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Noisy duel with retreat after the shots

by

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In the paper noisy duel is considered in which duelists (Players I and II) remove after firing all their bullets. Solved are cases of $m=1$, $n=1$; $m=2$, $n=1$; and $m=1$, $n=2$ where m, n are numbers of bullets of Players I and II, respectively.

1. Assumptions. Definition of the game

Assume that two players, Player I and Player II come up to each other. Player I moves with constant velocity v_1, Player II moves with constant velocity v_2, $v_1 \geq v_2$. Players I and II fight in a duel. Player I has m bullets (missiles), Player II has n bullets (missiles).

Without loss of generality we can suppose that at the moment $t=0$ players are in the distance 1 from each other and that $v_1+v_2=1$. Then if any of the players did not turn back and was not destroyed, they will meet each other at the moment $t=1$.

Denote by $P(t)$ the probability that Player I, II achieves a success (hits the opponent) if firing at distance corresponding to $1-t$. The function $P(t)$ is called accuracy function. We assume that $P(t)$ is increasing and continuous in $[0,1]$, twice differentiable (with continuous second derivative) in $(0,1)$ and that $P(0)=0, P(1)=1$.

Player I gains 1 if he is the only one who succeeds, and gains -1 if only Player II succeeds and gains 0 in the remaining cases. The duel is a zero-sum game.

It is assumed that duel is noisy - the player hears the shot of his opponent.

When one of the players fired all his bullets his motion in the direction of the opponent is no longer senseful. We shall assume, then, that a player evades after firing all his bullets.

The duel defined in such a way shall be denoted (m,n).

Let $v_1>v_2$ and suppose that Player I has fired all his bullets and evades. In this case Player II will do the best if he fires all his bullets immediately after the last shot of I. If, on the other hand, Player II has fired all his bullets and Player I has some bullets yet, the best he can do is to reach the opponent in a pursuit and thereby to surely achieve the success.

At the beginning let us consider the case when duel is carried in the interval $[0,1]$ and Players I and II have one bullet each. Let $K_0(s,t)$ be the expected gain for Player I in this duel when Players I and II fire at the moments s and t, respectively. Then under assumptions made we obtain in the limit, when Player II fires immediately after I, if he is not hit, that

$$K(s,t) = \begin{cases} P(s) - (1-P(s))P(s) & \text{if } s<t, \\ 0 & \text{if } s=t, \\ -P(t) + 1-P(t) & \text{if } s>t. \end{cases}$$

Then we have

$$K_0(s,t) = \begin{cases} P^2(s) & \text{if } s<t, \\ 0 & \text{if } s=t, \\ 1-2P(t) & \text{if } s>t. \end{cases}$$

Let a_{11} be a number such that

$$P^2(a_{11}) = 1-2P(a_{11}),$$

i.e. $P(a_{11}) = \sqrt{2}-1$. It is easy to show that an optimal (it means minimax) strategy of Player II is to fire at the moment $t=a_{11}$ (if Player I did not fire before) and that ε-optimal (it means ε-maximin) strategy of Player I is to choose at random, with an absolute continuous probability distribution (*ACPD*) the moment of his shot in the time interval $(a_{11}, a_{11}+\alpha(\varepsilon))$ (if Player II did not fire before), where $\alpha(\varepsilon) \to 0$ if $\varepsilon \to 0$.

The value of the game in this case is

$$v_{11} = 1-2P(a_{11}) = 3-2\sqrt{2}.$$

In the classical case, see [3,6,12,13,18], when players do not evade after firing all their bullets the optimal strategies of players are the same as above but the number a_{11} satisfies the equation $P(a_{11})=1/2$ and the value of the game is zero. Then, in the duel considered in this paper it is necessary to fire sooner than in the classical duel, and this which player has greater speed has substantial influence on the value of the game.

Suppose now that the duel is carried out in the interval $[a,1]$, meaning that the distance of Players at the beginning of the duel is $1-a$. This duel will be denoted by (1.1), $[a, 1]$. To simpify considerations we shall compute the time also from $t=a$. It is easy to see that if $a\le a_{11}$ ($P(a)\le\sqrt{2}-1$) then optimal strategies defined before remain the same.

Let now a be a number such that

$$\sqrt{2}-1<P(a)<1/2.$$

In this case an ε-optimal strategy of Player I is to choose at random the moment of the shot with an ACPD in the interval $(a,a+\alpha(\varepsilon))$ and the optimal strategy of Player II is to fire the shot at the moment a. The value of the game in this case is

$$v_{11}^{a}=1-2P(a).$$

Let $P(a)>1/2$. In this situation the strategy "fire at the moment a" is optimal for both players and the value of the game is 0.

As we see, in the last of the three cases considered Player I has none benefit from his greater speed.

