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OPTIMAL DISCOUNTED CONTROL OF THE BILINEAR DIFFUSION PROCESS AND ITS APPLICATION TO ECONOMICS[1]

by

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We solve the following stochastic control problem: to maximize the discounted total benefit

$$J(x,u)=E\int_0^\infty \exp(-rt)[x_t u_t-\phi(u_t)]\,dt,$$

subject to

$$dx_t=x_t[(A-Bu_t)dt+\delta dw_t],\ x_o=x>o,\ o\le u_t\le\bar{u},\ r>o,$$

where $\{w_t, t\ge 0\}$ is a Wiener process and $\phi(\cdot)$ is a nonnegative increasing concave, continuous function on $[o,u]$. All bounded, measurable and nonanticipating functionals $u(x)$ of the state process x are admissible as controls. Optimality of the bang-bang control is proved and the switching point is found. Applications of this result might include production policy of a firm aiming to maximize the expected profit $J(x,u)$, where u_t is the production rate and x_t is the price of the good produced.

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[1] Mathematical results were obtained by the author himself while application to economics was widely discussed with O.Gedymin. Thus this part of the paper is a common contribution.

1. Introduction

We consider the problem of optimal control of the bilinear diffusion process

$$dx_t=x_t[(\alpha-\beta u_t)dt+\sigma dw_t], \quad x_0=x, \quad t\geq 0 \tag{1.1}$$

where α, $\beta>0$, $\sigma>0$ are constants and $\{w_t, t\geq 0\}$ is a Wiener process on an appropriate probability space. There is a cost $\Phi(u_t)$ per unit time for using control u_t, where $\Phi(\cdot)$ is a nonnegative, increasing, concave, continuous function on $[o,\overline{u}]$. The controller has to choose a law $u_t(x)$ as a non-anticipative, measurable functional of the state process with values in $[o,\overline{u}]$ to maximize the discounted total benefit

$$J(x,u)=E\int_0^{\infty} e^{-rt}[x_t u_t-\Phi(u_t)]dt \tag{1.2}$$

This problem has a potential application to economics: u_t being the factory production rate at time t and x_t being the price of the good being produced (see section 5).

The economically obvious rule for our model is to produce with full capacity if x_t is bigger than some critical price δ, and to exert no control at all if $x_t<\delta$. Optimality of this law is proved and the cutoff point δ separating both regions of the price values is characterized in terms of parameters of the system.

General existence results for the problem of discounted stochastic control were given by Kushner , [1], and Bensousson, [2]. Beneš, Shepp and Witsenhausen, [3], and Karatzas, [4], proved the optimality of the bang-bang law for the control of the Wiener process, while Bensousson, Sethi, Vickson and Derzko, [5], showed that an optimal feedback solution exists for the LQG problem with control nonnegativity constraints.

This paper is organized as follows: we state the control problem in Section 2. An explicit solution of the Bellman equation is given in Section 3. Optimality of the bang-bang law is proved in Section 4. An economic application is presented in Section 5.

2. Stochastic control problem

Consider, as a basic probability space Ω, the space $C(\mathbb{R}^+)$ of continuous, real-valued functions in $\mathbb{R}^+$, and let $\mathcal{F}_t$, $t\geq 0$, denote the σ-algebra generated by $\{x_s;\ s\leq t\}$, $x\in\Omega$. Consider also the σ-algebra $\mathcal{M}$ generated by the Borel subsets M of $\mathbb{R}^+\times C(\mathbb{R}^+)$, with the property that each t-section of M_t of M belongs to $\mathcal{F}_t$ and each x-section M_x of M is Lebesgue measurable. A function $g:\mathbb{R}^+\times C(\mathbb{R}^+)\to\mathbb{R}$ is $\mathcal{M}$-measurable iff $g(t,\cdot)$ is $\mathcal{F}_t$-measurable for any $t\geq 0$ and $g(\cdot,x)$ is Lebesgue measurable for any $x\in C(\mathbb{R}^+)$.

