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Global existence of solutions of two-phase Stefan problems with nonlinear flux conditions described by time-dependent subdifferentials

by

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A global existence result for two-phase Stefan problems with nonlinear flux conditions with time-dependent subdifferentials on the fixed boundary is established. To this end, special energy estimates, holding for a large class of initial data, are proved.

1. Introduction

In our previous paper [13] we established a local existence result and a comparison result for the following two-phase Stefan problem in one-dimensional space:

$$\begin{cases} \rho(u)_t - u_{xx} = 0 \text{ in } Q_l^+(T) = \{(t,x);\ 0 < t < l(t),\ 0 < t < T\}, \\ \qquad \text{and in } Q_l^-(T) = \{(t,x);\ l(t) < x < 1,\ 0 < t < T\}, \end{cases} \tag{1.1}$$

$$u(0,x) = u_o(x) \text{ for a.e. } x \in [0, 1], \tag{1.2}$$

$$\begin{cases} u_x(t, 0+) \in \partial b_0^t(u(t,0)) \text{ for a.e. } t \in [0, T], \\ -u_x(t, 1-) \in \partial b_1^t(u(t, 1)) \text{ for a.e. } t \in [0, T], \end{cases} \tag{1.3}$$

$$u(t, l(t)) = 0 \text{ for any } t \in (0, T], \tag{1.4}$$

$$\begin{cases} l'(T)\left(= \dfrac{dl(t)}{dt}\right) = -u_x\ (t, l(t)-) + u_x, l(t)+) \text{ for a.e. } t \in [0, T], \\ l(0) = l_o, \end{cases} \tag{1.5}$$

where $u = u(t, x)$ and $x = l(t)$, $0 < l(t) < 1$, are the unknown functions which are respectively defined on $[0, T] \times [0, 1]$ and $[0, T]$; $\rho = \rho(r)$ is a given increasing function on R which vanishes at 0 and is bi-Lipschitz continuous on R; $\{b_i^t\} = \{b_i^t; t \geq 0\}$, $i = 0, 1$, are given families of proper lower semicontinuous (l.s.c.) convex functions $b_i^t(\cdot)$ on R and $\partial b_i^t(\cdot)$ stand for their subdifferentials in R; u_o, l_o are initial data given in $L^2(0, 1)$ and in the interval $(0, 1)$, respectively. This is a *two-phase Stefan problem with flux conditions controlled by time-dependent subdifferentials* $\partial b_i^t(\cdot)$ *on the fixed boundary* $x = i$, $i = 0, 1$

In the present paper we shall give an energy estimate for u and l by the same method as in Evans-Kotlow [4], and show that problem (1.1) — (1.5) is solvable for a larger class of initial functions u_o than that treated in [13]. Moreover, making use of the energy estimate, we shall investigate the behavior of the free boundary $x = l(t)$ as $t \uparrow T^*$, where T^* is the upper end of maximal interval where the solution exist. In particular, when $b_i^t(\cdot)$ is independent of time t, i.e. $b_i^t(\cdot) = b_i(\cdot)$, Stefan problems of the same type as above were completely solved by Yotsutani [21, 23]. In the time-dependent case, the energy estimate is of course more complicated than in the time-independent one. In fact, it depends essentially upon the smoothness of the mappings $t \rightarrow b_i^t(\cdot)$.

As to one or two-phase Stefan problems with nonlinear smooth flux conditions on the fixed boundary, many interesting results have been established. For instance, see Cannon-DiBenedetto [3], Fasano-Primicerio [5], Knabner [14], Niezgódka-Pawłow [16], Niezgódka-Pawłow-Visintin [17], Pawłow [18], and Visintin [19]. Also, see Yotsutani [20, 21,] and Kenmochi [11, 12] for related one-phase problems, and especially Magenes-Verdi-Visintin [14] and Bénilan-Crandall-Sacks [2] for the nonlinear semigroup approach.

Recently, an interesting problem of the free boundary control was proposed and has been studied by Hoffmann-Sprekels [6, 7, 8]. The free boundary control is very important from the mechanical point of view, and it can be done by controlling the flux of the temperature on the fixed boundary. The boundary condition (1.3) may be regarded as a simple mathematical description in such a context, though the expression is not so realistic in some practical respects.

2. Statement of results

In general, for a (real) Banach space V, we denote by $|\cdot|_V$ the norm in V, and use the symbols "$\rightarrow$" and "lim" to indicate strong convergence in V, unless otherwise stated.

Throughout this paper, for the sake of simplicity of notation we put

$$H = L^2(0, 1) \text{ and } X = W^{1,2}(0, 1) \ (\subset C([0, 1])).$$

We denote by $\mathrm{SP} = \mathrm{SP}(\rho; \{b_o^t\}, \{b_1^t\}, u_o, l_o)$ on $[0, T]$, $0 < T < \infty$, the problem (1.1) — (1.5) and say that $\{u, l\}$ *is a solution of* SP *on* $[0, T]$, if

$$u \in C([0, T]; H) \cap W^{1,2}_{\text{loc}}(0, T]); H) \cap L^2(0, T; X) \cap L^{\infty}_{\text{loc}}((0, T]; X),$$
$$b_i^{(\cdot)}(u(\cdot, i)) \in L^1(0, T) \cap L^{\infty}_{\text{loc}}((0, T]), i = 0, 1, \tag{2.1}$$
$$l \in C([0, T]) \cap W^{1,2}_{\text{loc}}((0, T]), 0 < l < 1 \text{ on } [0, T],$$

and (1.1) — (1.5) are satisfied. Also, we say that for $0 < T' \leq \infty$, $\{u, l\}$ *is a solution of* SP *on* $[0, T')$, if it is a solution of SP on $[0, T]$ for every $0 < T < T'$ in the above sense.

REMARK 2.1. In the above definition of solution $\{u, l\}$ to SP on $[0, T]$, we see from (2.1) that $u_x(\cdot, 0+)$, $u_x(\cdot, 1-)$ are in $L^2_{\text{loc}}((0, T])$, because $u_{xx} = \rho(u)_t \in L^2_{\text{loc}}((0, T]; L^2(I))$, $I = (0, \delta)$ or $(1 - \delta, 1)$, for a positive number δ with $\delta \leq l \leq 1 - \delta$ on $[0, T]$. Therefore in condition (1.5) the relation

$$l'(t) = -u_x(t, l(t)-) + u_x(t, l(t)+) \text{ for a.e. } t \in [0, T]$$

is equivalent to

$$l(t) = l(s) + \int_0^1 \rho(u)(s,x)\,dx - \int_0^1 \rho(u)(t,x)dx + \int_s^t \{u_x(\tau,1-) - u_x(\tau,0+)\}\,d\tau \tag{2.2}$$
$$\text{for any } 0 < s \leq t \leq T.$$

This equivalence can be easily proved with the aid of the formula of integration by parts. Moreover, if $u \in W^{1,2}(0, T; H)$, then (1.5) is equivalent to

$$l(t) = l_0 + \int_0^1 \rho(u_0)(x)\,dx - \int_0^1 \rho(u)(t,x)dx + \int_s^t \{u_x(\tau,1-) - u_x(\tau,0+)\}\,d\tau \tag{2.2$'$}$$
$$\text{for any } t \in [0, T];$$

note that $u_x(\cdot, 0+)$, $u_x(\cdot, 1-)$ are in $L^2(0, T)$ in this case.

Given two positive numbers C_1, C_2, we denote by $\Gamma(C_1, C_2)$ the class of all functions $\rho : R \to R$ such that $\rho(0) = 0$ and

$C_1(r - r') \leq \rho(r) - \rho(r') \leq C_2(r - r')$ for any $r, r' \in R$ with $r \geq r'$.

Also, given two functions $\alpha_0 \in W^{1,2}_{\text{loc}}(R_+)$, $\alpha_1 \in W^{1,2}_{\text{loc}}(R_+)$, we denote by $B(\alpha_0, \alpha_1)$ the class of all families $\{b^t; t \geq 0\}$ of proper l.s.c. convex functions $b^t(\cdot)$ on R having the following property (*):

(*) For any $s, t \in R_+$ with $s \leq t$ and any $r \in D(b^s)(= \{r \in R; b^s(r) < \infty\})$ there is $r \in D(b^t)$ such that

$$|\tilde{r} - r| \leq |\alpha_0(t) - \alpha_0(s)|(1 + |r| + |b^s(r)|^{1/2}),$$

$$b^t(\tilde{r}) - b^s(r) \leq |\alpha_1(t) - \alpha_1(s)|(1 + |r|^2 + |b^s(r)|).$$

For two proper l.s.c. convex functions $b_1(\cdot)$ and $b_2(\cdot)$ on R we indicate by "$b_1 \leq^* b_2$ on R" that

$$b_1(r \wedge r') + b_2(r \vee r') \leq b_1(r') + b_2(r') \text{ for any } r, r' \in R, \quad (2.3)$$

where $r \wedge r' = \min\{r, r'\}$ and $r \vee r' = \max\{r, r'\}$. It is easy to see that (2.3) implies

$$(r_1^* - r_2^*)(r_1 - r_2)^+ \geq 0 \text{ for any } r_1^* \in \partial b_1(r_i), \text{ i} = 1, 2. \quad (2.4)$$

In the existence theorem for SP $(\rho; \{b_o^t\}, \{b_1^t\}; u_o, l_o)$ which we shall prove in this paper, we postulate the strong (resp. weak) compatibility condition for the Stefan data $\{b_i^t\}$, $i = 0, 1$, $u_o \in X$ (resp. $u_o \in H$) and $0 < l_o < 1$, which consists of the following conditions (2.5), (2.6) and (2.7) (resp. (2.7)′):

$$\partial b_o^t(r) \subset (-\infty, 0) \text{ for any } t \in R_+ \text{ and } r < 0. \quad (2.5)$$

$$\partial b_1^t(r) \subset (0, +\infty) \text{ for any } t \in R_+ \text{ and } r > 0. \quad (2.6)$$

$$u_o \geq 0 \text{ on } [0, l_o], u_o \leq 0 \text{ on } [l_o, l], u_o(0) \in D(b_o^0) \text{ and } u_o(1) \in D(b_1^0). \quad (2.7)$$

$$u_o \geq 0 \text{ a.e. on } [0, l_o] \text{ and } u_o \leq 0 \text{ a.e. on } [l_o, 1]. \quad (2.7)'$$

The purpose of the present paper is to establish existence and uniqueness theorems for SP as well as the energy inequality for the solution under the weak compatibility condition for the Stefan data.

The first theorem is concerned with the existence of a solution to SP.

THEOREM 2.1. *Let $\rho \in \Gamma(C_1, C_2)$, $\{b_i^t\} \in B(\alpha_o, \alpha_1)$, $i = 0, 1$, $u_o \in H$ and $0 < l_o < 1$ be such that the weak compatibility condition holds. Then, for some positive number T, SP $(\rho; \{b_o^t\}, \{b_1^t\}; u_o, l_o)$ has a solution $\{u, l\}$ on* $[0, T]$ *such that*

$$t^{1/2}u' \in L^2(0, T; H), \; t^{1/2}u_x \in L^\infty(0, T; H),$$

and

$$t^{1/2}\, l' \in L^2(0, T),$$

where $u' = (d/dt)u$.

The next theorem is concerned with the comparison of solutions to SP associated with different Stefan data.

