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ON AN EXCHANGE - EXPANDING ECONOMIC MODEL

by

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The paper proposes an integrated Gale-von Neumann model of open economy. The main result presented consists in statement of existence of an equilibrium having the properties of both the expanding eqilibrium concept and the exchange equilibrium one.

1. Introduction

It is well known that Gale's [3] exchange model of an economy is concerned with pure exchange among its agents. On the other hand the von Neumann growth model [9] studies the evolution of a closed economy from a stationary point of view. Generalizations of both models in several directions have been studied by different authors. In particular, models of open economies applying the von Neumann model have been studied first by Morgenstern and Thompson and then, for instance, by L. Mardon and O. Moechlin.

In this paper we propose an integrated Gale - von Neumann's model to study an open economy. The main result that we obtain is the existence of an equilibrium which has the properties of both the expanding equilibrium concept and the exchange equilibrium one. The technique used to prove it is related to that introduced by E. Marchi in [3].

Now we are going to describe the model. Because we are here mainly concerned with the mathematical point of view, we shall give a brief economic description of the elements of the model.

The economy has m productive processes or activities which produce n different kinds of goods. The technological characteristics of these processes are described by two $m\times n$ non negative matrices $A=(a_{ij})$ and $B=(b_{ij})$. A is called the input matrix and B the output one. Each process is supposed to work with an intensity level which can be measured by a real non negative number. The intensity levels of all the processes can be arranged in a vector $X=(X_1,...,X_n)$. The i-th component X indicates the activity level of the i-th process. It is also assumed that a system of internal prices holds in the economy. Such a system can be described by a vector $Y=(Y_1,...,Y_n)$, where Y_j indicates the price of the j-th good. Moreover, we suppose that there exists a set of p exchange sectors. These sectors provide the link between the internal economy composed by the processes and the external economy. Some of these exchange sectors can be considered as consumption sectors. However, these exchange sectors not only can take out goods from the internal economy but also can put into it goods from the external economy.

The activity of the k-th exchange sector is described by a vector $Z_k=(Z_{kn})$, Z_{kn} whose positive components indicate the quantities of the goods put into the economy by this agent. The negative ones indicate the quantities of the goods taken out of the economy. The set of all admissible activities for the k-th sector is noted by Z^k, where each vector $Z\in Z^k$ represents a certain utility for the k-th exchange sector, and we assume that it can be measured by a real-valued function U_k defined on Z^k.

The goal of this paper is to obtain a vector $(\alpha,\beta,X,Y,Z_1,\dots,Z_p)$ belonging to $R^+\times R^+\times R^m_+\times R^m_+\times Z^1\times\dots\times Z^p$ satisfying the following relations

E1) $\sum_i b_{ij}\overline{x}_i \geq \alpha\sum_i a_{ij}\overline{x}_i+\sum_k \overline{z}_{kj}$ for all $j=1,\dots,n$.

E2) $\sum_j b_{ij}\overline{y}_j \geq \beta\sum_j a_{ij}\overline{y}$ for all $j=1,\dots,m$.

E3) $Z_k\in\phi_k(\overline{Y})$ and $U_k(\overline{Z}_k)\geq U_k(Z_k)$ for all $Z_k\in\phi_k(\overline{Y})=$

$$=\left\{Z_k\in\phi_k \mid \sum_j \overline{Y}_j Z_{kj}=0\right\}$$

E4) $\sum_{ij} b_{ij}\overline{X}_i\overline{Y}_j=\alpha\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j+\sum_{ij} \overline{Z}_{kj}\overline{Y}_j=\beta\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j>0.$

With R^+ we denote the set of positive real numbers and $R^m_+=\{X=(X_1,\dots,X_n)\ X_1\geq 0\}$. Similarly we define R^n_+.

A vector satisfying the conditions E1) to E4) will be called an exchange - expanding equilibrium.

In order to prove the existence theorem we make the following assumptions

AI) $A+B>0$.

AII) No row of A is zero.

