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A NOTE ON OPERATIONS INVOLVING M-MATRICES AND THEIR ECONOMIC APPLICATIONS[1]

by

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In this paper we investigate whether the class of nonsingular M-matrices is invariant under multiplication and addition with matrices that are in some sense closely related to *M*-matrices. We also discuss this topic by considering special kinds of sums and products that are different from the usual one, deriving some theorems on such matrices. Finally we illustrate the applicability of the previous theorems to a classical economic model.

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1. Introduction

The class of nonsingular M-matrices is extensively used in economic theory, biological sciences, numerical analysis and many other fields. Its great importance has led us to explore some matrix operations that can be performed on them especially in view of their economic applications.

The aim of this paper is to investigate whether the class of nonsingular M-matrices is invariant under multiplication and addition with matrices that are in some sense closely related to M-matrices. We also discuss this topic by considering special kinds of sums and products, which will be defined later on, that are different from the usual ones.

More precisely, in section 2 we give some basic definitions and derive some theorems on such matrices.

Section 3 is concerned with a special subclass of M-matrices (namely the Stieltjes matrices) and its properties. Finally, in section 4, a classical economic model is considered to illustrate some meaningful applications of the previous theorems.

2. Main theorems

We begin this section by introducing some definitions that will be used throughout the paper.

A real-valued square matrix $A = [a_{ij}]$ is called:

1) **positive** if $a_{ij} > 0$ for each i and j ;
   nonnegative if $a_{ij} \geq 0$ for each i and j ;
   semipositive if $a_{ij} \geq 0$ for each i and j and $A \neq [0]$;
2) Z**-matrix** if $a_{ij} \leq 0$ for each $i \neq j$;
3) **nonsingular** M**-matrix** (or briefly M-matrix) if it is a

Z-matrix which admits a semipositive inverse (or equivalently, if is a Z-matrix and a P-matrix or if is a Z-matrix and all its leading principal minors are positive).

4) **positive definite** if it is symmetric and all its leading principal minors are positive;
5) **Stieltjes matrix** if it is a symmetrix M-matrix;
6) ***P*-matrix** if all its principal minors are positive;
7) **inverse-*M*-matrix** if it is nonsingular and its inverse is an M-matrix;
8) **inverse-semipositive (positive)** if it is nonsingular and its inverse is semipositive (positive).

Let us denote the sets of Z,P and M matrices of size $n{\times}n$ by Z_n, P_n, M_n respectively.

After these preliminary definitions, we now give a theorem that exhibits the invariance of M_n under the usual product by some of the matrices defined before.

THEOREM 2.1. *Let M be an M-matrix, the following are also M-matrices;*

A1) *cM, where $c\in R^{*}$.*

A2) *MD, EM, EMD where E and D are any two diagonal matrices with positive diagonal elements.*

A3) *MB, BM, where B is nonsingular and semipositive such that $B^{-1}\geq M$.*

A4) *MN, where $M,N\in M_n$ and $MN\in Z_n$.*

A5) *PMP' where P is a permutation matrix.*

P r o o f.

A1) Trivial.

A2) Obviously both MD and EM are Z-matrices and M,D and E admit a semipositive inverse. It follows that MD, EM and EMD, being inverse-semipositive, are M-matrices.

A3) $(B^{-1}-M)>0$ and $M^{-1}\geq[0]$ implies $M^{-1}(B^{-1}-M)\geq[0]$. It follows that $M^{-1}B^{-1}\geq I$ and thus $(BM)^{-1}\geq[0]$. Furthermore, $B(B^{-1}-M)=I-BM\geq[0]$ implies that BM is a Z-matrix, being

$BM \leq I$. Hence the proof is complete. Similarly it can be proved that MB belongs to M_n.

A4) Trivial.

A5) PMP' is obviously a Z-matrix and also admits a semipositive inverse. In fact, since every permutation matrix is orthogonal, we have: $(PMP')^{-1}=PM^{-1}P' \geq [0]$. Then $PMP' \in M_n$. ∎

A very special kind of matrix product is M^r, r being any positive integer. This matrix power can be generalized to include also rational exponents. If r is a positive integer, we define $M^{1/r}$ as the matrix B such that $B^r = M$ Finally we define $M^{(k)}= [m^k]$, being $k \neq 0$ and integer.

Now we shall state the following.

THEOREM 2.2. *Let M be an M-matrix. Then the following are also M-matrices:*

B1) *M^r, where r is any positive integer and M^r is a Z-matrix.*

B2) *M^p, for $p=1/r$.*

B3) *$M^{(k)}$, where k is any odd integer and $M^{(k)}$ is a P-matrix.*

P r o o f. B1) Since $(M^r)^{-1}=(M^{-1})^r \geq [0]$ and M^r is a Z-matrix then it is an M-matrix.

