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THE GENERALIZED VON NEUMANN GROWTH MODEL

by

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In this paper we introduce consumption, labour, wages, savings and loans into the von Neumann growth model and study different models arising therefrom. Differently from other authors we introduce the new concepts according to a competitive point of view. Thus, relation with equilibrium points of a suitable game is obtained. We apply the existence theorem on equilibrium points for games with rational payoffs already formulated by the first author.

1. Introduction

In the subject of expanding economies introduced by the von Neumann's masterpiece [8], one can find a large bibliography. In particular during recents years, there has been a great interest in introducing new variables into the model. We may mention the works of Bawer [1], J.Łoś [4] and

Bromek [2]. On the other hand it is important to emphasize the important work done by Morgenstern and Thompson in [7], [8]. Moeschin has important contributions in the field of generalizing the open expanding economic model introduced by the previous authors. The authors mentioned try to introduce consumption, wages and labour in the von Neumann model. Some of their studies are based on the works of Morishima [7] and Malinvaud [5].

The last authors have already introduced some of the concepts mentioned extending the von Neumann models in different ways. However, none of these papers consider such concepts and their respective variables introduced in the competitive way.

In this paper we try to introduce the concepts of labour wages, consumption, saving and borrowing in the von Neumann model. This is done in a way resembling the papers mentioned, the main difference being the competitive aspect assigned to our variables. Therefore, we need to base our extensions on game theory. To be more specific we apply repeatedly a very recent result on equilibrium points of non-cooperative games with rational payoffs [6].

We will present three models which take into account different aspects of the new variables incorporated. The first model includes consumption and wages but not savings.

As a second consideration we will give a modification showing the relationship with a mathematical assumption about the matrices and coefficients involved. Finally we incorporate a model with the addition of savings and borrowings.

2. Consumption, labour and wages in the von Neumann model

We first introduce the basic version where consumption, labour and wages are taken into account.

Since we are most interested in mathematical aspects, we will only give an economic interpretation with a simple description. However the reader interested in such matters might develop them from the economic point of view.

We follow Burger's [3] treatment of the model.

As in the von Neumann model of an expanding economy, consider m processes or activities indexed by $i:1,..,m$. Each process consumes certain amounts of goods which are specified by the input $(a_{i1},...,a_{ij})$ where $a_{ij}\geq 0$ with $j:1,...,n$, is the input of the j-th commodity into i-th process. The output vector is $(b_{i1},...,b_{in})$ with $b_{ij}\geq 0$ being the amount of the j-th good produced by the same process. Similarly we assume that there is at least one labour class which consumes $c_{ij}\geq 0$ units of the j-th commodity of the i^{th} process, which operates at an arbitrary work intensity x_i, the input and output will be proportional. Moreover, if x_i is the consumption intensity of the process i, $\alpha>0$ is the economy expansion factor and $\gamma>0$ the consumption expansion factor, under assumption of a steady state economy, we have that the total production should be bigger than the required inputs and consumption at the expanded stage. Thus, we get the first inequality:

a) $$\sum_{i=1}^{m} b_{ij}x_i \geq \alpha \sum_{i=1}^{m} a_{ij}x_i+\gamma \sum_{i=1}^{m} c_{ij}z_i$$

for all commodities j.

On the other hand, on the financial side of the model, y_j is the price and w_j is the price factor of the commodity j, such that

$$\sum_{j=1}^{n} d_{ij}w_j$$

is the total wage for the labor force in the i-th processes. Here $d_{ij}\geq 0$ is the labor force demand of the j-th good in the i-th process. Then the monetary output value cannot exceed the expanded input value, times the interest factor, plus the demand of wages expanded by the salary factor, thus:

b) $$\sum_{j=1}^{n} b_{ij}y_j \leq \mathrm{B} \sum_{j=1}^{n} a_{ij}y_j+ \delta \sum_{j=1}^{n} d_{ij}w_j$$

for all processes i.

