, just because of the above considerations about speculation. There is no risk element in this class of investments (excepting the long run risk on the loss of value) that takes usually an important role in the speculator decisions (speculator chooses highly risky operations, by definition).

Temporary states have to be left out because of their own nature (as we have seen in the transition matrix T) and persistent states are to be reached from other classes: in this approach long run investments correspond to the classes of persistent states and they can preserve the original capital value. These classes of investments, both in the steady state (as we have seen above) and in a generalized state, are the optimum framework to develop speculative phenomena: in particular we can reconsider, from a financial point of view, the case of an increase dealing initially with one investment of a persistent class. This increase passes to all the states of the same class in the same percentage, and the same should occur also for the speculative component.

In this model, however, we consider a "ceteris paribus" hypothesis and then this model could work surely if (the sufficient condition "and only if" has to be verified) the model parameter set does not change. We can also observe that, in particular for any row of the stochastic matrix T, expecially in steady state hypothesis, there will be:

$$t_{ij}=\max_i \qquad \text{for } i=j$$

as the principal diagonal value assigns the investment probability to remain in the same state as in the previous period, instead of changing state.

This tendential "inertia" characteristic can be given its general economic causes: from this model point of view, this principal diagonal prevalent values are explained by the homogeneity in fluctuations hypothesis as, in a different case of erraticity, different kinds of models (see above) could be proposed.

The class identification in the stochastic matrix analysis is, however, simpler than a corresponding definition formulated from an economic-financial point of view. Perhaps there is some connection with the identification of group or "portfolios" of different investments in a definite set.

The investment classes and sets are identified according to similar charcteristics. The absorbing classes, in particular, (or the absorbing states in the limit case) have the unit value on the principal diagonal:

$$t_{jj} = 1$$

When reached, these states are never left and they could correspond to investments showing a maximum of "inertia" from the point of view of investors.

Concluding remarks

The evolution of investments, considered here as an aggregate in the long run, seems to be characterized by a Markov stochastic process, mainly if relative stability hypotheses are accepted in the considered economic system.

Then after introducing the process and its characteristic parameters, the relative transition matrix and the steady state hypothesis in their consequences as to the long run economy can be analysed.

This slow movement in the system for the steady state hypothesis is confirmed in the proposed approach both from the mathematical-statistical and the economic-financial point of view, with the possibility of further interesting applications to be developed in the theoretic and applied framework.

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PODEJŚCIE ŁAŃCUCHÓW MARKOWA DO DŁUGOTERMINOWYCH MODELI INWESTYCJI

Rozważa się podejście dotyczące długoterminowego planowania zagregowanych inwestycji w ujęciu stochastycznym, w którym wprowadza się modele procesów Markowa, by możliwie dobrze scharakteryzować długoterminowy rozwój inwestycji w ramach koncepcji zbliżonej do samuelsonowskich "nakładających się pokoleń". W pracy analizuje się własności macierzy przejścia I i ich konsekwencje, w celu opisania gospodarczo-finansowych cech względnych inwestycji. Wprowadza się hipotezę "stanu ustalonego", aby ograniczyć obszar rozważań i podkreślić skutki finansowe, w związku z efektem inercyjnym w długim horyzoncie czasowym.

ПОДХОД МАРКОВСКИХ ЦЕПЕЙ К ДОЛГОСРОЧНЫМ МОДЕЛЯМ КАПИТАЛОВЛОЖЕНИЙ

Рассматривается подход, касающийся долгосрочного планирования агрегированных капиталовложений в стохастическом виде, в котором вводятся модели марковских процессов, с целью получения по возможности достоверной характеристики долгосрочного развития капиталовложений в рамках концепции схожей самуэльсоновскому "наложению поколений". В работе анализируются свойства переходных матриц I и их последствия, с целью описания хозяйственно-финансовых относительных факторов капиталовложений. Вводится гипотеза "установившегося состояния", с целью ограничения области рассмотрения и увыпукления финансовых последствий связанных с инерционным эффектом в длительном временном горизонте.