B2) To prove that M^p is an M-matrix let us use its Taylor series expansion. We can assume that $M=sI-A$, being $A \geq [0]$ and $s>r(A)>0$, where $r(A)$ denotes the spectral radius of A so that $M^p=s^p(I-A/s)^p$. Hence applying the Taylor expansion to $(I-A/s)^p$, since the spectral radius of A is less than s, we obtain:

$$M^p=s^p(I-\sum C_k A^k) \text{ where } C_k=\binom{p}{k}(-1)^{k+1}A^k/s^k$$

But $0<p<1$ implies $C_k \geq 0$, from which it follows easily that M^p is a Z-matrix. Finally, $(M^p)^{-1}=(M^{-1})^p \geq [0]$ completes the proof of B2).

B3) Obviously $M^{(k)}$ is a Z-matrix, because any odd exponent doesn't change the sign pattern of M. Since

$M^{(k)} \in P_n$, the thesis follows. ∎

We shall now consider, as indicated earlier, the invariance of M_n under Fan, Hadamard and Direct Product and Sum with M_n and with other classes of matrices.

The "Fan-product" is defined by:

$$C=A\circ B=[c_{ij}],\quad c_{ij}=\begin{cases} -a_{ij}b_{ij} & \text{for} \quad i\neq j \\ a_{ij}b_{ij} & \text{for} \quad i=j \end{cases} \tag{2.1}$$

The "Hadamard-product" is defined by:

$$C=A*B=[c_{ij}],\qquad c_{ij}=[a_{ij}b_{ij}]. \tag{2.2}$$

The "Direct Product", sometimes referred to as the Kronecker product of matrices, is defined by:

$$C=A\times B=[c_{ij}],\qquad c_{ij}=Ab_{ij}. \tag{2.3}$$

where A and B are respectively $m\times n$ and $r\times s$ matrices and C is an $mr\times ns$ matrix.

The "Direct Sum" is defined by:

$$C=A\oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where A and B are respectively $m\times n$ and $n\times n$ matrices and C is a $(m+n)\times(m+n)$ block diagonal matrix.

From these definitions, we can deduce the following theorem:

THEOREM 2.3. *Let $A,B\in M_n$ and let D be a $n\times n$ diagonal matrix with positive diagonal. Then the following are also M-matrices:*

C1) *$A\circ B$.*

C2) *$A{*}N$, $N{*}A$, where N is an inverse M-matrix.*

C3) *$A\times D$, $D\times A$.*

C4) *$A{*}D$, $D{*}A$.*

C5) *$A\times G$, where G is a semipositive matrix with positive diagonal elements such that its inverse is also semipositive.*

C6) *$A\oplus B$.*

P r o o f.

C1) See [4] and Theorem 5.3 in [1].

C2) See Proposition 3 in [5].

C3) The result is an immediate consequence of (2.3) and of the following relationship:

$$(A\times B)^{-1}=A^{-1}\times B^{-1}. \tag{2.4}$$

C4) This follows from (2.2) and (2.4).

C5) GB being semipositive,we obtain from (2.3) that $A\times G\in Z_n$. Since G^{-1} is also semipositive, then $A\times G$ admits a semipositive inverse and thus it is an M-matrix.

C6) Clearly C is a Z-matrix. Since $C^{-1}=A^{-1}\oplus B^{-1}$ and A^{-1} and B^{-1} are both semipositive, then also $C^{-1}\geq[0]$. Therefore C is an M-matrix. ∎

Next we shall investigate whether the properties of M_n are preserved also by matrix addition (from D1) to D7)) or by a convex linear combination of two M-matrices (D8)) or finally by performing on it two or more operations previously defined (D9)).

To this purpose we state the following

THEOREM 2.4. *Let M be an M-matrix. Then the following are also M-matrices:*

D1) *$M+D$, where D is a diagonal matrix with positive diagonal.*

D2) *$M+T$, where M and T are both lower (upper) triangular M-matrices.*

D3) *$M+T$, where M and T are both lower (upper) triangular so that $(M+T)\in Z_n$ and $M+T$ has positive diagonal elements.*

D4) *$M+N$, where $N\in M_n$ and there exists an n-vector $w>O$ such that $Mw>O$ and $Nw>O$.*

D5) *$M+N$, where $N\in M_n$ and there exists an n-vector w such that $Mw\geq O$ and $Nw\geq O$.*

D6) *$M-B$, where $B\geq[O]$ and $r[M^{-1}B]<1$, $r(M^{-1}B)$ being the spectral radius of $M^{-1}B$.*

D7) *$M-B$, where $(M-B)\in Z_n$ and $[I-M^{-1}B]$ is inverse-semi-positive.*

D8) *$aM+(1-a)N$, where $N\in M_n$ such that $N\geq M$ and $O<a<1$.*

D9) *$M\circ D-(M-A)*(D-B)$, where $A,B,D\in M_n$ such that $M\geq A, D\geq B$.*

P r o o f D1) $M+D$ is obviously a Z-matrix. Since M is a P-matrix, it follows immediately that $M+D$ is also a P-matrix and thus it belongs to M_n.