On the other hand, in terms of commodities, the expanded total consumption cannot be greater than the total demand for each commodity, that is to say

c) $$\sum_{i=1}^{m} b_{ij}x_i \geq \gamma \sum_{i=1}^{m} c_{ij}z_i$$

for each j.

Finally, in financial terms for all the processes , we have that the total expanding demand of salaries cannot be greater than total consumption, or in other words

d) $$\delta \sum_{j=1}^{n} d_{ij}w_j \leq \sum_{j=1}^{n} c_{ij}y_j$$

for each process i.

We would like to emphasize that if the latter inequality were reversed it would turn out then that in each period there is a positive difference between the total expanded salary and the total consumption, that is to say there would exist accumulation of savings which we do not allow at present.

We also assume that the vectors x, y, w and z are normalized to one and non-negative.

An expanding equilibrium for our model is a vector $(\bar{\alpha}, \bar{B}, \bar{\gamma}, \bar{\delta}, \bar{x}, \bar{y}, \bar{w}, \bar{z})$ such that it satisfies a), b), c) and $0 < \bar{\alpha}, \bar{B}, \bar{\gamma}, \bar{\delta} < +\infty$. Moreover, if strict inequality holds in a) for a commodity j, then the price $y_j=0$, that is to say if production is bigger than the required inputs and consumption, the good is free. If in b) there appears a strict inequality for the process i, the process is profitless and hence the labour intensity x_i must be zero. Similarly for a commodity j, in c). Since saving is not allowed, $w_j=0$. Finally in d) if strict inequality holds for process i, total consumption is strictly greater than the expanded salary demand and then the consumption intensity z_i is zero.

We are going to relate an expanding equilibrium with an equilibrium point of a special game. We make the reference to the book by Burger [3]. But first we are going to consider such a reduction for a simplified model in its uniform formulation, which we are going to present next.

3. Uniform model

Here we consider a slightly different model which is more uniform with regard to variables than the one considered in paragraph 2. The equations now are

a') $$\sum_i b_{ij}x_i \geq \sigma \sum_i a_{ij}x_i + \gamma \sum_i c_{ij}x_i$$

for each commodity j,

b') $$\sum_j b_{ij}y_j \leq \gamma \sum_j a_{ij}x_i + \sum_j d_{ij}y_i$$

for each process i,

f') $$\sum_i d_{ij}z_i \geq \gamma \sum_i c_{ij}z_i$$

for each good j.

If strict inequality holds for j in a') then $y_j=0$, for i in b') - then $x_i=0$, for j in c') - then $w_j=0$ and finally in d') for i - then $z_i=0$.

One may give here an economic interpretation analogous to the one given in the first model presented. For this reason we do not repeat it here.Again an expanding equilibrium is a vector ($\overline{a}$, $\overline{B}$, $\overline{\gamma}$, $\overline{\delta}$, $\overline{x}$, $\overline{y}$, $\overline{w}$, $\overline{z}$) satisfying a'), b'), c') and d'), with the expanding coefficients as positive numbers.

We now relate it to equilibrium points of a suitable four-person game

$$\Gamma' = \{ X, Y, W, Z; B, -A, -C, -D\}$$

where the strategy sets are:

$$X = Z = \{ X \in R^m: x_i \geq 0, \sum_{i=1}^{m} x_i = 1\}$$

and:

$$Y = W = \{y \in R^n:\ y_j \geq 0,\ \sum_{j=1}^{n} y_j = 1\}.$$

The payoff functions are given by

$$A'(x, y, w, z) =$$

$$= \frac{(\sum_{i,j} b_{ij} x_i y_j)\ (\sum_{i,j} c_{ij}\ z_i w_j) - (\sum_{i,j} c_{ij}\ x_i y_j)(\sum_{i,j} d_{ij}\ z_i w_j)}{(\sum_{i,j} a_{ij}\ x_i y_j)(\sum_{i,j} c_{ij}\ z_i w_j)}$$