AIII) No column of β is zero.

AIV) All the utility functions U_k, $k=1,\dots,p$ are continuous and qasi concave.

AV) All the sets Z^k, $k=1,\dots,p$ are non empty, compact and convex, containing the zero vector. Moreover we ask that they have a non empty interior.

All these conditions are well known. Conditions AI), AII) and AIII) have been already used elsewhere by J.C.Cesco and, with slight difference, by J. Łoś [5], in order to prove existence theorems for generalizations of the von Neumann's model.

Assumption AV) can be relaxed as it is shown in E. Burger [1].

Now we are able to prove the existence result

THEOREM: *Under assumptions* AI) *to* AV) *the model described has an exchange - expanding equilibrium*

P r o o f. Let $\tilde{X}=\{X\in\mathbb{R}^m_+ \mid \sum_i x_i=1\}$ and $\tilde{Y}=\{Y\in\mathbb{R}^n_+ \mid \sum_j Y_j=1\}$.
and define the functions F and G by

$$F: \tilde{X}\times\tilde{Y}\times Z\times\tilde{Y}\longrightarrow\mathbb{R}$$

$$(\overline{X},\overline{Y},\overline{Z},Y)\longrightarrow(\sum_{ij} b_{ij}\overline{X}_i g_j-\sum \overline{Z}_{kj}\overline{Y}_j)(\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j)-$$

$$(\sum_{ij} b_{ij}\overline{X}_i\overline{Y}_j-\sum_k \overline{Z}_{kj}\overline{Y}_j)(\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j).$$

$$G: \tilde{X}\times\tilde{Y}\times Z\times\tilde{Y}\longrightarrow(\sum_{ij} b_{ij}\overline{X}_i\overline{Y}_j)(\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j)+(\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j)(\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j)$$

Here, Z denotes $\prod_{k=1}^{p} Z^k$.

Let φ be the correspondence defined by

$$\varphi: \tilde{X}\times\tilde{Y}\times Z\longrightarrow\tilde{X}\times\tilde{Y}\times Z$$

$$(X,Y,Z)\longrightarrow(\varphi_1(X,Y,Z),\varphi_2(X,Y,Z),\varphi_j(X,Y,Z))$$

where

$$\varphi_1(X,Y,Z)=\{\overline{X}\in\tilde{X} \mid G(X,Y,Z,\overline{X})\geq G(X,Y,Z,\tilde{X}) \text{ for all } \tilde{X}\in\tilde{X}\}$$

$$\varphi_2(X,Y,Z)=\{\overline{Y}\in\tilde{Y} \mid F(X,Y,Z,\overline{Y})\geq G(X,Y,Z,\tilde{Y}) \text{ for all } \tilde{Y}\in\tilde{Y}\}$$

$$\varphi_3(X,Y,Z)=\prod_{k=1}^{p}\{\tilde{Z}_k\in\phi_k(Y) \mid U_k(\overline{Z}_k)\geq U_k(\tilde{Z}_k) \text{ for all } \tilde{Z}\in\phi_k(Y)\}$$

Because the assumption AIV) holds, the correspondence is upper semicontinuous and because AV) holds, too, the correspondence is convex. Then, the Kakutani fixed point works and a point $(\overline{X},\overline{Y},\overline{Z})\in\tilde{X}\times\tilde{Y}\times Z$ exists such that it belongs to $\varphi(\overline{X},\overline{Y},\overline{Z})$, too. Such a vector satisfies the following inequalities

$$0=F(\overline{X},\overline{Y},\overline{Z},\overline{Y})\geq F(\overline{X},\overline{Y},\overline{Z},Y) \text{ for all } Y\in\tilde{Y} \tag{1}$$

$$0=G(\overline{X},\overline{Y},\overline{Z},\overline{X})\geq F(\overline{X},\overline{Y},\overline{Z},X) \text{ for all } X\in\tilde{X} \tag{2}$$

and

$$U_k(\overline{Z}_k)\geq U_k(Z_k) \text{ for all } Z_k\in\phi_k(\overline{Y}).$$

These last relations imply E3).