An admissible nonanticipative control u is an $\mathcal{M}$-measurable function

$$u:\mathbb{R}^+\times C(\mathbb{R}^+)\to[o,\bar{u}],\quad \bar{u}>o.$$

The class of such controls is denoted by $\mathcal{U}$. For any control law $u\in\mathcal{U}$ and any $x>0$ we can construct, by means of the Girsanov theorem, a probability space $(\Omega,\mathcal{F},P)$ and a pair of stochastic processes (y_t,w_t) on it, such that $\{w_t;\ t\geq 0\}$ is a Wiener process with respect to P and the *SDE*

$$dy_t=\ (\alpha+\frac{\delta^2}{2}-\mathrm{B}u_t)dt+\delta dw_t,\quad y_o=ln\ x, \tag{2.1}$$

is satisfied. Such a weak solution of (2.1) is known to be unique in the sense of the probability law (see [6]).
Now, from Itô's formula we see that $x_t=\exp y_t$ is a unique weak solution of the *SDE*

$$dx_t=x_t[(\alpha-\beta u)dt+\sigma dw_t],\quad x=x>0. \tag{2.2}$$

The control problem consists in finding a law $u^*\in\mathcal{U}$ that maximizes

$$J(x,u)=E\int_0^\infty e^{-rt}[x_tu_t-\Phi(u_t)]dt \tag{2.3}$$

when we start at x and use the control u, over all $u\in\mathcal{U}$, $x>0$. Here, E denotes expectation with respect to the probability measure P, $r>o$ is the discount factor, $\Phi(\cdot)$ is the running cost of control.
Now let $u_t=v=$ const, $v\in[o,\bar{u}]$, and let $x_t=x_t^v$ when the control v is applied. Then $Ex_t^v=x\exp(\alpha-\beta v)t$ and

$$J(x,v)=E\int_0^\infty e^{-rt}[vx_t^v-\Phi(v)]\,dt=xv\int_0^\infty e^{-(r-\alpha+Bv)t}dt-\frac{\Phi(v)}{r}=$$

$$=\begin{cases}\dfrac{xv}{r-\alpha+\beta v}-\dfrac{\Phi(v)}{r} & \text{if } r>A\\[2mm] +\infty, \quad \text{for } v\in(0,\dfrac{A-r}{\beta}) & \text{if } r<A\end{cases}$$

thus in the following we assume $r > A$.

3. The Bellman equation

To solve the Bellman equation

$$rV=\frac{\delta^2}{2}x^2V''+AxV'+\sup_{0\le u\le u}[xu-\Phi(u)-BuxV'] \tag{3.1}$$

we look for a number $\delta > 0$ and solution V, $|V(x)|\le M(1+|x|)$, to

$$x^2V''+\frac{2(A-B\bar{u})}{\delta^2}xV'-\frac{2r}{\delta^2}V-\frac{2(\bar{\Phi}-\bar{u}x)}{\delta^2}\;,\quad \bar{\Phi}\triangleq\bar{\Phi}(\bar{u}) \tag{3.2}$$

$$x(1-\beta V')>\frac{\bar{\Phi}-\Phi_0}{\bar{u}},\qquad x>\delta \qquad \Phi(o)=\bar{\Phi}_o$$

$$x^2V''+\frac{2\alpha}{\sigma^2}xV'-\frac{2r}{\sigma^2}V=\frac{2\Phi_o}{\sigma^2} \tag{3.3}$$

$$x(1-\beta V')<\;,\frac{\bar{\Phi}-\bar{\Phi}_o}{\bar{u}}\qquad x<\delta$$

which meet smoothly at $x=\delta$ to order 1.