$$B'(x, y, w, z) =$$

$$= \frac{(\sum_{i,j} b_{ij}\ x_i y_j)(\sum_{i,j} d_{ij}\ z_i w_j) - (\sum_{i,j} c_{ij}\ x_i y_j)(\sum_{i,j} d_{ij}\ z_i w_j}{(\sum_{i,j} a_{ij} x_i y_j)(\sum_{i,j} d_{ij} z_i w_j)}$$

$$C'(w, z) = \frac{\sum_{i,j} d_{ij} z_i w_j}{\sum_{i,j} c_{ij} z_i w_j}$$

$$D'(w, z) = \frac{\sum_{i,j} c_{ij} z_i w_j}{\sum_{i,j} d_{ij} z_i w_j}$$

We now have the first part of the intended reduction in the following result:

THEOREM 1. *Under the conditions that*

1) *For each* i *and* j $c_{ij} + d_{ij} > 0$.

2) *For all* i *and* j $a_{ij} + b_{ij} - \bar{\gamma} c_{ij} > o$ *or* < 0.

3) *For all* i *and* j $a_{ij} + b_{ij}\ \frac{1}{\bar{\gamma}}\ d_{ij} > 0$ *or* < 0.

an expanding equilibrium $(\bar{\alpha}, \bar{B}, \bar{\gamma}, \bar{\delta}, \bar{x}, \bar{y}, \bar{w}, \bar{z})$, *determines a positive equilibrium point* $(\bar{x}, \bar{y}, \bar{w}, \bar{z})$ *of* Γ' *such that* $\bar{\alpha} = \bar{A}'(\bar{x}, \bar{y}, \bar{w}, \bar{z})$, $\bar{B} = \bar{B}'(\bar{x}, \bar{y}, \bar{w}, \bar{z})$, $\bar{\gamma} = C'(\bar{w}, \bar{z})$ *and* $\delta = \frac{1}{\gamma} = D'(\bar{w}, \bar{z})$.

P r o o f. We mention that the second part of the game, namely C' and D' corresponds directly to a pure von Neumann model as can be seen in Burger [3]. Thus we omit the part which uses condition a). Moreover $\bar{\delta} = 1/\bar{\gamma}$.

Now, we multiply by y_j equation a') and sum over j'. Similarly we multiply by x_i b') and sum over i. Therefore, we have

a'') $\sum_{i,j} b_{ij}\bar{x}_i y_j - \bar{\gamma} \sum_{i,j} c_{ij} x_i y_j \geq \alpha \sum_{i,j} a_{ij}\bar{x}_i y_j$.

b'') $\sum_{i,j} b_{ij} x_i \bar{y}_j - \delta \sum_{i,j} c_{ij} x_i \bar{y}_j \leq B \sum_{i,j} a_{ij} x_i \bar{y}_j$.

From the first inequality, we derive that

$$\bar{\alpha} \leq \frac{\sum_{i,j} b_{ij}\bar{x}_i y_j - \bar{\gamma} \sum_{i,j} c_{ij} x_i y_j}{\sum_{i,j} b_{ij}\bar{x}_i y_j}$$

where the quotient is considered $+x$ if the denominator equals zero.

At this point we mention that condition 2) does not allow the denominator and the numerator to be both zero at any point y. Therefore we have from the expression of $\bar{\gamma}$: $0 < \bar{\alpha} = A'(\bar{x}, \bar{y}, \bar{w}, \bar{z}) \leq A'(\bar{x}, \bar{y}, \bar{w}, \bar{z})$ for any $y \in Y$. This is due to the condition expressed after d') regarding $y_j=0$.

Similarly for the first payoff function it is easy to derive that

$$0 < \bar{B} = \bar{B}'(\bar{x}, \bar{y}, \bar{w}, \bar{z}) \geq B'(\bar{x}, \bar{y}, \bar{w}, \bar{z}).$$

This is obtained using condition 3).