From (1) we obtain the following

$$(\sum_{ij} b_{ij}\overline{X}_i\overline{Y}_j-\sum_{kj}\overline{Z}_{kj}\overline{Y}_j)(\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j)\geq \tag{3}$$
$$(\sum_{ij} b_{ij}\overline{X}_i\overline{Y}_j-\sum_{kj}\overline{Z}_{kj}\overline{Y}_j)(\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j) \text{ for all } Y\in\tilde{Y}$$

Due to the fact that $Z_k\in\phi_k(Y)$ for all k, it follows that $\sum_{kj}\overline{Z}_{kj}\overline{Y}_j$ and (3) can be put in the following simpler from

$$(\sum_{ij} b_{ij}\overline{X}_i\overline{Y}_j)(\sum_{ij} b_{ij}\overline{X}_i\overline{Y}_j)\leq(\sum_{ij} b_{ij}\overline{X}_i\overline{Y}_j-\sum_{kj}\overline{Z}_{kj}\overline{Y}_j)(\sum_{ij} b_{ij}\overline{X}_i\overline{Y}_j) \tag{4}$$

for all $Y\in\tilde{Y}$.

Now we are going to prove that $\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j>0$. See that if this were not true, the right hand side of (4) would have to be equal to zero. On the other hand, A1) implies that $\sum_{ij} b_{ij}\overline{X}_i\overline{Y}_j>0$, and AII) - that there exists a vector $Y\in\tilde{Y}$ such that $\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j>0$. Thus, the left hand side of (4) is greater than zero for this choice of y. But this leads to a contradiction.

If we let Y be the vector $(0,\ldots,1,\ldots,0)$, with one in the j-th component and zero elsewhere, we obtain from (4) that

$$(\sum_{ij} b_{ij}\overline{X}_i\overline{Y}_j)(\sum_{ij} a_{ij}\overline{X}_i)\leq(\sum_{ij} b_{ij}\overline{X}_i-\sum_{kj}\overline{Z}_{kj})(\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j)$$

and if we put $\alpha=(\sum_{ij} b_{ij}\overline{X}_i\overline{Y}_j)/(\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j)$ we obtain E(1). Of course, it remains to show that $\overline{\alpha}>0$, but using (2) we obtain that

$$(\Sigma_{ij} b_{ij}\overline{X}_i\overline{Y}_j)(\sum_{ij} a_{ij}X_i\overline{Y}_i)\geq(\sum_{ij} b_{ij}X_i\overline{Y}_j)(\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j) \tag{5}$$

The assumption AIII) implies that there exists a point $X\in X$ such that $\sum_{ij} b_{ij}X_i\overline{Y}_j>0$ and because $\sum_{ij} a_{ij}\overline{X}_i\overline{Y}_j$ holds, we obtain that $\sum_{ij} b_{ij}X_iY_j>0$. This in turn implies that $\overline{\alpha}>0$. Moreover, letting $\overline{\beta}=\overline{\alpha}$ and X being the vector $(0,\ldots,1,\ldots,0)$ with one in the i-th component and zero elsewhere we can easily obtain E2) from (5). Relation E5) is obtained in a straightforward way and this completes the proof. ■

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O EKONOMICZNYM MODELU WYMIANY I ROZWOJU

W artykule zaproponowano zintegrowany model Gale'a i von Neumana otwartej gospodarki. Główny przedstawiony wynik to stwierdzenie istnienia równowagi mającej własności zarówno równowagi z rozwojem jak i równowagi z wymianą.

ОБ ЭКОНОМИЧЕСКОЙ МОДЕЛИ ОБМЕНА И РАЗВИТИЯ

В статье предлагается интегрированная модель Гейля и фон Ньюмена открытой экономики. Основной представленный результат состоит в утверждении существования равновесия имеющего свойства как равновесия развития так и равновесия обмена.