When $v_1=v_2$, one can fix, without any effect as to the value of the game, that both players behave in the same way after the shot of opponent as Player II before. Then, the playoff function has the form

$$K_0(s,t) = \begin{cases} P^2(s) & \text{if } s<t, \\ 0 & \text{if } s=t \\ -P^2(t) & \text{if } s>t. \end{cases}$$

The value of the game is 0, $v_{11}^{a}=0$, and optimal strategies for Players are "to fire the shot at the moment a" - at the beginning of the time interval during which the duel is carried out.

In all situations the value of the game depends only on the number a and on this which of the cases: $v_1<v_2$, $v_1=v_2$, $v_1>v_2$, occurs.

It is assumed in the following that a Player cannot fire two of his bullets at the same time.

For definitions and results in the game of timing theory see [1,2,4,5,7,10,15].

2. Duel (2,1) in the interval $[a,1]$

Let us consider the case in which Player I has two bullets and Player II has one bullet. Remaining assumptions made in Section 1 are the same. Then, a player, after firing all his shots, evades, and the players hear the shots of their opponent.

Let us suppose that $v_1 > v_2$. Without loss of generality, similarly as in the Section 1, to simplify the analysis, we assume that after the shot of Player II, Player I waits to benefit shoot to the moment of meeting, and after the second shot of Player I, Player II fires immediately after I.

Case 1. $P(a) \le P(a_{21}) = 1 - \frac{\sqrt{2}}{2}$.

Denote by ξ and η the following strategies of Players I and II.

Strategy of Player I: If Player II had not fired before, fire a shot at the moment a_{21}, and if Player II had not fired at a_{21}, play ε-optimally the duel (1,1).

Strategy of Player II: If Player I had not fired before, fire the shot according to an ACPD in the interval $(a_{21}, a_{21}+\alpha(\varepsilon))$. If he had fired, play optimally the duel (1,1).

Denote by $K(\hat{\xi},\hat{\eta})$ the expected gain of Player I for strategies $\hat{\xi}$ and $\hat{\eta}$ (may be randomized) of Players I and II. Since $a_{21} < a_{11}$ we have for strategies ξ and η.

$$K(\xi;\eta) \cong P(a_{21}) + (1-P(a_{21}))\, v_{11} \overset{df}{=} v_{21}, \qquad (1)$$

where approximate equality $\cong$ holds with accuracy to the constant ε if strategies ξ and η are ε-optimal, and v_{21} is the value of the game.

Assume that Player II fires at the moment a_{21} (this strategy shall be denoted simply by a_{21}) and that Player I fires after a_{21} according to a strategy $\hat{\xi}$. We have

$$K(\hat{\xi}:a_{21})=-P(a_{21})+1-P(a_{21}).$$

Determine the number a_{21} in such a way that $K(\hat{\xi};a_{21})=v_{21}$. In this case we obtain, by comparing equations (1) and (2)

$$P(a_{21})=\frac{1-v_{11}}{3-v_{11}}=1-\frac{\sqrt{2}}{2}$$

Moreover, from formula (1) we obtain that

$$v_{21}=\sqrt{2}-1.$$

We prove that strategies ξ and η are ε-optimal if the constant a_{21} satisfies condition (3).

Let a' be the moment of the shot of Player II. For $a'<a_{21}$ we have

$$K(\xi;a')=-P(a')+1-P(a')>1-2P(a_{21})=v_{21}.$$

Let $a'=a_{21}$. There is

$$K(\xi;a')=(1-P(a_{21}))^2=1/2>\sqrt{2}-1=v_{21}.$$

Let $a'>a_{21}$. In this case, denoting by K^1 the payoff function in the duel (1,1), and denoting by ξ_1 an ε-optimal strategy in this duel, we obtain

$$K(\xi;a')=P(a_{21})+(1-P(a_{21}))K^1(\xi_1;\alpha')\geq$$

$$\geq P(a_{21})+(1-P(a_{21}))(v_{11}-\varepsilon)\geq v_{21}-\varepsilon.$$

Then, Player I, by applying the strategy ξ in the duel (2,1) assures (in mean) for himself at least $v_{21}-\varepsilon$.

Let now $(a',\hat{\xi})$ be the strategy of Player I, in the duel (2,1), such that he fires at the moment a' and afterwards he plays according to the strategy $\hat{\xi}$. Let a_{21}^{ε} be the moment of the shot of Player II according to his (random) strategy η. We have, if $a'<a_{21}$,

$$K(a',\hat{\xi};\eta)=P(a')+(1-P(a'))K^1(\hat{\xi};\eta)\leq$$

$$\leq P(a')+(1-P(a'))(v_{11}+\varepsilon)<P(a_{21})+(1-P(a_{21}))v_{11}+\varepsilon=$$

$$=v_{21}+\varepsilon. \qquad (5)$$

When $a'>a_{21}+\alpha(\varepsilon)$ we obtain

$$K(a',\hat{\xi};\eta)\leq 1-2P(a_{21})+\varepsilon=v_{21}+\varepsilon. \qquad (6)$$

Here ε is a constant determined by the support of random variable a_{21}^{ε} and the function $P(t)$.