Thus, $(\bar{x}, \bar{y}, \bar{w}, \bar{z})$ is an equilibrium point of Γ' at which the payoff functions are positive . ■

Conversely, we have the next result:

THEOREM 2. *A positive equilibrium point, that is to say an equilibrium point* $(\bar{x}, \bar{y}, \bar{w}, \bar{z})$ *with*

$$0<\alpha=A'(\bar{x}, \bar{y}, \bar{w}, \bar{z}),\ 0<B'=B'(\bar{x}, \bar{y}, \bar{w}, \bar{z})<+\infty,\ 0<\bar{\gamma}=C'(\bar{w}, \bar{z})$$
$$0<\bar{\delta}=D'(\bar{w}, \bar{z})$$

determines an expanding equilibrium $(\bar{\alpha}, \bar{B}, \bar{\gamma}, \bar{\delta}, \bar{x}, \bar{y}, \bar{w}$

$\bar{z}$) *if the conditions* 1), 2), 3) *are fulfilled: and also:*

4) *For each i there is an j such that* $c_{ij} > 0$.

5) *For each j there is an i such that* $d_{ij} > 0$.

6) *For each i there is a j such that* $a_{ij} > 0$.

P r o o f . Since $(\bar{x}, \bar{y}, \bar{w}, \bar{z})$ is an equilibrium point of γ', then we have for the payoff C.

$$\bar{\gamma}= \frac{\sum_j \sum_i d_{ij}\bar{z}_i\bar{w}_j}{\sum_j \sum_i c_{ij}\bar{z}_i\bar{w}_j} - \frac{\sum_j \sum_i d_{ij}\bar{z}_i w_j}{\sum_j \sum_i c_{ij}\bar{z}_i w_j}$$

for all $w\in W$. Now by condition 1), it is impossible to have the indeterminate form 0/0, therefore if a denominator is zero, the corresponding numerator is a positive number. Consequently the ratio is $+\infty$. Now by condition 4), for some w, the ratio is finite, which implies that $\bar{\gamma} < =\infty$.

This on the other hand implies that c') is fulfilled, and $\bar{w}_j=0$ if the strict inequality holds in c') for j, because of the first equality given above.

In a similar manner it is possible to prove d') and $\bar{z}_j=0$ if the strict inequality holds. This can be done by using 1) and 5).

In an identical way, for the payoff B', we have

$$+\infty > \mathrm{B} > \frac{\sum_j \sum_i b_{ij}x_i\bar{y}_j - \delta_j \sum_i d_{ij}z_i\bar{y}_j}{\sum_j \sum_i a_{ij}x_i \bar{y}_j}$$

since $\bar{\delta}$ is indeed a positive number. By 3) the indeterminate form 0/0 cannot appear for any $x\in X$. Therefore since $\bar{\mathrm{B}}$ is a positive number the denominator can never be zero. Hence b') is a simple consequence of the previous inequality by taking x_i as pure strategies.

Finally for the remaining payoff function A', using 3) and 6) one gets a'). Essentially it is the same mechanism as the one just used q.e.d.). ■

Having the previous results, it is clear that in order to get more simplification, we can assume that the matrices A, C and D are positive. From an economic point of view this restriction is not troublesome since, instead of zero, an entry can take a very small amount without introducing a great modification in the model.

Now, we will present a first existence theorem which is an immediate consequence of the main result in [6].

THEOREM 3. *The game Γ' with positive matrices A, C and D has an equilibrium point.*

Hence, if such an equilibrium point is positive, it leads to an expanding equilibrium by Theorem 2. Indeed, giving some conditions about the positiveness of the numerators of the payoff functions A' and B' the positivenes of the equilibrium point may be obtained.