As a particular solution to (3.2) we find

$$\frac{\bar{u}x}{r+B\bar{u}-A}-\frac{\bar{\Phi}}{r}. \tag{3.4}$$

Now, let

$$\nu^{\pm}=\frac{1}{2}-\frac{\alpha-B\bar{u}}{\delta^2}\pm\sqrt{\left(\frac{1}{2}-\frac{\alpha-\beta\bar{u}}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}$$

$$\rho^{\pm}=\frac{1}{2}-\frac{\alpha}{\sigma^2}\pm\sqrt{\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}\,.$$

To get the general solution of (3.2) we add to (3.4) a solution

$$c_1^- x^{\nu^-} + c_1^+ x^{\nu^+}$$

of the homogeneous equation

$$x^2 V'' + \frac{2(\alpha-\beta \bar{u})}{\sigma^2} xV' - \frac{2r}{\sigma^2} V = 0.$$

Since $\nu^+ > 1$ for $r \geq \alpha$, the linear growth condition implies $c_1^+ = 0$, and we solve (3.2) by

$$V_1(x) = c_1^- x^{\nu-} + \frac{\bar{u}x}{r+\bar{u}-\alpha} - \frac{\bar{\Phi}}{r}, \qquad x > \delta. \tag{3.5}$$

A similar argument solves (3.3) as

$$V_2(x) = c_2^+ x^{\rho^+} - \frac{\Phi_o}{r}, \qquad 0 \leq x < \delta \tag{3.6}$$

Let us denote $\nu \triangleq \nu^-$, $\rho \triangleq \rho^+$.

Then we see that $\nu < 0$ and $r > \alpha$ implies

$$\rho = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + 2\frac{r-\alpha}{\delta^2}} > \frac{1}{2} - \frac{\alpha}{\sigma^2} + \left|\frac{1}{2} + \frac{\alpha}{\delta^2}\right| \geq 1.$$

We want the derivatives V_1' and V_2' to be equal to

$\frac{1}{\beta}\left(1 - \frac{\bar{\Phi}-\bar{\Phi}_o}{\bar{u}\delta}\right)$ at $x=\delta$, and $V_1(\delta) = V_2(\delta)$.

This determines δ, c_1^-, c_2^+ as

$$\delta = \frac{(r-\alpha+\beta\bar{u})[(\rho-\nu)r+\beta\bar{u}\rho\nu](\bar{\Phi}-\Phi_0)}{r\bar{u}[(\rho-\nu)(r-a)+\beta\bar{u}\nu(\rho-1)]} \tag{3.7}$$

$$c_1^- \triangleq c_1 = \frac{1}{\beta\nu}\left[\frac{(r-\alpha)\delta}{r-\alpha+\beta\bar{u}} - \frac{\bar{\Phi}-\Phi_0}{\bar{u}}\right]\delta^{-\nu} \tag{3.8}$$

$$c_2^+ \triangleq c_2 = \frac{1}{B\delta}\left[\delta - \frac{\bar{\Phi}-\Phi_o}{\bar{u}}\right]\delta^{-\rho} \tag{3.9}$$

To verify that the function (3.5), (3.6), (3.7), (3.8), (3.9) solves (3.2) and (3.3), we have to show that the following inequalities hold

$$\beta x V_2' > x - \frac{\Phi-\Phi_o}{\bar{u}} \quad \text{in} \quad 0 \le x < \delta \tag{3.10}$$

$$x V_1' < x - \frac{\Phi-\Phi_o}{\bar{u}} \quad \text{in } x > \delta. \tag{3.11}$$

But

$$\beta x V_2' = \left(\delta - \frac{\bar{\Phi}-\Phi_o}{\bar{u}}\right) \left(\frac{x}{\delta}\right)^{\rho} > x - \frac{\bar{\Phi}-\Phi_o}{\bar{u}} \quad \text{for } x \text{ small enough}$$

and the equation

$$\left(\delta - \frac{\bar{\Phi}-\Phi_o}{\bar{u}}\right) \left(\frac{x}{\delta}\right)^{\rho} = x - \frac{\bar{\Phi}-\Phi_o}{\bar{u}}$$

has one positive solution $x=\delta$. Thus (3.11) holds.

Now

$$\beta x V_1' = \frac{1}{r-\alpha+\beta\bar{u}} \left[\beta \bar{u} x + (r-\alpha) x \left(\frac{x}{\delta}\right)^{\nu-1}\right] - \frac{\bar{\Phi}-\Phi_o}{\bar{u}} < x - \frac{\bar{\Phi}-\Phi_o}{\bar{u}}$$

since $\left(\frac{x}{\delta}\right)^{\nu-1} < 1$ for $x > \delta$.