D2) Since $M+T$ is a triangular matrix with positive diagonal elements, its leading principal minors are positive. $M+T$ is also a Z-matrix and thus the thesis follows.

D3) See the proof of C2).

D4) Obviously $M+N$ is a Z-matrix. Since $Mw>0$ and $Nw>0$ we have that $(M+N)w>0$. Thus, as is well known, the existence of such a vector is a necessary and sufficient condition for the Z-matrix $M+N$ to be an M-matrix.

D5) As before $(M+N)\in Z_n$. Hence if we prove that all leading principal minors of $M+N$ are positive, we get $M+N\in M_n$. Let $M[k]$ and $N[k]$, $k=1,\dots,n$, the $k\times k$ leading principal minors of M and N, respectively.

Furthermore, let $w[k]$ be the k-vector obtained by taking the first k components of w.

Obviously $M[k]w[k]\geq[0]$ and $N[k]w[k]\geq[0]$. Applying to $M[k]$ and $N[k]$ a slightly modified version of Markham's determinant inequality due to R.L.Smith [12] and recalling that $M[k]$ and $N[k]$ are both M-matrices, we obtain:

$0<\det M[k] + \det N[k] \leq \det (M[k]+N[k])$ for $k=1,\dots,n$.

Thus, all leading principal minors of $M+N$ being positive, the thesis follows.

D6) Clearly, $M-B$ is a Z-matrix. We can write

$$(M-B)^{-1}=[M(I-M^{-1}B)]^{-1}=(I-M^{-1}B)^{-1}M^{-1}.$$

Since hypothesis assures that $(I-M^{-1}B)$ is also an M-matrix,and being $M^{-1}\geq[0]$,we get $(M-B)^{-1}\geq[0]$.

This completes the proof of D4).

D7) We can write $(M-B)^{-1}[I-M^{-1}B]^{-1}M^{-1}$.But since $(M-B)$ is a Z-matrix, M and $[I-M^{-1}B]$ are both semipositive-inverse, the thesis follows.

D8) $C=aM+(1-a)N$ is a Z-matrix. Since $N\geq M$ and $0<a<1$, we get immediately $C\geq M$. Hence, by Theorem 4.2 in [6], it follows that C is an M-matrix.

D9) See Corollary 5.3 in [1]. ■

3. A special class of M-matrices

We shall now discuss some invariance properties (under the operations defined above) of a proper subset of M_n, i.e the Stieltjes matrices or the symmetric M-matrices.

Before establishing the next theorem, it is useful to point out the property of a Stieltjes matrix to be positive definite. More precisely the following can be easily proved:

THEOREM 3.1. *A necessary and sufficient condition for a Z-matrix to be a Stieltjes matrix is for it to be positive definite.*

It is easy to verify that the class of Steltjes matrices is closed under the operations performed in A1),A5), B1),B2),B3),B4), C1),C2),C3),C4),C5),C6), D1), D4),D5),D7),D8).
This is not true for A2),A3),A4),D2),D3).

The next theorem exhibits a new set of properties satisfied by Stieltjes matrices.In fact the following holds:

THEOREM 3.2.*Let A,B be two n×n Stieltjes matrices. Then the following are also Stieltjes matrices:*
E1) $A+B$;
E2) $\Sigma a_i A_i$, *where* $\{A_i | i=1,k\}$ *is a collection of* $n \times n$ *Stieltjes matrices and* $\{a_i\}$ *is a set of positive scalars.*

P r o o f. E1) $A+B$ is obviously a Z-matrix. Since $(A+B)' = A'+B' = A+B$, it follows that $(A+B)$ is symmetric. By taking $0 \neq x \in R^n$ we obtain $x'(A+B)x = x'Ax + x'Bx > 0$ and thus the result follows,by Theorem 3.1.

E2) By A1) $a_i A_i$ is a Stieltjes matrix. Since by D1), the sum of Stieltjes matrices is also a Stieltjes matrix, the thesis follows ∎

4.Some economic applications

In this section we illustrate the applicability of the previous theorems to economics. To this purpose, let us point out that nonsingular M-matrices appear mostly in linear production models such as the classical Leontief input-output open model. Currently,hovewer, this model is being applied in many other areas as, e.g. studies of environmental pollution, some cost evaluation problems and cooperative planning problems.