By some simple manipulations, it is easy to get the following equality

$$\sum_i \sum_j (\bar{\delta} d_{ij} - \bar{\gamma} c_{ij}) \bar{x}_i \bar{y}_j = (\bar{B} - \bar{\alpha}) \sum_j \sum_i a_{ij} \bar{x}_i \bar{y}_j .$$

Thus if the first term is zero it turns out that $\bar{\alpha} = \bar{B}$. The expansion factor equals the interest rate.

4. Study of the original model

Coming back to the original model, we relate it to the game

$$\Gamma = \{X, Y, W, Z, B, -A, -C, -D\}$$

where the payoff functions are now given as follows:

$$A(x, y, w, z) = \frac{(\sum_{i,j} b_{ij} x_i y_j)(\sum_{i,j} c_{ij} z_i w_j) - (\sum_{i,j} c_{ij} x_i y_j)(\sum_{i,j} d_{ij} z_i w_j)}{(\sum_{i,j} a_{ij} x_i y_j)(\sum_{i,j} c_{ij} z_i w_j)}$$

$$B(x, y, w, z) = \frac{(\sum_{i,j} b_{ij} x_i y_j)(\sum_{i,j} d_{ij} z_i w_j) - (\sum_{i,j} c_{ij} x_i y_j)(\sum_{i,j} d_{ij} z_i w_j)}{(\sum_{i,j} a_{ij} x_i y_j)(\sum_{i,j} d_{ij} z_i w_j)}$$

$$C\ (w,\ z) = \frac{\sum_{i,j} d_{ij} z_i w_j}{\sum_{i,j} c_{ij} z_i w_j}$$

$$D\ (w,\ z) = \frac{\sum_{i,j} c_{ij} z_i w_j}{\sum_{i,j} d_{ij} z_i w_j}$$

Having the game associated to the expanding model, we have that an expanding equilibrium is related to the positive equilibrium points as it is indicated in the next result:

THEOREM 4. *Assuming that* A, C, *and* $D>0$, *an expanding equilibrium* $(\bar{\alpha}, \bar{B}, \bar{\gamma}, \bar{\delta}, \bar{x}, \bar{y}, \bar{w}, \bar{z})$ *determines univocally a positive equilibrium point of* Γ *with*

$$\bar{\alpha}= A(\bar{x},\ \bar{y},\ \bar{w},\ \bar{z}) \quad B= B(\bar{x},\ \bar{y},\ \bar{w},\ \bar{z}) \quad \gamma= C(\bar{x},\ \bar{y},\ \bar{w},\ \bar{z})$$
$$\textit{and} \quad \delta= D(\bar{x},\ \bar{y},\ \bar{w},\ \bar{z})$$

Conversely, a positive equilibrium point of Γ *determines an expanding equilibrium of the model.*

P r o o f . Multiplying c) by w_j and adding up over J, we have

$$0<\gamma\leq C(\bar{x},\ \bar{w},\ \bar{z})$$

for each $w\in W$ since $D>0$. But

$$\bar{\gamma}=C(\bar{x},\ \bar{w},\ \bar{z})$$

by what condition d) follows related with $w_j=0$. Therefore

$$0<C(\bar{x},\ w,\ \bar{z})\leq C(\bar{x},\ \bar{w},\ \bar{z})$$

for any $w\in W$. The proof of the corresponding inequality for payoff function D, is obtained in a similar way.