4. Optimal control

THEOREM 4.1. *Let $r > \alpha$. The optimal control for the problem (2.2), (2.3) is*

$$u_t^* = \begin{cases} 0, & \text{if } x_t^* < \delta \\ \bar{u}, & \text{if } x_t^* > \delta , \end{cases} \tag{4.1}$$

where δ is given by (3.7) and x_t^* solves

$$dx_t^* = x_t^*[(\alpha-\beta u_t^*)dt+\delta dw_t], \quad x_0 = x. \tag{4.2}$$

P r o o f. Introduce the process $M_t \triangleq e^{-rt}V(x_t) + \int_0^t e^{-rs}[x_s u_s - \Phi(u_s)]ds$. Jtô's formula yields the representation for $0 \le s \le t$

$$M_t = M_s + \int_s^t e^{-r\tau}\left[\frac{\sigma^2}{2} x_\tau^2 V''(x_\tau) + x_\tau(\alpha - \beta u_\tau)V'(x_\tau) + x_\tau u_\tau - \Phi(u_\tau) - rV(x_\tau)\right]d\tau + \sigma \int_s^t e^{-r\tau} x_\tau V'(x_\tau)dw_\tau \tag{4.3}$$

Taking conditional expectations relative to $\mathcal{F}_s^W = \sigma\{w_\tau, \tau \le s\}$ and using (3.1) we see that $E(M_t | \mathcal{F}_s^W) \le M_s$ and $EM_t \le EM_s$. But for $s=0$

$$EM_t = e^{-rt}EV(x_t) + E\int_0^t e^{-rs}[x_s u_s - \Phi(u_s)]ds \le EM_0 = V(x). \tag{4.4}$$

The linear growth condition for the Bellman function $|V(x)| \le M(1 + |x|)$, implies

$$E|V(x_t)| \le M(1 + x \exp \alpha t) \tag{4.5}$$

and $\lim_{t\to\infty} e^{-rt}V(x_t) = 0$.

Thus letting $t\to\infty$ in (4.4) we get

$$E\int_0^\infty e^{-rt}[x_t u_t - \Phi(u_t)]dt \le V(x).$$

This shows that the expected benefit of using $u \in \mathcal{U}$ is not bigger than $V(x)$. Consider now the law $u^* \in \mathcal{U}$. The integral following M_s in (4.3) has zero conditional expectation relative to the past of w, M is a martingale, and $EM_t = V(x)$, so that x_t^* achieves V. In other words, u_t^* is optimal. ∎

5. Application to monopolistic price adjustment under uncertainty[2]

Consider a firm which produces a homogeneous nonstorable commodity. We assume that the firm has a certain degree of monopolistic power and it makes price and output

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[2] For related problems see [9], [10].

decisions to maximize its discounted profit over time, while demand for its product is perturbed by the fluctuations of a haphazard nature. Our problem now is to build the price-output planning model and to find the optimal solutions. Proceeding in the spirit of Merton [7] we shall try to determine the equation describing the price dynamics.

Assume that the relative price rate $(x_{t+dt}-x_t)/x_t$, for t fixed, is a random variable S, such that (i) S is a sum of a large number n of random variables s_k, $S=\sum_{k=1}^{n} s_k$, (ii) s_k are stochastically independent, (iii) it is highly probable that each variable s_k is sufficiently small. Taking into account the above assumptions, and applying the central limit theorem, we obtain an asymptotical Gaussian distribution of the relative price rate, which has a mean $b(t)dt$ and variance $\sigma^2(t)dt$. Then passing to the limit as $n \to \infty$, we may write $S=b(t)dt+\sigma(t)dw_t$, where $\{w_t, t\geq 0\}$ is a Wiener process defined on an appropriate probability space, and finally the price dynamics equation is given by

$$dx_t=x_t[b(t)dt+\sigma(t)dw_t] \tag{5.1}$$

We are going now to specify the functions $b(\cdot)$, $\sigma(\cdot)$ in the case when the process (5.1) is to be controlled.