THEOREM 2.2. *Let $\rho \in \Gamma(C_1, C_2)$, $\{b_i^t\} \in B(\alpha_o, \alpha_1)$, $\{\hat{b}_i^t\} \in B(\alpha_o, \alpha_1)$, $i = 0, 1$, $u_o \in H$, $\hat{u}_o \in H$, $0 < l_o <$ and $0 < \hat{l}_o < 1$. Suppose*

$$b_i^t \leq^* \hat{b}_i^t \text{ on } R \text{ for any } t \in R_+ \text{ and } i = 0, 1, \quad (2.8)$$

Futher suppose that Stefan data $\{b_i^t\}$, $i = 0, 1$, u_o, l_o as well as $\{\hat{b}_i^t\}$, $i = 0, 1$, $\hat{u}_o$, $\hat{l}_o$ satisfy the weak compatibility condition. Let $\{u, l\}$ and $\{\hat{u}, \hat{l}\}$ be solutions of

$\mathrm{SP}(\rho; \{b_0^t\}, \{b_1^t\}; u_0, l_0)$ *and* $\mathrm{SP}(\rho; \{\hat{b}_0^t\}, \{\hat{b}_1^t\}, \hat{u}_0, \hat{l}_0)$ *on* $[0, T]$, *respectively. Then, for any* $0 \leq s \leq t \leq T$,

$$\begin{aligned} &|(\rho(u)(t) - \rho(\hat{u})(t))^+|_{L^1(0,1)} + (l(t) - \hat{l}(t))^+ \\ &+ \int_s^t (u_x(\tau, 0+) - \hat{u}_x(\tau, 0+))\sigma_0([u(\tau, 0) - \hat{u}(\tau, 0)]^+)\,d\tau \\ &- \int_s^t (u_x(\tau, 1-) - \hat{u}_x(\tau, 1-))\,\sigma_0([u(\tau, 1) - \hat{u}(\tau, 1)]^+)\,d\tau \\ &\leq |(\rho(u)(s) - \rho(\hat{u})(s))^+|_{L^1(0,1)} + (l(s) - \hat{l}(s))^+ \end{aligned} \tag{2.9}$$

where

$$\sigma_0(r) = \begin{matrix} 1 & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -1 & \text{if } r < 0. \end{matrix}$$

In particular, for any $0 \leq s \leq t \leq T$,

$$\begin{aligned} &|(\rho(u)(t) - \rho(\hat{u})(t))^+|_{L^1(0,1)} + (l(t) - \hat{l}(t))^+ \\ &\qquad \leq |(\rho(u)(s) - \rho(\hat{u})(s))^+|_{L^1(0,1)} + (l(s) - \hat{l}(s))^+ \end{aligned} \tag{2.10}$$

Moreover, if $b_i^t = \hat{b}_i^t$ *for any* $t \in R_+$ *and* $i = 0, 1$, *then for any* $0 \leq s \leq t \leq T$,

$$\begin{aligned} &|(\rho(u)(t) - \rho(\hat{u})(t)|_{L^1(0,1)} + (l(t) - \hat{l}(t)| \\ &\qquad \leq |\rho(u)(s) - \rho(\hat{u})(s)|_{L^1(0,1)} + (l(s) - \hat{l}(s)|. \end{aligned} \tag{2.11}$$

COROLLARY. *In addition to the assumptions of Theorem 2.2, assume that* $u_0 \leq \hat{u}_0$ *a.e. on* $[0, 1]$ *and* $l_0 \leq \hat{l}_0$. *Then,*

$$u \leq \hat{u} \text{ on } (0, T] \times [0, 1] \text{ and } l \leq \hat{l} \text{ on } [0, T].$$

The above corollary shows the uniqueness of solution to SP. The inequality (2.9) was proved in [13] for the solution $\{u, l\}$ under the strong compatibility condition. It is easily seen that the inequality remains true under the weak compatibility condition, too, so that the proofs of Theorem 2.2 and its corollary are omitted.

In this paper we shall establish the following theorem on the energy estimate for the solution of SP.

THEOREM 2.3. *Under the same assumptions as in Theorem* 2.1, *for the solution* $\{u, l\}$ *to* $SP(\rho; \{b_0^t\}, \{b_1^t\}; u_0, l_0)$ *on* $[0, T]$ *it holds that*

$$\chi(t, u(t)) + \frac{1}{C_2}\int_s^t |\rho(u)'(\tau)|_H^2 \, d\tau + \frac{1}{2}\int_s^t |l'(\tau)|^3 \, d\tau$$

$$\leq \chi(s, u(s)) + \int_s^t |\alpha_o'(\tau)|(|u_x(\tau, 0+)| + |u_x(\tau, 1-)|)\gamma(\tau, u(\tau))^{1/2} \, d\tau \quad (2.12)$$

$$+ \int_s^t |\alpha_1'(\tau)| \, |\gamma(\tau, u(\tau) d\tau \qquad \textit{for any } 0 < s \leq t \leq T,$$

where

$$\chi(t,z) = \frac{1}{2}|z_x|_H^2 + b_o^t(z(0)) + b_1^t(z(1)), \qquad z \in \chi$$

and

$$\gamma(t,z) = b_o^t(z(0)) + b_1^t(z(1)) + B_1(|z(0)| + |z(1)|) + B_2(\geq 0), \qquad z \in \chi,$$

with some positive constants B_1, B_2 determined only by T, $|\alpha_o'|_{L^1(0,T)}$, $|\alpha_1'|_{L^1(0,T)}$ and b_i^o, $i = 0, 1$. Moreover, we have

$$(t-s)\chi(t,u(t)) + \frac{1}{C_2}\int_s^t (\tau - s)|\rho(u)'(\tau)|_\kappa^2 \, d\tau + \frac{1}{2}\int_s^t (\tau - s)|l'(\tau)|^3 d\tau$$

$$\leq \int_s^t \chi(\tau, u(\tau))d\tau + \int_s^t (\tau - s)|\alpha_o'(\tau)|(|u_x(\tau, 0+)| + |u_x(\tau, 1-)|)\gamma(\tau, u(\tau))^{1/2} d\tau \quad (2.13)$$

$$+ \int_s^t (\tau - s)|\alpha_1'(\tau)|\gamma(\tau, u(\tau))d\tau \qquad \textit{for any } 0 \leq s < t \leq T.$$

REMARK 2.2 In the case the boundary condition (1.3) is of the usual Dirichlet type or b_i^t are independent of time t, i.e. $b_i^t(\cdot) = b_i(\cdot)$, the same kind of energy inequality as (2.12) was earlier obtained by Evans-Kotlow [4] and Yotsutani [21].

The next theorem is concerned with the convergence of solutions to SP.

THEOREM 2.4. *Let ρ, $\rho_n \in \Gamma(C_1, C_2)$, $\{b_i^t\}$, $\{b_{i,n}^t\} \in B(\alpha_o, \alpha_1)$, $i = 0, 1$, $u_o, u_{o,n} \in H$ and $l_o, l_{o,n} \in (0,1)$, $n = 1, 2, \cdots$. Suppose that Stefan data $\{b_i^t\}$, u_o, l_o as well as $\{b_{i,n}^t\}$, $u_{o,n}$, $l_{o,n}$ satisfy the weak compatibility condition for each $n = 1, 2, \cdots$, and suppose that*

$\rho_n \to \rho$ *uniformly on every compact subset of R,*
$u_{o,n} \to u_o$ *in H,*
$l_{o,n} \to l_o$ *in R*

and

$b_{i,n}^t \to b_i^t$ *on R in the sense of Mosco* (cf. [13]) *for each $t \in R_+$*

as $n \to \infty$. Furthermore assume that $SP = SP(\rho;\{b_o^t\};\{b_1^t\};u_o, l_o)$ has a solution

$\{u, l\}$ *on an interval* $[0, T]$, $0 < T < \infty$. *Then, for large n,* $SP_n = SP(\rho; \{b^t_{o,n}\}, \{b^t_{1,n}\}; u_{o,n}, l_{o,n})$ *has a solution* $\{u_n, l_n\}$ *on the same interval* $[0,T]$. *Moreover,*

$$\begin{aligned} &u_n \to u \text{ in } C([0, T]; H) \text{ and in } L^2(0, T; X),\\ &t^{1/2} u'_n \to t^{1/2} u' \text{ weakly in } L^2(0, T; H),\\ &t^{1/2} u_{n,x} \to t^{1/2} u_x \text{ weakly* in } L^\infty(0, T; H),\\ &l_n \to l \text{ in } C([0, T]) \end{aligned}$$

and

$$t^{1/3} l'_n \to t^{1/3} l' \text{ weakly in } L^3(0, T)$$
$$\text{as } n \to \infty.$$

Finally we shall show the behavior of the free boundary $x = l(t)$.

THEOREM 2.5. *Under the same assumptions as in Theorem* 2.1, *denote by* $T^* = T^*(\rho; \{b^t_o\}, \{b^t_1\}, u_o, l_o)$, $0 < T^* \leq \infty$, *so that* $[0, T^*]$ *is the maximal interval of existence of the solution* $\{u, l\}$ *to SP. Then, one and only one of the following cases* (a), (b), (c) *always holds:*

(a) $T^* = \infty$;
(b) $T^* < \infty$ *and* $\lim_{t \uparrow T^*} l(t) = 0$;
(c) $T^* < \infty$ *and* $\lim_{t \uparrow T^*} l(t) = 1$.

3. Known results and some lemmas

First of all we recall a local existence theorem for SP under strong compatibility condition, which has been proved in [13].

THEOREM 3.1 (cf. [13]). *Let* $\rho \in \Gamma(C_1, C_2)$, *and let* $\{b^t_i\} \in B(\alpha_o, \alpha_1)$, $i = 0, 1$, $u_o \in X$ *and* $0 < l_o < 1$ *be such that the strong compatibility condition holds. Then, for some positive number T,* $SP(\rho; \{b^t_o\}, \{b^t_1\}; u_o, l_o)$ *has a solution* $\{u, l\}$ *on* $[0, T]$ *such that*

$$u \in W^{1,2}(0, T; H) \cap L^\infty(0, T; X) \ (\subset C([0, T] \times [0, 1])),$$

$$b_i^{(\cdot)}(u(\cdot, i)) \in L^\infty(0, T),\ i = 0, 1,$$

and

$$l \in W^{1,2}(0, T).$$

In fact, it was proved in [13] that a local solution $\{u, l\}$ of SP can be constructed as that of the problem

$$\begin{cases} \rho(u)'(t) + \partial\Phi^t_l(u(t)) \ni 0 \text{ for a.e. } t \in [0, T], \\ (1.2) \text{ and } (1.5) \text{ hold,} \end{cases} \tag{3.1}$$

where $\rho(u)' = (d/dt)\rho(u)$, $\Phi_l^t(\cdot)$, $t \geq 0$, is a proper l.s.c. convex function on H defined by

$$\Phi_l^t(z) = \begin{cases} \frac{1}{2}|z_x|_H^2 + b_o^t(z(0)) + b_1^t(z(1)) & \text{if } z \in X \text{ and } z(l(t)) = 0, \\ \infty \text{ otherwise,} \end{cases}$$

and $\partial\Phi_l^t(\cdot)$ is the subdifferential of Φ_l^t in H. Simultaneously it was shown for the solution $\{u, l\}$ that for any $0 \leq s \leq t \leq T$

$$\begin{aligned} &\Phi_l^t(u(\mathrm{t})) + A_1 \int_s^t |\rho(u)'(\tau)|_H^2 d\tau \\ &\leq \Phi_l^s(u(s)) + A_2 \int_s^t \{|\alpha_o'(\tau)|^2 + |\alpha_1'(\tau)| + |l'(\tau)|^2\} (\Phi_l^\tau(u(\tau)) + A_3)\, d\tau, \end{aligned} \tag{3.2}$$

where A_i, $i = 1, 2, 3$, are positive constants depending only on $\rho \in \Gamma(C_1, C_2)$, $\{b_i^t\}$, $i = 0, 1$, u_o and l_o. From (3.2) it follows that

$$\lim_{t \downarrow s} \sup \Phi_l^t(u(t)) \leq \Phi_l^s(u(s)) \qquad \text{for any } 0 \leq s \leq T. \tag{3.3}$$

REMARK 3.1. As it is seen from checking carefully the construction of a solution in [13], the interval of existence of the solution u, can be chosen uniformly in $\rho \in \Gamma(C_1, C_2)$, $\{b_i^t\} \in B(\alpha_o, \alpha_1)$, $i = 0, 1$, $u_o \in X$ and l_o, as long as $|u_o|_X$, $b_i^0(u_o(i))$, $i = 0, 1$, vary in a bounded subset of R, l_o in a compact subset of (0, 1) and (2.5), (2.6) and (2.7) hold.