Take now a). Multiplying by y_j and summing up over j, we get

$$\sum_i \sum_j b_{ij}\bar{x}_i y_j \geq \alpha \sum_i \sum_j a_{ij}\bar{x}_i y_j + \bar{\gamma} \sum_i \sum_j c_{ij}\bar{z}_i y_j\ .$$

From here, replacing the value of $\bar{\gamma}$ by its corresponding value $C(\bar{x},\ \bar{w},\ \bar{z})$ and rearranging the inequality, we get:

$$0<\alpha\leq A(\bar{x},\ \bar{y},\ \bar{w},\ \bar{z}) =$$

$$= \frac{(\sum_{i,j} b_{ij}\bar{x}_i y_j)(\sum_{i,j} c_{ij}\bar{z}_i\bar{w}_j)-(\sum_{i,j} d_{ij}\bar{x}_i\bar{w}_j)(\sum_{i,j} c_{ij}\bar{z}_i y_j)}{(\sum_{i,j} a_{ij}\bar{x}_i y_j)(\sum_{i,j} c_{ij}\bar{z}_i\bar{w}_j)}$$

because $A>0$. On the other hand,

$$\alpha=A(\bar{x}, \bar{y}, \bar{w}, \bar{z})$$

by the condition related to $y_j=0$.

The inequality for the payoff functions B can be obtained following similar steps.

Conversely, following the ideas used in the proof for the unified model, it is easy to obtain the result.(q.e.d.)■

Again, as a simple consequence of the main theorem in [6], since the payoff functions in Γ are multilinear, we have

THEOREM 5. *The game Γ with A, C and $D>0$ has an equilibrium point.*

Accordingly, it is possible to study the behaviour of the numerator of the payoff functions A and B in order to get a positive equilibrium point.

Manupulating the amounts in the payoff functions it is easy to get the following equality

$$\sum_i \sum_j (d_{ij} - c_{ij})\bar{z}_i\bar{w}_j = (\bar{B}-\bar{\alpha}) \sum_i \sum_j a_{ij}\bar{x}_i\bar{y}_j$$

Thus, if the first term is zero it leads to $B=\alpha$ which again means that the expansion factor equals the interest rate. The condition given might be viewed as meaning that total demand equals total consumption in the entire economy.

5. Savings and loans in the model

In this section we shall incorporate the concept of savings and consequently the concept of loans into the model already studied with consumption, labour and wages. Thus we are further generalizing the von Neumann model.

As we have mentioned already after condition d), it is clear that if savings are allowed in the economy, supposing that there is no accumulation of goods and financial media, then the savings have to be reinvested as loans which again stimulate the economy. Thus, we introduce e_{ij} which can be interpreted as the amount saved by the process i of good j in the instance that the intensity of saving is unity for process i in a period, if the "lending price" is unit.

Therefore if μ denotes the loan expansion factor and ν the loan interest, it follows that because the total expanded loans cannot be greater than the total saving :

$$\text{e)} \qquad \sum_{i=1}^{m} e_{ij}u_i \geq \sum_{0=1}^{m} f_{ij}u_i$$

for each j, where u_i is the saving intensity of process i. Let ν_j be the lending price of good j. It is assumed that if the strict inequality holds in e) for a given j, then corresponding ν_j=0. This may be interpreted as meaning that the good will be free in the sense of borrowing.

Furthermore, on the financial part, the total amount in loans for any process i times the lending interest cannot be smaller than the financial savings:

$$\text{f)} \qquad \nu \sum_{i=1}^{m} f_{ij}v_j \geq \sum_{i=1}^{m} e_{ij}v_j$$

with u_i=0 for any i for which the strict inequality holds.

With this, condition d) is now transformed into

$$\delta \sum_{i=1}^{m} d_{ij}w_i - \sum_{i=1}^{m} c_{ij}y_i \leq \nu \sum_{i=1}^{m} f_{ij}v_j + \sum_{i=1}^{m} e_{ij}v_j$$

for each process i. Economically speaking, this condition tells us that from the financial point of view the expanded wages plus the expanded loans cannot exceed the total consumption plus the savings. This is due to the fact that no accumulation is permitted. In add i tion, z_i=0 if strict inequality holds for process i. This means that if the balance between the consumption plus savings exceeds the salary plus loans then the consumption intensity of the corresponding process decreases.