In the theory of the monopolistic firm it is usually accepted that the firm faces a price-output relation (demand curve) which is downward-sloping (see [8],[12]). One of the possible variants of such a relation (in the mean) could be given by the equation

$$\frac{dx}{dt}=(\alpha-\beta u)x, \quad \alpha<o \tag{5.2}$$

where $x_t=Ex_t$.

Comparing (5.1) and (5.2) we obtain equation (1.1).

We assume that the production cost $\Phi(u)$ is a nonnegative, increasing, concave, continuous function on $[0,\bar{u}]$, where $\bar{u}$ denotes the maximum production capacity of the firm. Thus the profit maximization problem for the firm can be stated as follows

$$\sup_{u\in\mathcal{U}} E\int_0^\infty e^{-rt}[x_t u_t-\Phi(u_t)]dt$$

subject to

$$dx_t=x_t[(\alpha-\beta u_t)dt+\sigma dw_t], \quad x_0=x>0.$$

The set $\mathcal{U}$ of admissible controls is defined in section 2.

6.Conclusion

(i) In the present paper a closed-form solution is given for the problem of controlling a bilinear SDE with control variable constraints.We used methods which follow the spirit of Beneš [3] and Karatzas [4].It turns out that the method originated by Beneš and his collaborators [3] can be adapted to the maximization problem studied in this paper.

(ii) From the economic point of view, this paper studies the optimal dynamic behaviour of the monopolistic firm under uncertainty. Theorem 4.1. shows that the bang-bang control, which is in fact the event planning (in terminology of Intriligator and Sheshinski, [11]), is superior to all nonanticipative controls, including time planning.

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STEROWANIE OPTYMALNE Z DYSKONTEM BILINIOWEGO PROCESU DYFUZJI I JEGO ZASTOSOWANIA EKONOMICZNE

W artykule rozwiązuje się zadanie sterowania stochastycznego, w którym maksymizuje się całkowity zdyskontowany zysk:

$$J(u,u)=E\int_{o}^{\infty}\exp(-rt)[x_t u_t-\phi(u_t)]dt$$

przy ograniczeniach

$$dx_t=x_t[(A-Bu_t)dt+\delta dw_t],\quad x_o=x>0,\quad 0\le u_t\le u,\quad r>0,$$

gdzie $\{w_t, t\ge 0$ jest procesem Wienera, zaś $\phi(\cdot)$ jest nieujemną rosnącą, wklęsłą i ciągłą funkcją na $[0, u]$. Wszystkie ograniczone, mierzalne i niewyprzedające funkcjonały $u_t(x)$ stanu procesu x_t są dopuszczalnymi sterowaniami. Dowodzi się optymalności sterowania typu bang-bang i determinuje się punkt przełączenia.

Zastosowania przedstawionych wyników mogą dotyczyć np. planów produkcyjnych firm zmierzających do maksymalizacji oczekiwanego zysku $J(x,u)$, gdzie u_t jest intensywnością produkcji a x_t jest ceną produktu.

ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ С ДИСКОНТИРОВАНИЕМ БИЛИНЕЙНОГО ПРОЦЕССА ДИФФУЗИИ И ЕГО ПРИМЕНЕНИЯ В ЭКОНОМИКЕ

В статье решается задача стохастического управления, в которой максимизируется полная дисконтированная прибыль:

$$J(u,u)=E\int_{0}^{\infty} \exp(-rt)[x_t u_t-\phi(u_t)]dt$$

при ограничениях

$$dx_t=x_t[(A-Bu_t)dt+\delta dw_t], \quad x_0=x>0, \quad 0\le u_t\le u, \quad r>0,$$

где: $\{w_t, t\ge 0$ является винеровским процессом, а $\phi(\cdot)$ является неотрицательной ,возрастающей вогнутой и непрерывной функцией на $[0,\bar{u}]$. Все ограниченные, измеряемые и неопережающие функционалы $u_t(x)$ состояния процесса x_t являются допускаемыми управлениями. Доказывается оптимальность управления типа банг-банг и определяется точка переключения.

Применения представленных результатов могут касаться например производственных планов фирм стремящихся к максимизации ожидаемой прибыли $J(x,u)$, где u_t является интенсивностью производства, а x_t цена продукта.