Next we list some useful inequalities in Sobolev spaces:

$$|v|_{L^\infty(0,\delta)} \leq C(\delta)|v|_{L^2(0,\delta)}^{1/2} |v|_{W^{1,2}(0,\delta)}^{1/2}, \; v \in W^{1,2}(0, \delta), \tag{3.4}$$

$$|v|_{L^\infty(0,\delta)} \leq \varepsilon|v_x|_{L^2(0,\delta)} + C(\delta,\varepsilon)|v|_{L^2(0,\delta)}, \; v \in W^{1,2}(0, \delta), \tag{3.4$'$}$$

where δ, ε are arbitrary positive numbers and $C(\delta)$ (resp. $C(\delta, \varepsilon)$) is a positive constant depending only on δ (resp. δ and ε). We note here that $C(\delta)$ and $C(\delta, \varepsilon)$ are chosen so as to be bounded in R_+, as long as δ and ε vary in any compact subset of $(0, \infty)$. Inequality (3.4)′ immediately follows from (3.4).

Aubin's compactness theorem, which is stated below, is very useful with inequalities (3.4) and (3.4)′ in this paper. Let Y_o, Y_1, Y_2 be three reflexive Banach spaces such that

$$Y_o \hookrightarrow Y_1 \hookrightarrow Y_2$$

and the injection from Y_o into Y_1 is compact. We put

$$W = \{v \in L^p(0, T; Y_o);\ v' \,(= (d/dt)v) \in L^q(0, T; Y_2)\},$$

where $0 < T < \infty$, $1 < p < \infty$, $1 < q < \infty$ are given numbers. The set W becomes a Banach space equipped with norm

$$|v|_W = |v|_{L^p(0,T;Y_o)} + |v'|_{L^q(0,T;Y_o)}.$$

Then Aubin's compactness theorem [1] shows that the injection from W into $L^p(0, T; Y_1)$ is compact. As a direct application of this result we prove:

LEMMA 3.1. *Let $0 < \delta < \infty$ and $0 < T < \infty$. Then we have:*

(i) *if $u_n \to u$ weakly in $L^2(0, T; L^2(0, \delta))$ as $n \to \infty$, $\{u_n\}$ is bounded in $L^2(0, T; W^{1,2}(0,\delta))$ and $\{u_n'\}$ is bounded in $L^q(0, T; W^{-1,2}(0,\delta))$, $1 < q < \infty$, then $u_n \to u$ in $L^2(0, T; L^2(0,\delta))$ as $n \to \infty$.*

(ii) *if $v_n \to v$ weakly in $L^2(0, T; W^{1,2}(0,\delta))$ as $n \to \infty$, $\{v_n\}$ is bounded in $L^2(0, T; W^{2,2}(0,\delta))$* and *$\{v_n'\}$ is bounded in $L^q(0, T; L^2(0,\delta))$, $1 < q < \infty$, then $v_n \to v$ in $L^2(0, ; W^{1,2}(0,\delta))$ as $n \to \infty$.*

Proof. Note that $W^{1,2}(0,\delta) \hookrightarrow L^2(0,\delta) \hookrightarrow W^{-1,2}(0,\delta)$ (= the dual space of $W^{1,2}_o(0,\delta)$) and the injection from $W^{1,2}(0,\delta)$ into $L^2(0,\delta)$ is compact. Hence assertion (i) is a direct consequence of Aubin's compactness theorem. Next, let $\{v_n\}$ be as in the statement (ii). Putting $u_n = v_{n,x}$, we see that $\{u_n\}$ satisfies the conditions in (i), so that $u_n \to u$, i.e. $v_{n,x} \to v_x$ in $L^2(0, T; L^2(0,\delta))$ (as $n \to \infty$). Thus $v_n \to v$ in $L^2(0, T; W^{1,2}(0,\delta))$, and thus (ii) holds.

■

LEMMA 3.2. *Let $0 < \delta < \infty$, $0 < T < \infty$, and $\{u_n\}$ be a bounded sequence in $W^{1,2}(0, T; L^2(0,\delta))$ and in $L^\infty(0, T; W^{1,2}(0,\delta))$. Suppose $u_n \to u$ weakly in $L^2(0, T; L^2(0,\delta))$ as $n \to \infty$. Then, $u \in C([0,T] \times [0,\delta])$ and*

$$u_n \to u \text{ in } C([0,T] \times [0,\delta]) \text{ as } n \to \infty. \tag{3.5}$$

In addition, if $\{u_{n,xx}\}$ is bounded in $L^2(0, T; L^2(0,\delta))$, then u is in $L^2(0, T; W^{2,2}(0,\delta))$, $u_n \to u$ in $L^2(0, T; W^{1,2}(0,\delta))$ as $n \to \infty$, and for any x_o in $[0, \delta]$,

$$u_{n,x}(\cdot, x_o) \to u_x(\cdot, x_o) \text{ in } L^2(0, T) \text{ as } n \to \infty. \tag{3.6}$$

Proof. By Ascoli-Arzela's theorem, we see that $u_n \to u$ in $C([0, T]; L^2(0,\delta))$ (as $n \to \infty$). Since $u_n \in C([0,T] \times [0,\delta])$ for each n, it follows from (3.4) that

$$|u_n(t) - u_m(t)|_{C([0,\delta])} \leq C(\delta)|u_n(t) - u_m(t)|^{1/2}_{L^2(0,\delta)} |u_n(t) - u_m(t)|^{1/2}_{W^{1,2}(0,\delta)}$$

for any $t \in [0,T]$ and positive integers n, m. This shows that $\{u_n\}$ is a Cauchy sequence in $C([(0,T] \times [0,\delta])$, whence (3.5) holds. Moreover, if $\{u_{n,xx}\}$ is bounded in $L^2(0, T; L^2(0,\delta))$, then (ii) of Lemma 3.1 implies that $u_n \to u$ in $L^2(0,T; W^{1,2}(0,\delta))$. By (3.4),

$$|u_{n,x}(t,x_o) - u_x(t,x_o)|^2 \leq C(\delta)^2 |u_n(t) - u_m(t)|_{W^{1,2}(0,\delta)} |u_n(t) - u_m(t)|_{W^{2,2}(0,\delta)},$$

from which (3.6) follows.

Finally we study the sign and comparison properties of solutions to the initial-boundary value problems (3.7) and (3.7)′ formulated below:

$$\begin{cases} \rho(u)_t - u_{xx} = 0 \quad \text{in} \quad D_o = \{(t,x);\, 0 < t < T,\, x_o < x < l(t)\}, \\ u(0,x) = u_o(x) \quad \text{for a.e. } x \in [x_o, x_1], \\ u_x(t,x_o+) \in \partial b_o^t(u(t,x_o)) \quad \text{for a.e. } t \in [0, T], \\ u(t,x) = 0 \quad \text{for } t \in (0, T] \text{ and } l(t) \leq x \leq x_1; \end{cases} \tag{3.7}$$

$$\begin{cases} \rho(u)_t - u_{xx} = 0 \quad \text{in} \quad D_1 = \{(t,x);\, 0 < t < T,\, l(t) < x < x_1\}, \\ u(0,x) = u_o(x) \quad \text{for a.e. } x \in [x_o, x_1], \\ -u_x(t,x_1-) \in \partial b_1^t(u(t,x_1)) \quad \text{for a.e. } t \in [0, T], \\ u(t,x) = 0 \quad \text{for } t \in (0, T] \text{ and } x_o \leq x \leq l(t), \end{cases} \tag{3.7$'$}$$

where $x = l(t)$ is a given curve in $C([0, T])$, x_o and x_1 $(x_o < x_1)$ are given reals such that $x_o < l(t) < x_1$ for any $t \in [0, T]$, and u_o is a given initial datum in $L^2(x_o, x_1)$.

LEMMA 3.3. *Let* $\rho \in \Gamma(C_1, C_2)$, $\{b_o^t\}$ *(resp.* $\{b_1^t\}$*)* $\in B(\alpha_o, \alpha_1)$ *and* $u_o \in L^2(x_o, x_1)$ *such that* (2.5) *(resp.* (2.6)*) holds together with the following* (3.8) *(resp.* (3.8)′*)*:

$$u_o \geq 0 \text{ a.e. on } [x_o, l(0)],\; u_o = 0 \text{ a.e. on } [l(0), x_1], \tag{3.8}$$

$$u_o \leq 0 \text{ a.e. on } [l(0), x_1],\; u_o = 0 \text{ a.e. on } [x_o, l(0)], \tag{3.8$'$}$$

Let u be a solution of (3.7) *(resp.* (3.7)′*) in* $C([0, T]; L^2(x_o, x_1)) \cap W^{1,2}_{\text{loc}}((0, T]; L^2(x_o,x_1)) \cap L^2_{\text{loc}}((0, T]; W^{1,2}(x_o,x_1))$. *Then,*

$$u \geq 0 \text{ on } D_o \; (\text{resp. } u \leq 0 \text{ on } D_1).$$

Proof. We prove the lemma for the case u is a solution of (3.7). Since $\rho(u)_t \in L^2_{\text{loc}}((0, T]; L^2(x_o,x_1))$ and $\rho(u)(t, \cdot) = 0$ on $[l(t), x_1]$, it follows that for a.e. $t \in [0, T]$

$$\frac{1}{2}\frac{d}{dt}\int_{x_o}^{x_1} |(-\rho(u)(t,x))^+|^2 dx$$

$$= -\int_{x_o}^{x_1} \rho(u)_t(t,x)\,(-\rho(u)(t,x))^+ dx$$

$$= -\int_{x_o}^{l(t)} u_{xx}(t,x)\,(-\rho(u)(t,x))^+ dx$$

$$= -\int_{\{u(t,x)<0\}} u_x(t,x)\rho(u)_x(t,x)dx + u_x(t,x_o+)(-\rho(u)(t,x_o))^+ \leq 0,$$

because

$$-\int_{\{u(t,x)<0\}} u_x(t,x)\rho(u)_x(t,x)dx \geq C_1\int_{\{u(t,x)<0\}} |u_x(t,x)|^2 dx \geq 0,$$

and by (2.5)

$$u_x(t,x_o+)(-\rho(u)(t,x_o))^+ \leq 0 \text{ for a.e. } t \in [0, T].$$

Therefore, noting $\rho(u_o) \geq 0$ a.e. on $[x_o,x_1]$ by (3.8), we derive that for any $t \in [0, T]$

$$\int_{x_o}^{x_1} |(-\rho(u)(t,x)^+|^2 dx \leq 0.$$

Thus $\rho(u)(t,\cdot)) \geq 0$ a.e. on $[x_o,x_1]$ for any $t \in [0, T]$, which implies that $u \geq 0$ on D_o.