From the formulae (5) and (6) it follows that Player II, applying the strategy η, will lose at most $v_{21}+\varepsilon$ (in mean).

Then, the strategies ξ and η of Players I and II are ε-optimal. The value of the game is given by (4).

Case 1, considered above, occurs when constant a in the interval $[a,1]$ satisfies the condition

$$P(a)\leq P(a_{21})=1-\frac{\sqrt{2}}{2} \qquad (7)$$

CASE 2. $1-\frac{\sqrt{2}}{2}<P(a)\leq\sqrt{2}-\sqrt{4-2\sqrt{2}}$.

Let us consider the following strategies ξ, η of Players I and II.

Strategy of Player I: Fire a shot at a and if Player II did not fire at a play ε-optimally the duel (1,1).

Strategy of Player II: Fire the shot at a^{ε}. If Player I had fired before play optimally the duel (1,1).
Here a^{ε} is a random moment defined similarly as a_{21}^{ε}.

We show that above strategies are ε-optimal. We have

$$K(\xi;\eta)\cong P(a)+(1-P(a))v_{11}\overset{df}{=}v_{21}$$

where the equality $\cong$ holds with accuracy to the constant ε.

Let $a'=a$. We have

$$K(\xi;a')=(1-P(a))^2\geq v_{21}^{a}$$

if

$$P(a)\le\sqrt{2-\sqrt{4-2\sqrt{2}}}\ . \tag{8}$$

Let $a'>a$. In this case

$$K(\xi;a')\ge P(a)+(1-P(a)(v_{11}-\varepsilon)\ge v^{a}_{21}-\varepsilon.$$

Then, if constant a fulfills the condition (8) Player I assures for himself the value $v^{a}_{21}-\varepsilon$.

Consider the strategy η of Player II. Let $a'<a$. Then

$$K(a',\hat{\xi};\eta)\le P(a)+(1-P(a))v_{11}+v^{a}_{21}+\varepsilon.$$

In the case $a'>a+\alpha(\varepsilon)$ we have

$$K(a',\hat{\xi};\eta)\le 1-2P(a)+\varepsilon.$$

There should be

$$1-2P(a)\le P(a)+(1-P(a))v_{11}=v^{a}_{21},$$

meaning that

$$P(a)\ge\frac{1-v_{11}}{3-v_{11}}=P(a_{21})=1-\frac{\sqrt{2}}{2}\ .$$

Now the strategy ξ is ε-maximin and η is ε-minimax if

$$0.29289\ldots=1-\frac{\sqrt{2}}{2}<P(a)\le\sqrt{2}-\sqrt{4-2\sqrt{2}}=0.33182\ldots\ .$$

CASE 3. $\sqrt{2}-\sqrt{4-2\sqrt{2}}\le P(a)<\sqrt{2}-1$.

Let us consider the strategies ξ and η of Players I and II.

Strategy of Player I: Fire a shot at a and if Player II did not fire at a play ε-optimally the duel (1,1).

Strategy of Player II: Fire the shot at a.

Now

$$K(\xi;\eta) = (1-P(a))^2 \stackrel{df}{=} v^a_{21}.$$

Now

$$K(\xi;\eta)=(1-P(a))^2\stackrel{df}{=}v^a_{21}.$$

Let $a'>a$. We have

$$K(\xi;a')\geq P(a)+(1-P(a))(v_{11}-\varepsilon)\geq(1-P(a))^2-\varepsilon=v^a_{21}-\varepsilon$$

if

$$P(a)\geq\sqrt{2}-\sqrt{4-2\sqrt{2}}\,.$$

On the other hand, for $a'>a$

$$K(a',\hat{\xi};\eta)=1-2P(a)\leq(1-P(a))^2.$$

However, the strategy ξ can be realized if $P(a)<P(a_{11})$. Then strategy ξ is ε-optimal and η is optimal when

$$0.33182...=\sqrt{2}-\sqrt{4-2\sqrt{2}}\leq P(a)<\sqrt{2}-1=0.41421...\ .$$

CASE 4. $P(a)\geq\sqrt{2}-1$.

To determine optimal strategies in the remaining case $P(a)\geq\sqrt{2}-1$ we introduce new notations and assumptions. In further part of the paper we shall assume that between successive shots of the same player the time period $\hat{\varepsilon}$ has to elipse.