On the other hand, c) becomes

$$\text{c)} \qquad \phi \sum_{i=1}^{m} d_{ij}x_i - \sum_{i=1}^{m} c_{ij}z_i \geq \sum_{i=1}^{m} e_{ij}u_i - \mu \sum_{i=1}^{m} f_{ij}u_i$$

for each good j. This condition tells us that, in terms of commodities, the total demand plus the expanded loans cannot be less than the expanded consumption plus the savings. Again $w_j=0$, if the equality does not hold for process j.

Finally, it is clear that the loans originated by the savings will go back to the economy stimulating it. Thus, we can consider now that the inputs and outputs depend upon the loans and savings. In addition a_{ij} and b_{ij} are continuous functions of u and v.

Conditions a) and b) are now kept the same.

We again define an expanding equilibrium for the economy as a set $(\bar{\alpha}, \bar{B}, \bar{\gamma}, \bar{\delta}, \bar{\mu}, \bar{\nu}, \bar{x}, \bar{y}, \bar{u}, \bar{w}, \bar{z})$ where the coefficients are positive and all of them satisfy a), b), c), d), e) and f).

Here we also assume that $u \in U = Z$ and $v \in V = W$.

We note that even though it seems that an expanding equilibrium in the complete model under study might generate an expanding equilibrium in the original model, this is not so.

We again are going to relate this more complete model with a strategy game, namely

$$\Gamma = \{X,\ Y,\ U,\ W,\ Z;\ \bar{B},\ -\bar{A},\ \bar{E},\ -\bar{E},\ -\bar{C},\ -\bar{D}\}$$

The payoff functions are defined respectively as follows

$$A(x,\ u,\ u,\ v,\ w,\ z) =$$

$$= \frac{(\sum_i \sum_j b_{ij}(u,v) x_i y_j)(\sum_i \sum_j c_{ij} z_i w_j) - (\sum_i \sum_j d_{ij} x_i w_j)(\sum_i \sum_j c_{ij} z_i y_j)}{(\sum_i \sum_j a_{ij}(u,v) x_i y_j)(\sum_i \sum_j d_{ij} z_i w_j)}$$

$$B(x,\ u,\ u,\ v,\ w,\ z) =$$

$$= \frac{(\sum_i \sum_j b_{ij}(u,v) x_i y_j)(\sum_i \sum_j c_{ij} z_i w_j) - (\sum_i \sum_j c_{ij} x_i w_j)(\sum_i \sum_j d_{ij} x_i w_j)}{(\sum_i \sum_j a_{ij}(u,v) x_i y_j)(\sum_i \sum_j d_{ij} z_i w_j)}$$

$$C(x,\ u,\ u,\ v,\ w,\ z) =$$

$$= \frac{(\sum_i \sum_j d_{ij} x_i w_j)(\sum_i \sum_j f_{ij} u_i v_j) + (\sum_i \sum_j e_{ij} u_i v_j)(\sum_i \sum_j f_{ij} u_i w_j)}{(\sum_i \sum_j f_{ij} u_i v_j)(\sum_i \sum_j c_{ij} z_i w_j)}$$

$$-\frac{(\sum_i \sum_j e_{ij}u_i v_j)(\sum_i \sum_j f_{ij}u_i w_j)}{(\sum_i \sum_j f_{ij}u_i v_j)(\sum_i \sum_j c_{ij}z_i w_j)}$$

$$D(x,\ u,\ u,\ v,\ w,\ z) =$$

$$=\frac{(\sum_i \sum_j c_{ij}z_i u_j)(\sum_i \sum_j f_{ij}u_i v_j)+(\sum_i \sum_j f_{ij}u_i v_j)(\sum_i \sum_j e_{ij}z_i v_j)}{(\sum_i \sum_j d_{ij}z_i w_j)(\sum_i \sum_j c_{ij}z_i w_j)}$$

$$-\frac{(\sum_i \sum_j e_{ij}u_i v_j)(\sum_i \sum_j f_{ij}z_i v_j)}{(\sum_i \sum_j d_{ij}z_i w_j)(\sum_i \sum_j f_{ij}u_i v_j)}$$

$$E(u,\ v,) = \frac{(\sum_i \sum_j e_{ij}u_i v_j)}{(\sum_i \sum_j f_{ij}u_i v_j)}$$

Having the game $\bar{\Gamma}$, we now present the following result relating the expansion equilibrium in the economic model with the concept of positive equilibrium point of $\bar{\Gamma}$.