LEMMA 3.4. *Let $\rho \in \Gamma(C_1,C_2)$, l, $\{b_o^t\}$ (resp. $\{b_1^t\}$), $u_o \in L^2(x_o,x_1)$ be as in Lemma* 3.3; *condition* (2.5) (*resp.* (2.6)) *and* (3.8) (*resp.* (3.8)′) *are as well assumed. Now, let $\hat{l} \in C([0, T])$ with $x_o < \hat{l} < x_1$ on* $[0, T]$, $\{\hat{b}_o^t\}$ (*resp.* $\{\hat{b}_1^t\}$), $\hat{u}_o \in L^2(x_o, x_1)$, *and suppose that conditions* (2.5) (*resp.* (2.6)) *and* (3.8) (*resp.* (3.8′) *are satisfied for* $\{\hat{b}_o^t\}$ (*resp.* $\{\hat{b}_1^t\}$), $\hat{l}$ *and* $\hat{u}_o$. *Further suppose that*

$$b_o^t \leq^* \hat{b}_o^t \ (\textit{resp. } b_1^t \leq^* \hat{b}_1^t) \textit{ on } R \textit{ for any } t \in R_+,$$

$$l \leq \hat{l} \textit{ on } [0, T]$$

and

$$u_o \leq \hat{u}_o \textit{ a.e. on } [x_o,x_1].$$

Let u be a solution of (3.7) (*resp.* (3.7)′) *and $\hat{u}$ be a solution of* (3.7) (*resp.* (3.7)′), *corresponding to* $\rho, \hat{l}, \hat{u}_o, \{\hat{b}_o^t\}$ (*resp.* $\{\hat{b}_1^t\}$), *in* $C([0, T]; L^2(x_o,x_1)) \cap W^{1,2}_{\mathrm{loc}}((0, T]; L^2(x_o,x_1)) \cap L^2_{\mathrm{loc}}((0, T]; W^{1,2}(x_o,x_1))$. Then,

$$u \leq \hat{u} \text{ on } (0, T] \times [x_o,x_1].$$

P r o o f. Consider the lemma in the case of problem (3.7). We first note that $\sigma_o([\rho(u) - \rho(\hat{u})]^+) = \sigma_o([u - \hat{u}]^+)$ for the same function σ_o in Theorem 2.2, and by Lemma 3.3 that $[u - \hat{u}]^+ = 0$ on $\{0 < t \leq T, \ l(t) \leq x \leq x_1\}$. Therefore, for a.e. $t \in [0, T]$ we have

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{x_o}^{x_1} |(\rho(u)(t,x) - \rho(\hat{u})(t,x))^+| dx$$

$$= \int_{x_o}^{x_1} (\rho(u)_t(t,x) - \rho(\hat{u})_t(t,x))\sigma_o([\rho(u)(t,x) - \rho(\hat{u})(t,x)]^+)dx$$

$$= \int_{x_o}^{l(t)} (u_{xx}(t,x) - \hat{u}_{xx}(t,x))\,\sigma_o([u(t,x) - \hat{u}(t,x)]^+)\,dx.$$

Here, we approximate σ_o by a sequence of smooth functions σ_n on R such that $\sigma_n(0) = 0, -1 \leq \sigma_n \leq 1, \sigma_n' \geq 0$ on R and $\sigma_n \to \sigma_o$ pointwise on R (as $n \to \infty$). With this function σ_n we observe that

$$\int_{x_o}^{l(t)} (u_{xx}(t,x) - \hat{u}_{xx}(t,x))\,\sigma_o([u(t,x) - \hat{u}(t,x)]^+)\,dx$$

$$= \lim_{n\to\infty} \int_{x_o}^{l(t)} (u_{xx}(t,x) - \hat{u}_{xx}(t,x))\,\sigma_n([u(t,x) - \hat{u}(t,x)]^+)\,dx$$

$$= \lim_{n\to\infty} \{ - \int_{x_o}^{l(t)} (u(t,x) - \hat{u}(t,x))_x\,\sigma_n([u(t,x) - \hat{u}(t,x)]^+)_x\,dx$$

$$- (u_x(t,x_o+) - \hat{u}_x(t,x_o+))\,\sigma_n([u(t,x_o) - \hat{u}(t,x_o)]^+)$$

$$\leq \lim_{n\to\infty} - \int_{\{u(t,x) - \hat{u}(t,x) > 0\}} |u_x(t,x) - \hat{u}_x(t,x)|^2\,\sigma_n'([u(t,x) - \hat{u}(t,x)]^+)\,dx$$

$$\leq 0 \qquad \text{for a.e. } t \in [0, T],$$

because $(u_x(t,x_o+) - \hat{u}_x(t,x_o+))\,\sigma_n([u(t,x_o) - \hat{u}(t,x_o)]^+) \geq 0$ for a.e. $t \in [0, T]$, by (2.4). Therefore,

$$\frac{d}{dt}\int_{x_o}^{x_1} |(\rho(u)(t,x) - \rho(\hat{u})(t,x))^+|dx \leq 0 \text{ for a.e. } t \in [0, T],$$

whence

$$\int_{x_o}^{x_1} |(\rho(u)(t,x) - \rho(\hat{u})(t,x))^+|dx \leq \int_{x_o}^{x_1} |(\rho(u_o(x)) - \rho(\hat{u}_o(x))^+|dx = 0$$

for all $t \in [0, T]$. This shows that $\rho(u) \leq \rho(\hat{u})$ a.e. on $[0, T] \times [x_o, x_1]$. Hence $u \leqslant \hat{u}$ on $(0, T] \times [X_o, X_1]$.

4. Proof of Theorem 2.3

In this section, let $\rho \in \Gamma(C_1, C_2)$, $\{b_i^j\} \in B(\alpha_o, \alpha_1)$, $i = 0, 1, 0 < l_o <$ and $u_o \in X$ so that the strong compatibility condition holds. Let $\{u, l\}$ be the solution of

$SP = SP(\rho; \{b_o^t\}, \{b_1^t\}; u_o, l_o)$ on $[0, T]$, $0 < T < \infty$ such that $u \in W^{1,2}(0, T; H) \cap L^\infty(0, T; X)$, $b_i^{(\cdot)}(u(\cdot, i)) \in L^\infty(0, T)$, $i = 0, 1$, and $l \in W^{1,2}(0, T)$ (cf. Theorem 3.1). We are going to prove the inequality (2.12) for $0 \le s \le t \le T$.

We choose a positive number δ in (0, 1) so that

$$\delta < l(t) < 1 - \delta \text{ for any } t \in [0, T],$$

and so that the functions $u(\cdot, \delta)$ and $u(\cdot, 1-\delta)$ are in $W^{1,2}(0, T)$. For simplicity we put

$$f_o(t) = u(t,\delta), f_1(t) = u(t, 1-\delta),$$

$$E_o(t) = \frac{1}{2}\int_0^\delta |u_x(t,x)|^2\, dx + b_o^t(u(t,0)),$$

$$E_1(t) = \frac{1}{2}\int_{1-\delta}^1 |u_x(t,x)|^2\, dx + b_1^t(u(t, 1)),$$

$$E(t) = E_o(t) + E_1(t),$$

$$F_o(t) = b_o^t(u(t,0)) + B_1' |u(t,0)| + B_2' \ (\ge 0),$$

$$F_1(t) = b_1^t(u(t,1)) + B_1' |u(t,1)| + B_2' \ (\ge 0), \text{ and}$$

$$F(t) = F_o(t) + F_1(t),$$

where B_2', B_2' are non-negative constants to be determined later.

The purpose of this section is to prove the following lemma:

LEMMA 4.1. *For any* $0 \le s \le t \le T$,

$$E(t) + \frac{1}{C_2}\int_s^t \int_0^\delta |\rho(u)_\tau(\tau,x)|^2 dx d\tau + \frac{1}{C_2}\int_A^t \int_{1-\delta}^{\prime} |\rho(u)_\tau(\tau,x)|^2 dx d\tau$$

$$\le E(s) + \int_s^t |\alpha_0'(\tau)| (|u_x(\tau,0+)| + |u_x(\tau,1-)|) F(\tau)^{1/2}\, d\tau + \int_s^t |\alpha_1'(\tau)| F(\tau)\, d\tau$$

$$+ \int_s^t \{u_x(\tau,\delta) f_0'(\tau) - u_x(\tau,1-\delta) f_1'(\tau)\}\, d\tau. \tag{4.1}$$

For each $\lambda \in (0, 1]$ and $t \in R_+$ we consider the Yosida approximation $b_{i,\lambda}^t$ of b_i^t:

$$b_{i,\lambda}^t(r) = \inf_{r' \in R}\left\{\frac{1}{2\lambda}|r - r'|^2 + b_i^t(r')\right\}, r \in R.$$

It is known (cf. [9; section 1.5]) that there are positive constants B_1', B_2', determined only by T, the norms of α_o', α_1' in $L^1(0, T)$ and

$$R_i(b_i^o) \equiv \inf\{r \geq 0; \exists\, s \in D(b_i^o), |s| \leq r, |b_i^o(s) \leq r\},\ i = 0, 1, \quad (4.2)$$

such that

(i) $b_{i,\lambda}^t(r) B_1' |r| + B_2' \geq 0$ for any $\lambda \in (0,1]$, $\tau \in [0, T]$, $r \in R$, $i = 0,1$;
(ii) for each $\lambda \in (0, 1]$ and $t \in R$ the functions $t \to b_{i,\lambda}^t(r)$, i $= 0, 1$, are of bounded variation of $[0, T]$ and

$$b_{i,\lambda}^t(r) - b_{i,\lambda}^s(r) \leq \int_s^t \frac{d}{dt} b_{i,\lambda}^\tau(r)\, d\tau \text{ for any } 0 \leq s \leq t \leq T,$$

$$\frac{d}{dt} b_{i,\lambda}^t(r) \leq |\alpha_0'(t)|\,|\partial b_{i,\lambda}^t(r)|\,(b_{i,\lambda}^t(r) + B_1'|r| + B_2')^{1/2}$$

$$+ |\alpha_1'(t)|(b_{i,\lambda}^t(r) + B_1'|r| + B_2') \qquad \text{for a.e. } t \in [0, T]. \quad (4.3)$$

In order to prove Lemma 4.1 we approximate u on $[0, T] \times [0, \delta]$ and $[0, T] \times [1 - \delta, 1]$ by the function u_λ which is the solution of the following problems (4.4) and (4.4)′:

$$\begin{cases} \rho(u_\lambda)_t - u_{\lambda,xx} = 0 & \text{in } (0, T) \times (0, \delta), \\ u_\lambda(t, \delta) = f_o(t) & \text{for } 0 \leq t \leq T, \\ u_{\lambda,x}(t, 0+) = \partial b_{o,\lambda}^t(u_\lambda(t, 0)) & \text{for a.e. } t \in [0, T], \\ u_\lambda(0, x) = u_o(x) & \text{for } 0 \leq x \leq \delta; \end{cases}$$

$$\begin{cases} \rho(u_\lambda)_t - u_{\lambda,xx} = 0 & \text{in } (0, T) \times (0-, \delta, 1), \\ u_\lambda(t, 1-\delta) = f_1(t) & \text{for } 0 \leq t \leq T, \\ -u_{\lambda,x}(t, 1-) = \partial b_{1,\lambda}^t(t, 1)) & \text{for a.e. } t \in [0, T], \\ u_\lambda(0, x) = u_o(x) & \text{for } 1-\delta \leq x \leq 1; \end{cases}$$

According to a result in [9; section 2.8] (or [10; Theorem 1.1]), problems (4.4) and (4.4)′ have unique solutions u_λ in $W^{1,2}(0, T; L^2(0, \delta)) \cap L^\infty(0, T, W^{1,2}(0, \delta)) \subset C([0, T] \times [0, \delta])$ and in $W^{1,2}(0, T; L^2(1-\delta, 1)) \cap L^\infty(0, T; W^{1,2}(1-\delta, 1)) \subset C([0, T] \times [1-\delta, 1])$.

Next we derive the energy inequalities for the approximate solutions u_λ.

For a.e. $\tau \in [0, T]$ and any $s \in [0, T]$ with $s \leq \tau$ we observe that

$$\int_0^\delta \rho(u_\lambda)_\tau(\tau,x)(u_\lambda(\tau,x)-u_\lambda(s,x))dx = \int_0^\delta u_{\lambda,xx}(\tau,x)(u_\lambda(\tau,x)-u_\lambda(s,x))dx$$

$$= -\int_0^\delta u_{\lambda,x}(\tau,x)(u_\lambda(\tau,x)-u_{\lambda,x}(s,x))dx + u_{\lambda,x}(\tau,\delta)(f_o(\tau)-f_o(s))$$

$$- u_{\lambda,x}(\tau,0+)(u_\lambda(\tau,0)-u_\lambda(s,0)). \tag{4.5}$$

Using (4.3), we have

$$- u_{\lambda,x}(\tau,0+)(u_\lambda(\tau,0)-u_\lambda(s,0)) \leq b^\tau_{o,\lambda}(u_\lambda(s,0)) - b^\tau_{o,\lambda}(u_\lambda(\tau,0))$$

$$= b^s_{o,\lambda}(u_\lambda(s,0)) - b^\tau_{o,\lambda}(u_\lambda(\tau,0)) + b^\tau_{o,\lambda}(u_\lambda(s,0)) - b^s_{o,\lambda}(u_\lambda(s,0))$$

$$\leq b^s_{o,\lambda}(u_\lambda(s,0)) - b^\tau_{o,\lambda}(u_\lambda(\tau,0)) \tag{4.6}$$

$$+ \int_s^\tau |\alpha_0'(\sigma)||\partial b^\sigma_{o,\lambda}(u_\lambda(s,0))|(b^\sigma_{o,\lambda}(u_\lambda(s,0)) + B_1'|u_\lambda(s,0)| + B_2')^{1/2}d\sigma$$

$$+ \int_s^\tau |\alpha_1'(\sigma)|(b^\sigma_{o,\lambda}(u_\lambda(s,0)) + B_1'|u_\lambda(s,0)| + B_2')\,d\sigma$$