Let (m,n), $[a,1]$ be the duel defined similarly as the duel $(1,1)$, $[a,1]$. We say that Player I assures in limit the value u_1 in this duel if for each $\varepsilon>0$, $\hat{\varepsilon}>0$ he has strategy $\xi_{\varepsilon\hat{\varepsilon}}$ such that

$$K(\xi_{\varepsilon\hat{\varepsilon}};\hat{\eta})\geq u_1-k_1(\varepsilon,\hat{\varepsilon})$$

for any strategy $\hat{\eta}$ of Player II, where $k(\varepsilon,\hat{\varepsilon})\to 0$ when $\varepsilon\to 0$, $\hat{\varepsilon}\to 0$.

Similarly, Player II assures in limit the value u_2 if for each $\varepsilon>0$, $\hat{\varepsilon}>0$ he has a strategy $\eta_{\varepsilon\hat{\varepsilon}}$ such that

$$K(\hat{\xi};\eta_{\varepsilon\hat{\varepsilon}})\le u_2+k_2(\varepsilon,\hat{\varepsilon})$$

for any strategy $\hat{\xi}$ of Player I, where $k_2(\varepsilon,\hat{\varepsilon})\to 0$ when $\varepsilon\to 0$, $\hat{\varepsilon}\to 0$.

Assume that Players I and II assure in limit the same value v^a_{mn} in the duel (m,n), $[a,1]$. The number v^a_{mn} will be called the limit value of the game.

Suppose that there is a strategy $\xi_{\hat{\varepsilon}}$ of Player I assuming in the limit, in the duel (m,n), $[a,1]$, the value v^a_{mn} where $k_1(\varepsilon,\hat{\varepsilon})=k_1(\hat{\varepsilon})$. This strategy $\xi_{\hat{\varepsilon}}$ we shall call optimal in the limit.

If, however instead of condition $k_1(\hat{\varepsilon})\to 0$ for $\hat{\varepsilon}\to 0$ we have

$$\lim_{\hat{\varepsilon}\to 0} k_1(\hat{\varepsilon})\le\varepsilon$$

then such strategy $\xi_{\varepsilon\hat{\varepsilon}}$ is called ε-optimal in the limit.

Similarly are defined the optimal and ε-optimal in the limit strategies of Player II.

Let (m,n), $[a+c,a;1]$, $0<c\le\hat{\varepsilon}$, be the duel in which Player I has m bullets, Player II has n bullets but if $c<\hat{\varepsilon}$ Player I can fire his bullets from the moment $a+c$ and Player II from the moment a. If $c=\hat{\varepsilon}$ rule is the same with the only exception that Player II is not allowed to fire a shot at a.

Similarly we define the duel (m,n), $[a,a+c;1]$.

If in the duel (m,n), $[a,1]$, $m>1$, Player I fires as the first the bullet at the moment $a'\ge a$ and Player I does not fire at this moment then the game (m,n), $[a,1]$ reduces to the game $(m-1,n)$, $[a'+\hat{\varepsilon},a';1]$.

Similarly as in the duel (m,n), $[a,1]$ we define strategies optimal (ε-optimal) in the limit and limit value of the game in the duel (m,n), $[a+c,a;1]$. Denote this limit value by $\ddot{v}^a_{mn}$.

Let us consider the following strategies ξ and η in the game $(1,1)$, $[a+\hat{\varepsilon},a;1]$.

Strategy of Player I: If Player II had not fired before, fire the shot at the moment $(a+\hat{\varepsilon})^{\varepsilon}$.

Strategy of Player II: Fire the shot at $\ddot{a}$, $a\le\ddot{a}<a+\hat{\varepsilon}$.

We have

$$K(\xi;\eta)=1-2P(\ddot{a})=1-2P(a)-\varepsilon_1\overset{df}{=}v_{11}^{a}-\varepsilon_1.$$

Let $a'<a+\hat{\varepsilon}$. We have

$$K(\xi;a')\ge 1-2P(a).$$

Let $a'>a+\hat{\varepsilon}+\alpha(\varepsilon)$. We obtain

$$K(\xi;a')\ge P(a+\hat{\varepsilon})-(1-P(a+\hat{\varepsilon}))P(a+\hat{\varepsilon})-\varepsilon\ge P^2(a)-\varepsilon-\varepsilon_1.$$

when $\varepsilon_1\to 0$ if $\hat{\varepsilon}\to 0$.

On the other hand, since in the duel the moment a' of the shot of Player I is greater than $\ddot{a}$ we have

$$K(a',\hat{\xi};\eta)=1-2P(\ddot{a}).$$

Then, strategies defined above are ε-optimal in the limit and $\ddot{v}_{11}^{a}=1-2P(a)$ is the limit value of the game if

$$P^2(a)\ge 1-2P(a)$$

that is, if

$$P(a)\ge\sqrt{2}-1.$$

Let us study what will happen if the strategy ξ of Player I, mentioned above, will be changed to the following strategy ξ'.