THEOREM 6. *Under the conditions that* $A(u,\ v)$, C, D *and* $F>0$ *the expansion equilibrium* $(\bar{\alpha},\ \bar{B},\ \bar{\gamma},\ \bar{\delta},\ \bar{\mu},\ \bar{\nu},\ \bar{x},\ \bar{y},\ \bar{u},\ \bar{v},\ \bar{w},\ \bar{z})$ *determines a positive equilibrium point* $(\bar{x},\ \bar{y},\ \bar{u},\ \bar{w},\ \bar{z})$ *of* Γ *with* $\bar{\alpha}=\bar{A}(\bar{x},\ \bar{y},\ \bar{u},\ \bar{v},\ \bar{w},\ \bar{z})$, $\bar{B}=\bar{B}(\bar{x},\ \bar{y},\ \bar{u},\ \bar{v},\ \bar{w},\ \bar{z})$ $\bar{\gamma}=\bar{C}(\bar{x},\ \bar{y},\ \bar{u},\ \bar{v},\ \bar{w},\ \bar{z})$, $\bar{\delta}=\bar{D}(\bar{x},\ \bar{y},\ \bar{u},\ \bar{w},\ \bar{z})$, $\bar{\mu}=\bar{\nu}=\bar{E}(\bar{u},\ \bar{v})$, *and conversely.*

The proof can be carried out as in the previous case and therefore is omitted.

Related with this result, we have again as a consequence of the main theorem in [6] the following existence theorem.

THEOREM 7. *The game* $\bar{\Gamma}$ *has an equilibrium point.*

If this point is positive then it is related by the previous result with the expanding equilibrium in the economic model.

6. Final remarks

We point out as a first remark that the classical von Neumann model is obtained from the one of ours when $C = A = 0$. Similarly, it results from the model with consumption, labour, and wages when taking $C = D = 0$. An analogous behaviour for $E = F = 0$ in the complete model.

Finally, let us mention that in the complete model, the savings and loans go to work directly to the salary and consumptions sector and from there stimulate the economy. However, one may consider on the other hand that they would act directly on the productive process of the economy and from there on the wages and consumptions. The structure would not be altered by this assumption.

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UOGÓLNIENIE MODELU WZROSTU VON NEUMANNA

W artykule wprowadzono do modelu wzrostu von Neumanna konsumpcję, siłę roboczą, płace, oszczędności i pożyczki, i przeanalizowano wynikające stąd różne sformułowanie modeli. W odróżnieniu od prac innych autorów, pewne pojęcia zostały wprowadzone z uwzględnieniem konkurencji. W ten sposób otrzymano związki z punktami równowagi odpowiednich gier. Zastosowano twierdzenie o istnieniu punktów równowagi gier z wypłatami rzeczywistymi, uprzednio sformułowane przez pierwszego z autorów.

ОБОБЩЕНИЕ МОДЕЛИ РОСТА ФОН НЕЙМАНА

В статье вводится в модель роста фон Неймана потребление, рабочая сила, зарплата, сбережения и заемы, а также анализируются вытекающие отсюда разные формулировки модели. В отличие от работ других авторов, некоторые понятия вводились с учетом конкуренции. Таким образом получены связи с точками равновесия соответствующих игр. Используется теорема о существовании точек равновесия игр с действительными выигрышами, ранее сформулированная первым из авторов.