Hence it follows from (4.5) and (4.6) that

$$\int_0^\delta \rho(u_\lambda)_\tau(\tau,x)(u_\lambda(\tau,x)-u_\lambda(s,x))dx + E_{o,\lambda}(\tau) - E_{o,\lambda}(s)$$

$$\leq \int_s^\tau |\alpha_0'(\sigma)||\partial b^\sigma_{o,\lambda}(u_\lambda(s,0))|(b^\sigma_{o,\lambda}(u_\lambda(s,0) + B_1'|u_\lambda(s,0)| + B_2')^{1/2}\,d\sigma \tag{4.7}$$

$$+ \int_s^\tau |\alpha_1'(\sigma)|(b^\sigma_{o,\lambda}(u_\lambda(s,0) + B_1'|u_\lambda(s,0)| + B_2')^{1/2}\,d\sigma + u_{\lambda,x}(\tau,\delta)(f_o(\tau)-f_o(s)),$$

where

$$E_{o\lambda}(\tau) = \frac{1}{2}\int_0^\delta |u_{\lambda,x}(\tau,x)|^2dx + b^\tau_{o,\lambda}(u_\lambda(\tau,0)).$$

Note here (cf. [9; Proposition 0.3.5, Lemma 1.21]) that

$$|\partial b^\tau_{o,\lambda}(u_\lambda(s,0)) - \partial b^\tau_{o,\lambda}(u_\lambda(\tau,0))| \leq \frac{1}{\lambda}|u_\lambda(s,0) - u_\lambda(\tau,0)|$$

and

$$|b^\tau_{o,\lambda}(u_\lambda(s,0)) - b^\tau_{o,\lambda}(u_\lambda(\tau,0))| \leq A_\lambda(|u_\lambda(s,0)|+|u_\lambda(\tau,0)|)|u_\lambda(s,0) - u_\lambda(\tau,0)|$$

for every $\tau \in [s, t]$, where A_λ is a constant depending only on λ. Therefore, dividing (4.7) by $\tau - s$, letting $s \uparrow \tau$ and noting the relation $u_{\lambda,x}(\tau,0+) = \partial b^\tau_{o,\lambda}(u_\lambda(\tau,0))$, we see that for a.e. $\tau \in [0, T]$

$$\int_0^\delta \rho(u_\lambda)_\tau(\tau,x)u_{\lambda,\tau}(\tau,x)dx + \frac{d}{d\tau}E_{o,\lambda}(\tau)$$

$$\leq |\alpha_0'(\tau)||u_{\lambda,x}(\tau,0+)|F_{o,\lambda}(\tau)^{1/2} + |\alpha_1'(\tau)|F_{o,\lambda}(\tau) + u_{\lambda,x}(\tau,\delta)f_0'(\tau), \qquad (4.8)$$

where

$$F_{o,\lambda}(\tau) = b_{o,\lambda}^\tau(u_\lambda(\tau,0)) + B_1'|u_\lambda(\tau,0)| + B_2';$$

it should be noted that $E_{o\lambda}$ is of bounded variation on $[0, T]$ and

$$E_{o,\lambda}(t) - E_{o,\lambda}(s) \leq \int_s^t \frac{d}{d\tau}E_{o,\lambda}(\tau)d\tau \qquad \text{for } 0 \leq s \leq t \leq T.$$

Hence

$$\frac{1}{C_2}\int_s^t\int_0^\delta |\rho(u_\lambda)_\tau|^2 dxd\tau + E_{o,\lambda}(t) - E_{o,\lambda}(s)$$

$$\leq \int_s^t |\alpha_0'(\tau)||u_{\lambda,x}(\tau,0+)|F_{o,\lambda}(\tau)^{1/2}d\tau + \int_s^t |\alpha_1'(\tau)|F_{o,\lambda}(\tau)d\tau \qquad (4.9)$$

$$+\int_s^t u_{\lambda,x}(\tau,\delta)f_0'(\tau)\, d\tau.$$

Similarly we can obtain

$$\frac{1}{C_2}\int_s^t\int_{1-\delta}^1 |\rho(u_\lambda)_\tau|^2 dxd\tau + E_{1,\lambda}(t) - E_{1,\lambda}(s)$$

$$\leq \int_s^t |\alpha_0'(\tau)||u_{\lambda,x}(\tau,1-)|F_{1,\lambda}(\tau)^{1/2}d\tau + \int_s^t |\alpha_1'(\tau)|F_{1,\lambda}(\tau)d\tau \qquad (4.10)$$

$$-\int_s^t u_{\lambda,x}(\tau,1-\delta)f_1'(\tau)d\tau.$$

where

$$E_{1,\lambda}(t) = \frac{1}{2}\int_{1-\delta}^1 |u_{\lambda,x}(t,x)|^2 dx + b_{1,\lambda}^t(u_\lambda(t,1))$$

and

$$F_{1\lambda}(t) = b_{1,\lambda}^t(u_\lambda(t,1)) + B_1'|u_\lambda(t,1)| + B_2'.$$

Proof of LEMMA 4.1. Since $|u_\lambda(\tau,0+)| \leq C_\delta(|u_{\lambda,x}|_{L^2(0,\delta)} + |f_o(\delta)|)$ for a constant C_δ depending only on δ, we see that

$$F_{o,\lambda}(t) \leq E_{o,\lambda}(t) + C_\delta'', \qquad t \in [0, T], \lambda \in (0, 1], \qquad (4.11)$$

with a constant C_δ'' independent of λ and t. Besides, since $\rho(u_\lambda)_t = u_{\lambda xx}$ in $(0, T) \times (0, \delta)$, we have with the aid of (3.4)

$$|u_{\lambda,x}(t)|_{L^\infty(0,\delta)} \le \varepsilon|\rho(u_\lambda)_t(t)|_{L^2(0,\delta)} + C(\delta,\varepsilon)|u_{\lambda,x}(t)|_{L^2(0,\delta)} \tag{4.12}$$

for a.e. $t \in [0, T]$, where ε is any positive number. Taking $\varepsilon > 0$ small enough we derive from (4.9) together with (4.11) and (4.12) that

$$L_1 \int_s^t |\rho(u_\lambda)_\tau(\tau)|^2_{L^2(0,\delta)} d\tau + E_{o,\lambda}(t) - E_{o,\lambda}(s)$$

$$\le L_2 \int_s^t (|\alpha_0'(\tau)|^2 + |\alpha_1'(\tau)| + 1)(E_{o,\lambda}(\tau) + L_3)\, d\tau + L_4 \int_s^t |f_o'(\tau)|^2\, d\tau \tag{4.13}$$

for any $\lambda \in (0, 1]$ and $0 \le s \le t \le T$, where $L_1, \cdots, L_4$ are positive constants independent of $\lambda \in (0, 1]$ and $s, t \in [0, T]$. Therefore, applying Gronwall's inequality to (4.13), we obtain that $\{u_\lambda;\ 0 < \lambda \le 1\}$ is bounded in $W^{1,2}(0, T; L^2(0,\delta)) \cap L^\infty(0, T; W^{1,2}(0,\delta))$ and $\{b_{o,\lambda}^{(\cdot)}(u_\lambda(\cdot,0));\ 0 < \lambda \le 1\}$ is bounded in $L^\infty(0, T)$. Accordingly, by Lemma 3.2 it follows that for a suitable sequence $\{\lambda_n\}$ with $\lambda_n \downarrow 0$ (as $n \to \infty$),

$$u_{\lambda_n} \to u \text{ in } C([0, T] \times [0, \delta]) \cap L^2(0, T; W^{1,2}(0, \delta)),$$

$$u_{\lambda_n,x}(\cdot,0+) \to u_x(\cdot,0+) \qquad \text{in } L^2(0, T),$$

$$u_{\lambda_n,x}(\cdot,\delta) \to u_x(\cdot,\delta) \qquad \text{in } L^2(0, T),$$

and

$$\rho(u_{\lambda_n})_t \to \rho(u)_t \text{ weakly in } L^2(0, T; L^2(0,\delta)).$$

Also, by some standard techniques in the subdifferential operator theory we may conclude that

$$b_{o,\lambda_n}^t(u_{\lambda_n}(t,0)) \to b_o^t(u(t,0)) \text{ for a.e. } t \in [0, T],$$

$$\liminf_{n\to\infty} b_{o,\lambda_n}^t(u_{\lambda_n}(t,0)) \ge b_o^t(u(t,0)) \text{ for any } t \in [0, T]$$

and

$$E_{o,\lambda_n}(0) \to E_o(0).$$

Taking $\lambda = \lambda_n$ in (4.9), we get by passing to the limit in n that

$$\frac{1}{C_2} \int_s^t \int_0^\delta |\rho(u)_\tau|^2 dx d\tau + E_o(t)$$

$$\leq E_o(0) + \int_0^t |\alpha_0'(\tau)||u_x(\tau,0+)|F_o(\tau)^{1/2}\,d\tau + \int_0^t |\alpha_1'(\tau)|F_o(\tau)d\tau \quad (4.14)$$
$$+ \int_0^t u_x(\tau,\delta)\,f_o'(\tau)\,d\tau \qquad \text{for any } t \in [0, T].$$

Similarly it follows from (4.10) that

$$\frac{1}{C_2}\int_s^t \int_{1-\delta}^1 |\rho(u)_\tau|^2\,dxd\tau + E_1(t)$$
$$\leq E_1(0) + \int_0^t |\alpha_0'(\tau)||u_x(\tau,1-)|F_1(\tau)^{1/2}\,d\tau + \int_0^t |\alpha_1'(\tau)|F_1(\tau)\,d\tau \quad (4.15)$$
$$- \int_0^t u_x(\tau,1-\delta)\,f_1'(\tau)\,d\tau \qquad \text{for any } t \in [0, T].$$

Adding (4.12) and (4.13) yields (4.1) with $s = 0$ and any $t \in [0, T]$. By taking $s \in (0, T]$ as the initial time and repeating the same argument as above, we obtaing (4.1) for any $0 < s \leq t \leq T$. ■

5. Proof of Theorem 2.3 (continued)

For the moment we continue our discussion under the assumptions of section 4; let $\{u, l\}$ and $0 < \delta < 1$ be as in that section.