Strategy of Player I: Fire the shot at $a+\varepsilon$ if Player I had not fired before.

$$K(\xi';\eta)=1-2P(a')$$

and $K(\xi';a')\ge 1-2P(a)-\varepsilon$ for $a'<a+\hat{\varepsilon}$; $K(\xi';a')\ge P^2(a)-\varepsilon_1$ for $a'>a+\hat{\varepsilon}$ and for each $a'\in[a+\hat{\varepsilon},1]$ $K(a',\hat{\xi};\eta)=1-2P(\ddot{a})$, but now for $a'=a+\hat{\varepsilon}$

$$K(\xi';a')=0.$$

Then ξ' is the optimal in the limit strategy of Player I if $1-2P(a)\le 0$, that is, if

$P(a) \geq 1/2$.

Let us return to the duel (2,1), $[a,1]$, to the situation when $P(a) \geq \sqrt{2}-1$. Let us consider the strategies ξ and η of Players I and II.

Strategy of Player I: Fire a shot at a and if Player I did not fire a shot at a, play optimally or (ε-optimally) the duel (1,1), $[a+\hat{\varepsilon}, a; 1]$.

Strategy of Player II: Fire at a.

Now

$$K(\xi;\eta)=(1-P(a))^2 \overset{df}{=} v_{21}^a$$

and for $a'>a$ we have

$$K(\xi;a') \geq P(a)+(1-P(a))(\ddot{v}_{11}^a-\varepsilon_1-\varepsilon) \geq$$

$$\geq P(a)+(1-P(a))(1-2P(a))-\varepsilon_1-\varepsilon \geq (1-P(a))^2-\varepsilon_1-\varepsilon.$$

On the other hand, for $a'>a$

$$K(a',\hat{\xi};\eta)=1-2P(a) \leq (1-P(a))^2.$$

Then $v_{21}^a=(1-P(a))^2$ is the limit value of the game, the strategy η is optimal in the limit and strategy ξ is ε-optimal in the limit strategy in this game.

Let $v_1=v_2$. It is easy to show that in the duel (2,1), $[a,1]$ the optimal in the limit strategies of Players I and II are:

Strategy of Player I: Fire a shot at a and $a+\hat{\varepsilon}$.

Strategy of Player II: Fire the shot at a.

3. Duel (1,2) in the interval $[a,1]$

Now we shall have to do with the duel in which Player I has one bullet, Player II has two bullets and the game begins at the moment $t=a$. Let $v_1>v_2$. The duel (1,2) and generally (m,n), $m<n$, has some peculiarities. At first, when $a=0$, Player I assures the value zero for himself simply by evading at the moment zero. Then the value of the game v_{mn}, $m<n$, has to be nonnegative, nonetheless Player I has less bullets. Secondly, as it can be shown in the duel

$(1,2)$, $[a,1]$, Player II has infinitely many strategies optimal in the limit.

CASE 1. $P(a)<P_0\cong 0.148066$.

Let us consider the following strategies ξ and η of Player I and II in the duel $(1,2)$, $[a,1]$.

Strategy of Player I: If Player II had not fired a shot before, fire at random with an ACPD in the interval $(a,a+\alpha(\varepsilon))$. If he had fired, play ε-optimally the duel $(1,1)$.

Strategy of Player II: If Player I had not fired the shot before, fire at the moment $\hat{a}$ satisfying the equation

$$Q^3(\hat{a})-(3+v_{11})Q(\hat{a})+2=0 \qquad Q(\hat{a})=0.780539\ldots , \tag{9}$$

and if Player I did not fire at this moment play optimally the duel $(1,1)$.

Let a^ε be the randon moment of the shot, in the interval $(a,a+\alpha(\varepsilon))$, chosen according to the strategy ξ. If $a+\alpha(\varepsilon)<\hat{a}$ we obtain

$$K(\xi;\eta)=E(P(a^\varepsilon)-(1-P(a^\varepsilon))(1-(1-P(a^\varepsilon))(1-P(a^\varepsilon+\check{\varepsilon}))))=$$

$$=E(1-2Q(a^\varepsilon)+Q^2(a^\varepsilon)Q(a^\varepsilon+\check{\varepsilon})=1-2Q(a)=Q^3(a)+ \tag{10}$$

$$+k(\varepsilon,\hat{\varepsilon})\overset{df}{=}v^a_{12}+k(\varepsilon,\hat{\varepsilon}).$$

Here we denote by

E - the operator of expected value,

$\hat{\varepsilon}$ - the shortest time which has to pass between two successive shots of Player II, $\check{\varepsilon}=(v_1-v_2)\hat{\varepsilon}$,

$Q(t)=1-P(t)$,

$k(\varepsilon,\hat{\varepsilon})$ - a function which tends to 0 if $\varepsilon\to 0$, $\hat{\varepsilon}\to 0$.