Now, let us consider an open Leontief input-output model that, as is very well known, can be described by the following equation:

$$[I-A]x=d \tag{4.1}$$

where A is the $n \times n$ technological coefficient matrix (also called input-output matrix), x is the n-vector denoting the gross output of each industry and d is the n-vector representing the final demand. The problem here is to determine under which conditions a solution $x \geq [0]$ exists for every $d \geq [0]$.

The economic model just described also has an associated dual model (the price-valuation system) described by:

$$p'[I-A]=v' \tag{4.2}$$

where p denotes the price vector of the commodities, supposed labour-commanded and v is the value added vector.

From the economic point of view, the solvability of (4.1) ((4.2)) in the nonnegative unknown $x \geq [0]$ ($p \geq [0]$) means that the technology is productive (or profitable).

To this purpose, the theory of nonnegative matrices shows that as long as $M=[I-A]$ is a nonsingular M-matrix, (4.1) and(4.2) have the required solution.

Now, a typical problem often arising in input-output analysis concerns the effects of technological changes on the price system defined by (4.2).

In fact, after technological change, (4.2) will appear as:

$$p'[I-(A+\Delta A)]=v' \tag{4.3}$$

where ΔA is the $n \times n$ matrix describing the variation of the input-output coefficients.

Hence one of the main questions is whether $A+\Delta A$ is still profitable or not. This is equivalent to testing whether $[I-(A+\Delta A)]$ is a nonsingular M-matrix.

Depending on the technological change, the resulting matrix $M^{\circ}=I-(A+\Delta A)=M-\Delta A$ can take different patterns as those exhibited in A1), A2), D1), D3), D6) and D7).

Next we show the interpretations of the above mentioned theorems in term of the open Leontief model:

A1) In this case we have $M^{\circ}=cM$, c being a positive real number. Then, this could be interpreted as a new technology affecting proportionally all input-output coefficients. There will be a technological progress if $0<c<1$ and a regress if $c>1$, but, by theorem 2.1 A1), the resulting technology will be, in any case, feasible.

A2) If we consider, for instance, $M^{\circ}=MD$, then the technical change influences the input-output coefficients of the industries in the same way.

D1) $M^{\circ}=M+D$ shows a technological progress that perturbates only the diagonal coefficients, i.e. those expressing the amount of the ith commodity needed as input in industry i. Clearly, we deal with a progress because the technological coefficients decrease.

D3) As in D1), the technical change influences only some coefficients. More precisely, those lying over (under) the main diagonal. Let us point out that D3) has a meaningful economic interpretation, whenever the input-output matrix can be transformed into a lower triangular matrix. As is well known, the triangularization exhibits a whole hierarchical correlation. This implies that industry i buys inputs from industry $(i+k)$ $(k=1,n-i)$ and sells outputs to industry $(i-h)$ $(h=1,i-1)$.

The above described transformation, if it exists, can be easily performed by multiplying the matrix $[I-A]$ by P on the right and P' on the left, P being a suitable permutation matrix. Owing to theorem 2.1, A5), the resulting matrix will be also an M-matrix.

Therefore, in this case, as long as $M-T$ is a

Z-matrix, the new technology will be still feasible.
D6) and D7) generalize the previous propositions by taking into account technological changes involving all industries. More precisely, D6) refers to a progress while D7) is concerned with arbitrary changes of the input-output coefficients.

Finally C6) can be interpreted in terms of industrial interdependence. In fact C6) shows that if the industries can be aggregated in two independent and productive blocks *A* and *B* then the whole technology is itself productive.

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UWAGI O OPERACJACH NA M-MACIERZACH I ICH ZASTOSOWANIACH EKONOMICZNYCH

W artykule rozważa się zagadnienie czy klasa M-macierzy jest inwariantna względem mnożenia i dodawania z macierzami, które w pewnym sensie są blisko związane z M-macierzami. Zagadnienie to jest również rozważane dla specjalnych rodzajów sum i iloczynów, różnych od standardowych, i dla tych warunków udowodniono odpowiednie twierdzenia. W zakończeniu pokazano zastosowanie udowodnionych twierdzeń w klasycznym modelu ekonomicznym.

ЗАМЕЧАНИЯ ОВ ОПЕРАЦИЯХ НА М-МАТРИЦАХ И ИХ ПРИМЕНЕНИЯХ В ЭКОНОМИКЕ

В статье рассматривается вопрос: является ли класс М-матриц инвариантным по отношению к перемножению и сложению с матрицами, которые в некотором смысле являются близко связанными с М-матрицами. Этот вопрос рассматривается также для случая особых видов сумм и произведений, отличающихся от стандартных, и для этих условий доказаны соответствующие теоремы. В заключении показано применение доказанных теорем в классической экономической модели.