First Step. In the first step we assume that ρ is a smooth function in $\Gamma(C_1, C_2)$, and prove:

LEMMA 5.1. *For any* $0 \leq s \leq t \leq T$,

$$\frac{1}{C_2}\int_s^t \int_\delta^{1-\delta} |\rho(u)_\tau(\tau,x)|^2 dxd\tau + \frac{1}{2}\int_s^t |l'(\tau)|^3 d\tau + \frac{1}{2}\int_\delta^{1-\delta} |u_x(t,x)|^2 dx \quad (5.1)$$
$$\leq \frac{1}{2}\int_\delta^{1-\delta} |u_x(s,x)|^2 dx - \int_s^t \{u_x(\tau,\delta)\,f_o'(\tau) - u_x(\tau,1-\delta)\,f_1'(\tau)\}d\tau.$$

P r o o f. According to Evans-Kotlow [4; Theorems 1, 2] we see that

$$l' \in H^\Theta([s, T]) \text{ for any } s \in (0, T),$$

$$u \in H2+\Theta 1+\theta/2\,(Q^+),\ u \in H2+\Theta,1+\theta/2\,(Q^-),$$

and

$$u_{xt} \in L^2(Q^+),\ u_{xt} \in L^2(Q^-),$$

for any region Q with $\overline{Q} \subset (0, T] \times (0, 1)$, where $0 < \Theta < 1$ and

$$Q^+ = Q \cap Q_l^+(T),\ Q^- = Q \cap Q_l^-(T),$$

Let us differentiate $u(t, l(t)) = 0$ with respect to $t \in [0, T]$. Then,

$$u_t(t, l(t) \pm) = -u_x(t, l(t) \pm) l'(t) \qquad \text{for } t \in (0, T). \tag{5.2}$$

Also, we have for any $0 < s \le t \le T$

$$\frac{1}{C_2}\int_s^t\int_\delta^{l(\tau)} |\rho(u)_\tau|^2\,dxd\tau \le \int_s^t\int_\delta^{l(\tau)} \rho(u)_\tau u_\tau\,dxd\tau = \int_s^t\int_\delta^{l(\tau)} u_{xx}u_\tau dxd\tau$$

$$= -\int_s^t\int_\delta^{l(\tau)} u_x u_{x\tau}\,dxd\tau + \int_s^t u_x(\tau,l(\tau)-)\,u_\tau(\tau,l(\tau)-)d\tau - \int_s^t u_x(\tau,\delta)\,f_o'(\tau)d\tau$$

$$= -\frac{1}{2}\int_\delta^{l(t)} |u_x(t,x)|^2dx + \frac{1}{2}\int_\delta^{l(s)} |u_x(s,x)|^2dx + \frac{1}{2}\int_s^t |u_x(\tau,l(\tau)-)|^2\,l'(\tau)d\tau$$

$$+\int_s^t u_x(\tau,l(\tau)-)\,u_\tau(\tau,l(\tau)-)\,d\tau - \int_s^t u_x(\tau,\delta)\,f_o'(\tau)d\tau$$

$$= -\frac{1}{2}\int_\delta^{l(t)} |u_x(t,x)|^2dx + \frac{1}{2}\int_\delta^{l(s)} |u_x(s,x)|^2dx - \frac{1}{2}\int_s^t |u_x(\tau,l(\tau)-)|^2\,l'(\tau)\,d\tau$$

$$-\int_s^t u_x(\tau,\delta)\,f_o'(\tau)\,d\tau$$

We have used (5.2) in order to get the last equality. Thus, for any $0 < s \le t \le T$,

$$\frac{1}{C_2}\int_s^t\int_\delta^{l(\tau)} |\rho(u)_\tau|^2\,dxd\tau + \frac{1}{2}\int_s^t |u_x(\tau,l(\tau)-)|^2\,l'(\tau)d\tau + \frac{1}{2}\int_\delta^{l(t)} |u_x(t,x)|^2dx$$
$$\le \frac{1}{2}\int_\delta^{l(s)} |u_x(s,x)|^2dx - \int_s^t u_x(\tau,\delta)\,f_o'(\tau)d\tau. \tag{5.3}$$

Similarly, for any $0 < s \le t \le T$,

$$\frac{1}{C_2}\int_s^t\int_{l(\tau)}^{1-\delta} |\rho(u)_\tau|^2\,dxd\tau - \frac{1}{2}\int_s^t |u_x(\tau,l(\tau)+)|^2\,l'(\tau)\,d\tau + \frac{1}{2}\int_{l(t)}^{1-\delta} |u_x(t,x)|^2dx$$
$$\le \frac{1}{2}\int_{l(s)}^{1-\delta} |u_x(s,x)|^2dx + \int_s^t u_x(\tau,1-\delta)\,f_1'(\tau)d\tau.$$

Since

$$u_x(\tau,l(\tau)-)^2\,l'(\tau) - u_x(\tau,l(\tau)-)^2\,l'(\tau)$$

$$= -(-u_x(\tau,l(\tau)-) + u_x(\tau,l(\tau)+))(u_x(\tau,l(\tau)-) + u_x(\tau,l(\tau)+))l'(\tau)$$

$$= -|l'(\tau)|^2(u_x(\tau,l(\tau)-)+u_x(\tau,l(\tau)+))$$

$$\leq |l'(\tau)|^3 \qquad \text{for a.e. } \tau \in [0, T],$$

we infer (5.1) for $0 < s \leq t \leq T$ from (5.3) and (5.4).

Combining Lemma 4.1 with Lemma 5.1 we see that the inequality (2.12) with $B_1 = B_1'$ and $B_2' = 2B_2'$ holds for any $0 < s \leq t \leq T$. Besides, on account of (3.3), (2.12) holds for $s = 0$, too.

Second Step. In the general case of $\rho \in \Gamma(C_1, C_2)$ we take a sequence ρ_n of smooth functions in $\Gamma(C_1, C_2)$ such that

$$\rho_n \to \rho \text{ uniformly on each bounded subset of } R.$$

Now, let $\{u_n, l_n\}$ be the solution of $SP(\rho_n; \{b_o^t\}, \{b_1^t\}; u_o, l_o)$ on $[0, T_n^*)$, where $[0, T_n^*)$ is the maximal interval of existence. In view of the first step we have

$$\chi(t, u_n(t)) + \frac{1}{C_2}\int_0^t\int_0^1 |\rho_n(u_n)_\tau|^2 dx d\tau + \frac{1}{2}\int_0^t |l_n'|^3 d\tau$$

$$\leq \chi(0, u_o) + \int_0^t |\alpha_o'(\tau)|(|u_{n,x}(\tau,0+)| + |u_{n,x}(\tau,1-)|)\gamma(\tau,u_n(\tau))^{1/2}d\tau \quad (5.5)$$

$$+ \int_0^t |\alpha_1'(\tau)|\gamma(\tau,u_n(\tau))d\tau \qquad \text{for any } t \in [0, T_n^*).$$

For each n we put

$$T_n = \sup\{t < \min\{T, T_n^*\}; \delta \leq l_n \leq 1-\delta \text{ on } [0, T]\}.$$

LEMMA 5.2. *There is a constant $M_1 \geq 0$, independent of n, such that*

$$|u_n|_{L^2(0,T_n;X)} \leq M_1,\ |u_n(t)|_H \leq M_1 \text{ for any } T \in [0, T_n),$$

$$\left|\int_0^{T_n}\{b_o^t(u_n(\tau,0)) + b_1^\tau(u_n(\tau,1))\}d\tau\right| \leq M_1.$$

P r o o f. First, choose functions $h_i \in W^{1,2}(0,T)$, $i = 0, 1$, such that $b_i^{(\cdot)}(h_i(\cdot))$ are bounded on $[0, T]$; in fact, we can take as h_i as solutions of $h_1'(t) + \partial b_i^t(h_i(t)) \ni 0$ on $[0, T]$ (cf. [8; Chapter 1]). Next, consider the function
$h = h(t,x)$ on $[0, T] \times [0, 1]$, given by

$$h(t,x) = \begin{cases} h_o(t)\left(1-\dfrac{x}{\delta}\right) & \text{for } (t,x) \in [0, T] \times [0, \delta], \\ 0 & \text{for } (t,x) \in [0, T] \times [\delta, 1-\delta], \\ h_1(t)\left(1-\dfrac{1}{\delta}+\dfrac{x}{\delta}\right) & \text{for } (t,x) \in [0, T] \times [1-\delta, 1]. \end{cases}$$

Then we observe that for any $t \in [0, T_n)$

$$\int_0^t \int_0^1 \rho_n(u_n)_\tau (u_n - h)\,dxd\tau = \int_0^t \int_0^{l_n(\tau)} u_{n,xx}(u_n - h)\,dxd\tau + \int_0^t \int_{l_n(\tau)}^1 u_{n,xx}(u_n - h)\,dxd\tau$$

$$= -\int_0^t \int_0^1 u_{n,x}(u_{n,x} - h_x)\,dxd\tau - \int_0^t u_{n,x}(\tau,0+)(u_n(\tau,0) - h_o(\tau))d\tau$$

$$+ \int_0^t u_{n,x}(\tau,1-)(u_n(\tau,1) - h_1(\tau))d\tau$$

$$\leq \frac{1}{2}\int_0^t \int_0^1 |h_x|^2 dxd\tau - \frac{1}{2}\int_0^t \int_0^1 |u_{n,x}|^2 dxd\tau + \int_0^t \{b_o^\tau(h_o(\tau)) + b_1^\tau(h_1(\tau)\}d\tau$$

$$- \int_0^t \{b_o^\tau(u_n(\tau,0)) + b_1^\tau(u_n(\tau,1))\}d\tau.$$

On the other hand, for any $t \in [0, T_n)$,

$$\int_0^t \int_0^1 \rho_n(u_n)_\tau (u_n - h)\,dxd\tau = \int_0^t \int_0^1 \rho_n(u_n)_\tau u_n dxd\tau - \int_0^t \int_0^1 \rho_n(u_n)_\tau h dxd\tau$$

$$= \int_0^1 \rho_n^*(u_n(t,x))\,dx - \int_0^1 \rho_n^*(u_o(x))\,dx + \int_0^t \int_0^1 \rho_n(u_n) h_\tau dxd\tau$$

$$+ \int_0^1 \rho_n(u_o(x)) h(0,x)\,dx - \int_0^1 \rho_n(u_n(t,x)) h(t,x)dx,$$

where

$$\rho_n^*(r) = \tilde{\rho}_n(\rho_n(r)) \text{ with } \tilde{\rho}_n(s) = \int_0^s \rho_n^{-1}(\sigma)d\sigma.$$

Noting that

$$k_1|r|^2 \leq \rho_n^*(r) \leq k_2|r|^2,\ r \in R,$$

for some constants $k_1, k_2 > 0$ independent of n, we have for any $0 \leq t \leq T_n$

$$\int_0^t \int_0^1 \rho_n(u_n)_\tau (u_n - h)\,dxd\tau \geq k_3 \int_0^1 |\,u_n(t,x)^2\,|\,dx - k_4 \int_0^t \int_0^1 |u_n|\,dxd\tau - k_5,$$

where $k_3, k_4, k_5 > 0$ are positive constants independent of n and t. Hence,

$$\frac{1}{2}\int_0^t \int_0^1 |u_{n,n}|^2 dxd\tau + \int_0^t \{b_o^\tau(u_n(\tau,0)) + b_1^\tau(u_n(\tau,1))\}d\tau + k_3 \int_0^t |u_n(t,x)|^2 dx \quad (5.6)$$

$$- k_4 \int_0^t \int_0^1 |u_n|\,dxd\tau \leq M_2' \qquad \text{for any } t \in [0, T_n),$$

where M_1' is a positive constant independent of n and t. Also, for any $0 \leq t < T_n$,

$$b_o^t(u_n(t,0)) + b_1^t(u_n(t,1)) \geq -B_1(|u_n(t,0)| + |u_n(t,1)|) - B_2$$
$$\geq -\frac{1}{4}|u_{n,x}(t)|_H^2 - k_6|u_n(t)|_H^2 - k_7, \tag{5.7}$$

where k_6, k_7 are constants independent of n and t. In order to get the last inequality we have used (3.4)′. Accordingly, (5.6) and (5.7) yield an inequality of the following form:

$$\frac{1}{4}\int_0^t |u_{n,x}(\tau)|_H^2 d\tau + k_3|u_n(t)|_H^2 \leq k_8 \int_0^t |u_n(\tau)|_H^2 d\tau + k_9 \qquad \text{for } 0 \leq t < T_n, \tag{5.8}$$

where k_8, k_9 are some constants independent of n and t. Applying Gronwall's inequality to (5.8), we get the desired estimates for a constant $M_1 > 0$ independent of n. ■

LEMMA 5.3. *There is a constant $M_2 \geq 0$ independent of n, such that*

$$|\rho_n(u_n)|_{W^{1,2}(0,T_n;H)} \leq M_2, \qquad |u_n|_{L^\infty(0,T_n;H)} \leq M_2,$$
$$|b_i^{(\cdot)}(u_n(\cdot,i))|_{L^\infty(0,T_n)} \leq M_2,\ i = 0, 1,\ |l_n'|_{L^3(0,T_n)} \leq M_2.$$

Proof. We use the following inequalities which are derived from (3.4):

$$|u_n(t,0)| + |u_n(t,1)| \leq \varepsilon|u_{n,x}(t)|_H + C(\varepsilon)|u_n(t)|_H$$

and

$$|u_{n,x}(t,0+)|^2 + |u_{n,x}(t,1-)|^2 \leq \varepsilon(|u_{n,xx}(t)|^2_{L^2(0,\delta)} + |u_{n,xx}(t)|^2_{L^2(1-\delta,1)})$$
$$+ C(\delta,\varepsilon)|u_{n,x}(t)|_H^2 \leq \varepsilon|\rho_n(u_n)'(t)|_H^2 + C(\delta,\varepsilon)|u_{n,x}(t)|_H^2.$$

By the estimates in Lemma 5.2 together with the above inequalities it follows from (5.5) that for any $t \in [0, T_n)$

$$\chi(t,u_n(t)) + k_{10}\int_0^t \int_0^1 |\rho_n(u_n)_\tau|^2 dx d\tau + \frac{1}{2}\int_0^t |l_n'(\tau)|^3 d\tau$$
$$\leq \chi(0, u_o) + k_{11}\int_0^t (\alpha_o'(\tau)|^2 + |\alpha_1'(\tau)|)(\chi(\tau,u_n(\tau)) + k_{12})d\tau + k_{13}, \tag{5.9}$$

where $k_{10},\cdots,k_{13}$ are positive constants independent of n and t. It is easy to see from (5.9) that the required inequalities hold. ■

LEMMA 5.4. $T_n = T$ *for all large n.*
Proof. By Lemma 5.3, $u_n \in W^{1,2}(0, T_n; H) \cap L^\infty(0, T_n; X)$ and $l_n \in W^{1,3}(0, T_n)$ for each n. We also have

$$l_n(T_n) = \delta \text{ or } 1-\delta, \text{ if } T_n < T. \tag{5.10}$$

In fact, if $T_n < T$ and $\delta < l_n(T_n) < 1-\delta$, then $u_n(T_n) = \lim_{t \uparrow T_n} u_n(t)$ exists in H and weakly in X. By Lemma 3.3, the data $\{b_i^t; t \geq T_n\}, i = 0, 1, u_n(T_n), l_n(T_n)$ satisfy the strong compatibility condition with T_n as the initial time. Hence, the local existence theorem (Theorem 3.1) yields a contradiction to the definition of T_n.