Let $a'<a$. We have

$$K(\xi;a',\hat{\eta})=-P(a')+(1-P(a'))K^1(\xi;\hat{\eta})\geq$$

$$\geq -1+Q(a)(1+v_{11})-k_1(\varepsilon,\hat{\varepsilon}), \tag{11}$$

where K^1 is the payoff function in the duel $(1,1)$ and $k_1(\varepsilon,\hat{\varepsilon})\to 0$ if $\varepsilon\to 0$, $\hat{\varepsilon}\to 0$, $\hat{\eta}$ is a strategy in the duel $(1,1)$.

From (10) and (11) it follows that there has to be

$$Q^3(a)-(3+v_{11})Q(a)+2\le 0. \qquad (12)$$

Let $a'>a+\alpha=(\omega)$. Then

$$K(\xi;a',\hat{\eta})\ge v^{a}_{12}-k_2(\varepsilon,\hat{\varepsilon})$$

where $k_2(\varepsilon,\hat{\varepsilon})\to 0$ when $\varepsilon\to 0$, $\hat{\varepsilon}\to 0$.

Then Player I assures for himself in the limit the value $v^{a}_{12}=1-2Q(a)+Q^3(a)$ when

$$Q(a)>Q(\hat{a})=0.780539\ldots\ . \qquad (13)$$

The number $Q(\hat{a})$ is the root of the multinomial on the left side of (12).

On the other hand, for $a'<\hat{a}$ we have

$$K(a';\eta)=1-2Q(a')+Q^3(a')+k(\hat{\varepsilon}).$$

The above function of the variable Q has the only minimum in the point

$$Q_1=\sqrt{\frac{2}{3}}=0.8165\ldots$$

and then always

$$1-2Q(a')-Q^3(a')\le 1-2Q(a)+Q^3(a)$$

when $Q_1\le Q(a')\le Q(a)$. The above condition holds for all a', $\hat{a}<a'<\hat{a}$, if $Q(a)\ge Q(\hat{a})$ and

$$1-2Q(a)+Q^3(a)\ge 1-2Q(\hat{a})+Q^3(\hat{a}).$$

Let $a'=\hat{a}$.We obtain

$$K(a';\eta)=-Q^2(\hat{a})(1-Q(\hat{a}+\check{\varepsilon})).$$

Let $a'>\hat{a}$. We obtain

$$K(a';\eta)\le -1+(1+v_{11})Q(\hat{a}).$$

Then the strategy η will be optimal in the limit if

$$1-2Q(a)+Q^3(a)>$$

$$>max(1-2Q(\hat{a})+Q^3(\hat{a}),-Q^2(\hat{a})(1-Q(\hat{a})),-1+(1+v_{11})Q(\hat{a}))=$$

$$\overset{df}{=}max(S_1(Q),S_2(Q),S_3(Q)). \qquad (14)$$

There always is $S_1(Q) \geq S_2(Q)$. Taking that into account it is easy to prove that the function of the variable Q at the right hand side of the inequality (14) will be the smallest when

$$Q^3(\hat{a})-(3+v_{11})Q(\hat{a})+2=0, \tag{15}$$

that is, when $Q(\hat{a}) \cong 0.780539$. For the number $Q(\hat{a})$ chosen in such a way the right hand side of (14), $z=-0.0855416$. By solving the inequality

$$Q^3(a)-2Q(a)+1 \geq z$$

with respect to $Q(a)$, $Q(a)>Q(\hat{a})$, we obtain that strategy ξ is ε-optimal in the limit and η is optimal in the limit when

$$Q(a)>Q_0 \cong 0.851934. \tag{16}$$

If this condition holds then the number

$$v_{12}^{a}=Q^3(a)-2Q(a)+1$$

is the limit value of the game (for $\hat{\varepsilon} \to 0$).

CASE 2. $0.148066 \cong P_0<P(a)<P(\hat{a}) \cong 0.219461$.

Let now $Q(a) \leq Q_0$. Let us consider the strategies ξ and η of Players I and II.

Strategy of Player I: If Player II had not fired a shot before, fire at random at $\hat{a}^{\varepsilon}$ according to an ACPD in the time interval $(\hat{a},\hat{a}+\alpha(\varepsilon))$. If Player II had fired, play ε-optimally the duel (1,1).

Strategy of Player II: If Player I had not fired the shot before, fire at the moment $\hat{a}$, and if Player I did not fire at $\hat{a}$ play optimally the duel (1,1).

As in previous case the constant $\hat{a}$ is the root of the equation (15).