Next, suppose that $T_{n_k} < T$ for a subsequence $\{n_k\}$ of $\{n\}$, and $T_{n_k} \to T'$ as $k \to \infty$. We then note by Remark 3.1 that $T' > 0$. On account of Lemma 5.3, given $\varepsilon > 0$, there exists $T'_\varepsilon < T'$ such that

$$T' - \varepsilon \leq T'_\varepsilon \leq T_{n_k}, \quad |l_{n_k}(T'_\varepsilon) - l_{n_k}(T_{n_k})| \leq \varepsilon \text{ for all large } k. \tag{5.11}$$

Moreover, in view of Lemmas 3.1, 3.2 and Lemma 5.3 we may assume that

$$u_{n_k} \to \tilde{u} \text{ in } C([0, T'_\varepsilon] \times [0, 1]) \text{ and weakly* in } L^\infty(0, T'_\varepsilon; X),$$

$$\rho_{n_k}(u_{n_k}) \to \rho(\tilde{u}) \text{ in } C([0, T'_\varepsilon] \times [0, 1]) \text{ and weakly in } W^{1,2}(0, T'_\varepsilon; H),$$

$$u_{n_k, x}(\cdot, 0+) \to \tilde{u}_x(\cdot, 0+),\ u_{n_k}(\cdot, 1-) \to \tilde{u}_x(\cdot, 1-) \text{ in } L^2(0, T'_\varepsilon)$$

and

$$l_{n_k} \to \tilde{l} \text{ in } C([0, T'_\varepsilon]) \text{ and weakly in } W^{1,3}(0, T'_\varepsilon).$$

By the expression (2.2)′ of the free boundary we have

$$l_{n_k}(t) = l_o + \int_0^1 \rho_{n_k}(u_o)(x)\,dx - \int_0^1 \rho_{n_k}(u_{n_k})(t,x)\,dx$$
$$+ \int_0^t \{u_{n_k,x}(\tau, 1-) - u_{n_k,x}(\tau, 0+)\}\,d\tau \quad \text{for any } t \in [0, T'_\varepsilon].$$

Letting $k \to \infty$ in this inequality yields

$$\tilde{l}(t) = l_o + \int_0^1 \rho(u_o)(x)\,dx - \int_0^1 \rho(\tilde{u})(t,x)\,dx +$$
$$\int_0^t \{\tilde{u}_x(\tau, 1-) - \tilde{u}_x(\tau, 0+)\}\,d\tau$$

for any $t \in [0, T_\varepsilon']$. Therefore, $\{\tilde{u}, \tilde{l}\}$ is the solution of SP $(\rho; \{b_o^t\}, \{b_l^t\}; u_o, l_o)$ on $[0, T_\varepsilon']$, and by the uniqueness of solution we have

$$\tilde{u} = u \quad \text{on} \quad [0, T_\varepsilon'] \times [0, 1], \tilde{l} = l \quad \text{on} \quad [0, T_\varepsilon'].$$

Moreover, it follows from (5.10) and (5.11) that

$$|l(T_\varepsilon') - \delta| \leq \varepsilon \quad \text{or} \quad |l(T_\varepsilon') - (1 - \delta)| \leq \varepsilon$$

for an arbitrary $\varepsilon > 0$. But this contradicts that $\delta < l < 1 - \delta$ on $[0, T]$.

Q.E.D. ■

From Lemma 5.4 it follows that (5.5) holds for any $t \in [0, T]$ and large n. Therefore, the passage to the limit in n yields that (2.12) is valid for any $t \in [0, T]$ and $s = 0$. Repeating the same argument as above in the case of initial time $s \in (0, T)$, we see that (2.12) holds for any $0 \leq s \leq t \leq T$.

We now accomplish the proof of Theorem 2.3.

Proof of THEOREM 2.3: On account of Lemma 3.2, we easily see under the assumptions of Theorem 2.3 that for $s \in (0, T)$, $\{b_i^t\}_{t \geq s}$, $i = 0, 1$, $l(s)$ and $u(s)$ satisfy the strong compatibility condition with s as initial time. Hence by the above fact, inequality (2.12) holds for $0 < s \leq t \leq T$.

Next, it should be noted from (2.12) that the function $t \to X(t, u(t))$ is of bounded variation on each compact subset of $(0, T]$,

$$\chi(t, u(t)) - \chi(s, u(s)) \leq \int_s^t \frac{d}{d\tau} \chi(\tau, u(\tau))\, d\tau \quad \text{for any} \quad 0 < s \leq t \leq T$$

and

$$\frac{d}{d\tau} \chi(\tau, u(\tau)) + \frac{1}{C_2} |\rho(u)'(\tau)|_H^2 + \frac{1}{2} |l'(\tau)|^3$$

$$\leq |\alpha_0'(\tau)|(|u_x(\tau, 0+)| + |u_x(\tau, 1-)|)\gamma(\tau, u(\tau))^{1/2} + |\alpha_1'(\tau)|\gamma(\tau, u(\tau)) \quad (5.12)$$

$$\text{for a.e. } \tau \in [0, T].$$

Multiplying both sides of (5.12) by $(\tau - s)$ and integrating them over $[s, t]$, $0 \leq s \leq t \leq T$, we have (2.13) for any $0 \leq s \leq t \leq T$. ■

6. Some remarks on one-phase Stefan problems

One-phase problem is regarded as a special case of two-phase problem. Given $\rho \in \Gamma(C_1, C_2)$, $\{b^t\} \in B(\alpha_o, \alpha_1)$, $u_o \in L^2(0, \infty)$ and $l_o > 0$, we denote by SP$_o$ $(\rho;$

$\{b^t\}; u_o, l_o)$ on $[0, T]$ the problem of finding $u = u(t, x)$ on $[0, T] \times [0, \infty)$ and $x = l(t) > 0$ on $[0, T)$ such that

$$\rho(u)_t - u_{xx} = 0 \quad \text{in } Q_l^+(T) = \{(t, x); 0 < t < T, \quad 0 < x < l(t)\}, \tag{6.1}$$
$$u(0, x) = u_o(x) \quad \text{for a.e.} \quad x \geq 0, \tag{6.2}$$
$$u_x(t, 0+) \in \partial b^t(u(t, 0)) \quad \text{for a.e.} \quad t \in [0, T], \tag{6.3}$$
$$u(t, x) = 0 \quad \text{for any} \quad t \in (0, T] \quad \text{and} \quad x \geq l(t), \tag{6.4}$$
$$\begin{cases} l'(t) = -u_x(t, l(t)-) & \text{for a.e.} \quad t \in [0, T], \\ l(0) = l_o. \end{cases} \tag{6.5}$$

A pair $\{u, l\}$ is called a solution of SP_o on $[0, T]$, if $u \in C([0, T]; L^2(0, \infty)) \cap W^{1,2}_{loc}((0, T]; L^2(0, \infty)) \cap L^2(0, T; W^{1,2}(0, \infty)) \cap L^\infty_{loc}((0, T]; W^{1,2}(0, \infty))$, $l \in C([0, T]) \cap W^{1,2}_{loc}((0, T])$, $b^{(\cdot)}(u(\cdot, 0)) \in L^1(0, T) \cap L^\infty_{loc}((0, T])$ and (6.1) – (6.5) hold. A solution of SP_o on R_+ is defined in a similar way to the two-phase problem.

In view of Lemma 3.3, it is easy to see that $\{u, l\}$ is a solution of SP_o on $[0, T]$ if and only if it is a solution of the two-phase problem SP $(\rho; \{b_o^t\}, \{b_1^t\}; u_o, l_o)$, with $b_o^t = b^t$ and $b_1^t(r) = 0$ for $r = 0$ and $= \infty$ otherwise, on $[0, T]$, provided that $0 < l < 1$ on $[0, T]$ and $u_o = 0$ a.e. on $[l_o, \infty)$.

In such a sense one-phase problem is regarded as a special case of a two-phase problem. Therefore, as far as the local existence and uniqueness of solution to SP_o are concerned, similar results to Theorems 3.1, 2.2 and 2.3 are valid, even though $l_o \geq 1$ and the problem is formulated on the half line $[0, \infty)$ of the spatial variable x. Moreover, we prove:

PROPOSITION 6.1. *Let $\rho \in \Gamma(C_1, C_2)$ and $\{b^t\} \in B(\alpha_o, \alpha_1)$ such that*
$$\partial b^t(r) \subset (-\infty, 0] \textit{ for any } t \leq 0 \textit{ and } r < 0. \tag{6.6}$$
Let $l_o > 0$ and $u_o \in L^2(0, \infty)$ such that
$$u_o \geq 0 \text{ a.e. on } 0, l_o] \text{ and } u_o \leq 0 \text{ a.e. on } l_o, \infty). \tag{6.7}$$