We have

$$K(\xi;\eta)=-1+(1+Q(\hat{a}))v_{11}+k(\varepsilon)=$$

$$\stackrel{df}{=} v_{12}^{a}+k(\varepsilon) \cong -0.0855416+k(\varepsilon)$$

where $k(\varepsilon) \to 0$ if $\varepsilon \to 0$.

Let $a'<\hat{a}$. We obtain

$$K(\xi;a',\hat{\eta}) \geq -P(a')+(1-P(a'))_{11}-\varepsilon \geq$$

$$\geq -P(a)+(1-P(a))v_{11}-k_1(\varepsilon)=v_{12}-k_1(\varepsilon),$$

where $k_1(\varepsilon) \to 0$ when $\varepsilon \to 0$.

Let $a'>a+\alpha(\varepsilon)$. We have

$$K(\xi;a',\hat{\eta})\geq 1-2Q(\hat{a})+Q^2(\hat{a})Q(\hat{a}+\hat{\varepsilon})-k_2(\varepsilon)\geq v_{12}^a-k_2(\varepsilon,\hat{\varepsilon}),$$

where $k_2(\varepsilon)\to 0$, $k_2(\varepsilon,\hat{\varepsilon})\to 0$ when $\varepsilon\to 0$, $\hat{\varepsilon}\to 0$.

Then Player I assures for himself in the limit the value v_{12}^a.

On the other hand, since the strategy of Player I remained the same as in the previous case we obtain that to assure for Player II in the limit the value $v_{12}^a=-1+(1+Q(\hat{a}))v_{11}$ there has to be

$$v_{12}^a=-1+(1+Q(\hat{a})v_{11}\geq$$

$$\geq max(Q^3(a')-2Q(a')+1,\ -Q^2(\hat{a})(1-Q(\hat{a})),\ -1+(1+Q(\hat{a}))v_{11})$$

for each a', $a<a'<\hat{a}$.

By solving the inequality

$$1-2Q(a')+Q^3(a')\leq -1+(1+Q(\hat{a}))v_{11}$$

we obtain $Q(a)\leq Qa'\leq Q_0$ and moreover

$-0.0855416...=-1+(1+Q(\hat{a}))v_{11}> \quad -Q^2(\hat{a})(1-Q(\hat{a}))=-0.1337047...$

From above it follows that if $Q(\hat{a})\leq Q(a)<Q_0$ then strategy ξ is ε-optimal in the limit and η is optimal in the limit.

For $Q(a)$ from the above interval the limit value of the game, $v_{12}^a=-1+(1+Q(\hat{a}))v_{11}$ is independent of a.

CASE 3. $0.219461\cong P(\hat{a})\leq P(a)<P_2\cong 0.269158$.

Let ξ and η be defined as follows

Strategy of Player I: If Player II had not fired a shot before, fire at random in the interval $(a,a+\alpha(\varepsilon))$ according to an ACPD. If Player II had fired play ε-optimally the duel $(1,1)$.

Strategy of Player II: Fire a shot at a and if Player I did not fire at a play optimally the duel $(1,1)$.

Now

$$K(\xi;\eta)=-1+(1+v_{11})Q(a)+k(\varepsilon)\ \stackrel{df}{=}\ v_{12}^a+k(\varepsilon), \tag{17}$$

where $k(\varepsilon)\to 0$ when $\varepsilon\to 0$

We obtain for $a'=a$

$$K(\xi;a',\hat{\eta})=-P(a')+(1-P(a'))(v_{11}-\varepsilon)\geq v_{12}^{a}-k_1(\varepsilon),$$

where $k_1(\varepsilon)\to 0$ when $\varepsilon\to 0$.

Similarly, for $a'>a+\alpha(\varepsilon)$

$$K(\xi;a',\hat{\eta})\geq 1-2Q(a)+Q^2(a)Q(a+\check{\varepsilon})-k_2(\varepsilon)\geq$$

$$\geq 1-2Q(a)+Q^3(a)-k_2(\varepsilon,\hat{\varepsilon})$$

where $k_2(\varepsilon)\to 0$, $k_2(\varepsilon,\hat{\varepsilon})\to 0$, when $\varepsilon\to 0$, $\hat{\varepsilon}\to 0$.

Then Player I assures for himself in the limit the value $v_{12}^{a}=-1+(1+v_{11})Q(a)$ if $Q(a)\geq Q(a_{11})$ and

$$1-2Q(a)+Q^3(a)\geq -1+(1+v_{11})Q(a). \qquad (18)$$

On the other hand, for $a'=a$

$$K(a';\eta)=-Q^2(a)(1-Q(a+\check{\varepsilon})).$$

Then if $Q(a)>Q(a_{11})$ and

$$-Q^2(a)(1-Q(a))<-1+(1+v_{11})Q(a) \qquad (19)$$

also Player II assures in the limit the value v_{12}^{a}.
By solving the inequalities (18) and (19) we obtain

$$0.730842\cong Q_2<Q(a)\leq Q(\hat{a})\cong 0.780539.$$

For these a, ξ is ε-optimal and η optimal in limit.

CASE 4. $0.269158\cong P_2\leq P(a)\leq\sqrt{2}-1\cong 0.414214$.

Strategy of Player I: Fire a shot at a.

Strategy of Player II: Fire a shot at a and if Player I did not fire at a play optimally the duel (1,1).

Now

$$K(\xi;\eta)=-Q^2(a)(1-Q(a+\check{\varepsilon})).$$

Let $a'>a$. We obtain

$$K(\xi;a',\hat{\eta})=1-2Q(a)+Q^2(a)Q(a+\check{\varepsilon})\geq -Q^2(a)(1-Q(a+\check{\varepsilon})).$$

On the other hand, if $a'>a$

$$K(a';\eta)\leq -1+(1+v_{11})Q(a)+k_1(\hat{\varepsilon})\leq$$

$$\le -Q^2(a)(1-Q(a))+k_1(\hat{\varepsilon})$$

if

$$Q^3(a)-Q^2(a)-(1+v_{11})Q(a)+1\ge 0$$

i.e. if $Q(a)\le Q_2\cong 0.730842$. Moreover, strategy of Player II can be realized if $P(a)\le\sqrt{2}-1$. Then for $2-\sqrt{2}\le Q(a)\le Q_2$ strategies ξ and η are optimal in the limit and value of the game is $v_{12}^a=-Q^2(a)(1-Q(a))$.

CASE 5. $P(a)\ge P_3\cong 0.347296$.

Consider, at the end, the following strategies of Player I and II.

Strategy of Player I: Fire the shot at a.

Strategy of Player II: Fire the shots at a and at $a+\hat{\varepsilon}$.

Now

$$K(\xi;\eta)=-Q^2(a)(1-Q(a+\check{\varepsilon})).$$

Let $a'>a$. We obtain

$$K(\xi;a',\hat{\eta})=1-2Q(a)+Q^2(a)Q(a+\check{\varepsilon} \ge -Q^2(a)(1-Q(a=\check{\varepsilon})).$$

On the other hand, if $a<a'<a+\hat{\varepsilon}$ then

$$K(a',\eta)\le -P(a)+(1-P(a))P(a)-(1-P(a))^2P(a)+k_1(\hat{\varepsilon})\le$$
$$\le -Q^2(a)(1-Q(a))+k_1(\hat{\varepsilon}).$$

where $k_1(\hat{\varepsilon})\to 0$ if $\varepsilon\to 0$.

If $a'=a+\hat{\varepsilon}$ then

$$K(a';\eta)=-P(a)\le -Q^2(a)(1-Q(a)).$$

If, at the end, $a'>a+\hat{\varepsilon}$ then

$$K(a';\eta)\le -1+2Q^2(a)+k_2(\hat{\varepsilon}),$$

where $k_2(\hat{\varepsilon})\to 0$ if $\hat{\varepsilon}\to 0$.

It is only in the third case that we obtain a bound on a. There has to be

$$-1+2Q^2(a)\le -Q^2(a)(1-Q(a))$$

or

$$Q^3(a)-3Q^2(a)+1\geq 0$$

This condition holds if

$$Q(a)\leq Q_3\cong 0.652704 \qquad (20)$$

Since it is only a bound a then under (20) the strategies ξ and η are optimal in the limit.

The same results can be obtained with the help of the game $\hat{(1,1)},[a,a+\varepsilon;1]$.

In the duel considered the (limit) value of the game $v_{12}^a=0$, when $a=0$, but for $0<a<1$, v_{12}^a is negative.

For other results concerning duels see [5,8,9,11,14, 15,16,17].

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GŁOŚNY POJEDYNEK Z WYCOFANIEM SIĘ PO ODDANIU STRZAŁÓW

W pracy rozważa się głośny pojedynek, w którym pojedynkujący się (gracze: I i II) usuwają się po wystrzeleniu wszystkich pocisków. Rozwiązano przypadki $m=1$, $n>1$; $m=2$, $n=1$, oraz $m=1$, $n=2$, gdzie m i n są ilościami pocisków, graczy: pierwszego i drugiego.

ИЗВЕСТНЫЙ ПОЕДИНОК С УХОДОМ ПОСЛЕ ВЫСТРЕЛОВ

В работе рассматривается известный поединок, в котором участники дуэли (игроки I и II) уходят с поля после использования всех патронъ. Решаются случаи: м=1, н=1; м=2, н=1, а также м=1, н=2, где м и н являются количеством патронов соответствующе первого и второго игроков.