Then there exists one and only one solution $\{u, l\}$ of $SP_o(\rho; \{b^t\}, u_o, l_o)$ on R_+ such that l is non-decreasing on R_+ and

$$t^{1/3} l' \in L^3_{loc}(R_+),$$

and such that function $t \to t|u_x(t)|^2_{L^2(0, \infty)}$ is locally bounded on R_+ and

$$t^{1/2} u' \in L^2_{loc}(R_+; L^2(0, \infty)).$$

Proof. (First Step) In case

$$u_o \in W^{1,2}(0, \infty),\ u_o(0) \in D(b^0),\ u_o \geq 0 \text{ on } [0, l_o] \text{ and } u_o = 0 \text{ on } [l_o, \infty), \tag{6.8}$$

by Theorems 3.1 and 2.2 problem SP_0 has a unique solution $\{u, l\}$ on $[0, T^*)$, $T^* > 0$, where $[0, T^*)$ is the maximal interval of existence. Since $u \geq 0$ on $Q_l^+(T^*)$ by Lemma 3.3, $u_x(t, l(t)-) \leq 0$ for a.e. $t \in [0, T^*)$, so that $l' \geq 0$ a.e. on $[0, T^*)$, that is, l is non-decreasing on $[0, T^*)$. We are going to show $T^* = \infty$. To the contrary, suppose $T^* < \infty$. Then, by Theorem 2.3,

$$\chi_o(t, u(t)) + \frac{1}{C_2}\int_0^t |\rho(u)'(\tau)|^2_{L^2(0,\infty)}\, d\tau + \frac{1}{2}\int_0^t |l'(\tau)|^3\, d\tau \tag{6.9}$$

$$\leq \chi_o(0, u_o) + \int_0^t |\alpha_o'(\tau)||u_x(\tau, 0+)|\gamma_o(\tau, u(\tau))^{1/2}\, d\tau +$$

$$+ \int_0^t |\alpha_1'(\tau)|\gamma_o(\tau, u(\tau))\, d\tau$$

and

$$t\chi_o(t, u(t)) + \frac{1}{C_2}\int_0^t \tau|\rho(u)'(\tau)|^2_{L^2(0,\infty)}\, d\tau + \frac{1}{2}\int_0^t \tau|l'(\tau)|^3\, d\tau$$

$$\leq \int_0^t \chi_o(\tau, u(\tau))\, d\tau + \int_0^t \tau|\alpha_o'(\tau)||u_x(\tau, 0+)|\gamma_o(\tau, u(\tau))^{1/2}\, d\tau \tag{6.10}$$

$$+ \int_0^t \tau|\alpha_1'(\tau)|\gamma_o(\tau, u(\tau))\, d\tau$$

for any $t \in [0, T^*)$, where

$$\chi_o(t, z) = \frac{1}{2}|z_x|^2_{L^2(0,\infty)} + b^t(z(0)),\ \gamma_o(t, z) = b^t(z(0)) + B_1|z(0)| + B_2$$

for $z \in W^{1,2}(0, \infty)$, and B_1, B_2 are positive constants determined by T^*, the norms of α_o', α_1' in $L^1(0, T^*)$ and b^0. Since $(0, T^*) \times (0, \delta) \subset Q_l^+(T^*)$ for $0 < \delta < l_o$, it follows from (6.9), with the aid of inequality (cf. (3.4)′)

$$|z_x(0+)| \leq \varepsilon|z_{xx}|_{L^2(0,\delta)} + C(\delta, \varepsilon)|z_x|_{L^2(0,\delta)},$$

that $\rho(u)' \in L^2(0, T^*; L^2(0, \infty))$, $u \in L^\infty(0, T^*; W^{1,2}(0, \infty))$, $b^{(\cdot)}(u(\cdot, 0)) \in L^\infty(0, T^*)$ and $l \in W^{1,3}(0, T^*)$. Hence, $u(T^*) \in W^{1,2}(0, \infty)$, $u(T^*, 0) \in D(b^{T^*})$, $u(T^*, \cdot) \geq 0$ on $[0, l(T^*)]$ and $u(T^*, \cdot) = 0$ on $[l(T^*), \infty)$. This implies that $\{u, l\}$ can be extended beyond T^*, which contradicts the definition of T^*. Therefore T^* must be infinite.

(Second Step) In the general case of u_o we take a sequence $u_{o,n}$ satisfying (6.8) such that $u_{o,n} \to u_o$ in $L^2(0, \infty)$, and denote by $\{u_n, l_n\}$ the solution of SP $(\rho; \{b^i\}; u_n, l_o)$ on R_+. We note that the free boundary $x = l_n(t)$ admits the representation (cf. [11]):

$$l_n(t)^2 = l_o^2 + 2\int_0^\infty x\rho(u_o)(x)\,dx - 2\int_0^\infty x\rho(u_n)(t, x)\,dx + 2\int_0^t u_n(\tau, 0)\,d\tau,$$

and inequality (6.10) holds for each n. Also, just as in Lemma 5.2, we can prove that $\{u_n\}$ is bounded in $L^\infty_{loc}(R_+; L^2(0, \infty))$ and in $L^2_{loc}(R_+; W^{1,2}(0, \infty))$. Therefore, using the above expression for the free boundary, we see by a slight modification of the proof of Theorem 1.1 in [11] that there is a subsequence of $\{n\}$, denoted by $\{n\}$ again, such that

$$u_n \to u \quad \text{in} \quad C([0, T]; L^2(0, \infty)) \quad \text{and in} \quad L^2(0, T; W^{1,2}(0, \infty))$$

and

$$l_n \to l \quad \text{in} \quad C([0, T])$$

for every finite $T > 0$, and the limit $\{u, l\}$ is the solution of $\mathrm{SP}_o(\rho; \{b^i\}; u_o, l_o)$ on R_+ having the desired properties. ∎

Employing the same technique as in the second step of the proof of Proposition 6.1, we can prove:

PROPOSITION 6.2. *Let ρ, $\{b^i\}$, u_o and l_o be as in Proposition 6.1. Also, let $l_{o,n} > 0$ and $u_{o,n} \in L^2(0, \infty)$ such that for each $n = 1, 2, \ldots$, the same type of condition as* (6.7) *holds and*

$$l_{o,n} \to l_o \text{ and } u_{o,n} \to u_o \text{ in } L^2(0, \infty) \text{ (as } n \to \infty).$$

Then, the solutions $\{u_n, l_n\}$ of $\mathrm{SP}_o(\rho; \{b^i\}; u_{o,n}, l_{o,n})$ on R_+ converge to the solution $\{u, l\}$ of $\mathrm{SP}_o(\rho; \{b^i\}; u_o, l_o)$ on R_+ in the sense that

$$u_n \to u \text{ in } C([0, T]; L^2(0, \infty)) \cap L^2(0, T; W^{1,2}(0, \infty))$$

and

$$l_n \to l \quad \text{in} \quad C([0, T])$$

for every finite $T > 0$.

In the proof of Theorems 2.1 and 2.5, we need another one-phase Stefan problem. Given $\rho \in \Gamma(C_1, C_2)$, $\{b^i\} \in B(\alpha_o, \alpha_1)$, $u_o \in L^2(-\infty, 1)$ and $l_o < 1$, $\mathrm{SP}_1(\rho; \{b^i\}; u_o, l_o)$ on $[0, T]$ is the problem of finding $u = u(t, x)$ on $[0, T] \times (-\infty, 1]$ and $x = l(t) < 1$ on $[0, T]$ such that

$$\rho(u)_t - u_{xx} = 0 \quad \text{in} \quad Q_l^-(T) = \{(t,x); 0 < t < T, l(t) < x < 1\}, \tag{6.1'}$$
$$u(0,x) = u_o(x) \quad \text{for a.e.} \quad x \le 1, \tag{6.2'}$$
$$-u_x(t,-1) \in \partial b^t(u(t,1)) \quad \text{for a.e.} \quad t \in [0,T], \tag{6.3'}$$
$$u(t,x) = 0 \quad \text{for any} \quad t \in (0,T] \quad \text{and} \quad x \le l(t), \tag{6.4'}$$
$$\begin{cases} l'(t) = u_x(t, l(t)+) \quad \text{for a.e.} \quad t \in [0,T], \\ l(0) = l_o. \end{cases} \tag{6.5'}$$

We say that $\{u, l\}$ is a solution of SP_1 on $[0, T]$, if $u \in C([0,T]; L^2(-\infty, 1)) \cap W^{1,2}_{loc}((0, T]; L^2(-\infty, 1)) \cap L^2(0, T; W^{1,2}(-\infty, 1)) \cap L^\infty_{loc}((0, T); W^{1,2}(-\infty, 1))$, $b^{(\cdot)}(u(\cdot, 1)) \in L^1(0, T) \cap L^\infty_{loc}((0, T])$, $l \in C([0, T]) \cap W^{1,2}_{loc}((0, T])$ and (6.1)′ – (6.5)′ are satisfied. A solution of SP_1 on R_+ is defined in a way similar to the problem SP_o on R_+.

As for problem SP_1 we have:

PROPOSITION 6.1′. *Let $\rho \in \Gamma(C_1, C_2)$ and $\{b^t\} \in B(\alpha_o, \alpha_1)$ such that*

$$\partial b^t(r) \subset [0, \infty) \quad \textit{for any} \quad t \ge 0 \quad \textit{and} \quad r > 0.$$

Let $l_o < 1$ and $u_o \in L^2(-\infty, 1)$ such that

$$u_o \le 0 \quad \text{a.e. } \textit{on} \quad l_o, 1] \quad \textit{and} \quad u_o = 0 \quad \text{a.e. } \textit{on} \quad (-\infty, l_o].$$

Then there exists one and only one solution $\{u, l\}$ of $SP_1(\rho; \{b^t\}; u_o, l_o)$ on R_+ such that l is non-increasing on R_+ and $t^{1/3} l' \in L^3_{loc}(R_+)$, and such that the function $t \to t|u_x(t)|^2_{L^2(-\infty,1)}$ is locally bounded on R_+ and $t^{1/2}u'$ belongs to $L^2_{loc}(R_+; L^2(-\infty, 1))$.

Concerning problem SP_1, the convergence result similar to Proposition 6.2 holds, too.

7. Proofs of Theorems 2.1, 2.4 and 2.5

We begin with the following lemma.

LEMMA 7.1. *Let ρ, $\{b_i^t\}$, $i = 0, 1$, l_o and u_o be as in Theorem 2.1, and let $\{u, l\}$ be the solution of $SP(\rho; \{b_o^t\}, \{b_i^t\}; u_o, l_o)$ on an interval $[0, T_o)$, $0 < T_o < \infty$. Suppose that*

$$0 < \inf_{t \in [0, T_o)} l(t) \le \sup_{t \in [0, T_o)} l(t) < 1. \tag{7.1}$$

Then we have

$$\begin{cases} u \in W^{1,2}(T_o - \varepsilon, T_o; H) \cap L^\infty(T_o - \varepsilon, T_o; X), \\ l \in W^{1,3}(T_o - \varepsilon, T_o), \\ b_i^{(\cdot)}(u(\cdot, i)) \cap L^\infty(T_o - \varepsilon, T_o), \ i = 0, 1, \end{cases} \tag{7.2}$$

for some $0 < \varepsilon < T_o$, and the solution $\{u, l\}$ is extendable beyond the time T_o.
Proof. By assumption (7.1), there is a constant $\delta > 0$ such that

$$\delta < l < 1 - \delta \text{ on } [0, T_o).$$

Hence we obtain (7.2) from (2.12) with inequalities (3.4) and (3.4)'. Besides, (7.2) and Lemma 3.3 imply that $u(T_o) \in X$, $u(T_o, i) \in D(b_i^{T_o})$, $u(T_o, \cdot) \geq 0$ on $[0, l(T_o)]$ and $u(T_o, \cdot) \leq 0$ on $[l(T_o), 1]$, so that on account of Theorem 3.1, SP $(\rho; \{b_o^t\}, \{b_i^t\}; u(T_o), l(T_o))$ has a solution on a certain interval $[T_o, T_o']$, $T_o' > T_o$. The continuation of $\{u, l\}$ by this solution gives a solution of SP on $[0, T_o']$. ■

LEMMA 7.2. *Let ρ, $\{b_i^t\}$, $i = 0, 1$, u_o, l_o be as in Theorem 2.1, and let $\{u, l\}$ be the solution of SP $(\rho; \{b_o^t\}, \{b_i^t\}; u_o, l_o)$ on $[0, T^*)$, where $[0, T^*)$ is the maximal interval of existence. Suppose $T^* < \infty$. Then, for each $\varepsilon \in (0, T^*)$, u is bounded on $[T^* - \varepsilon, T^*) \times [0, 1]$.*
Proof. Consider the problem

$$\begin{cases} \rho(v)_t - v_{xx} = 0 \quad \text{in } (0, \infty) \times (0, 1), \\ v(0,x) = \begin{cases} u_o(x) & \text{for a.e. } x \in [0, l_o], \\ 0 & \text{for } x \in (l_o, 1], \end{cases} \\ v_x(t, 0+) \in \partial b_o^t(v(t, 0)) \quad \text{for a.e. } t \geq 0, \\ v(t, 1) = 0 \quad \text{for } t > 0. \end{cases} \tag{7.3}$$

It is well known (cf. [9, 10]) that problem (7.3) has a unique solution v in $C(R_+; H) \cap W^{1,2}_{\text{loc}}((0,\infty); H) \cap L^{\infty}_{\text{loc}}((0,\infty); X)$ $(\subset C((0,\infty0 \times [0, 1]))$. Comparing u with v on $Q_l^+(T^*)$, we have by Lemmas 3.3 and 3.4

$$0 \leq u \leq v \qquad \text{on } Q_l^+(T^*).$$

Since v is continuous on $[T^* - \varepsilon, T^*] \times [0, 1]$, it follows that u is bounded on $Q_l^+(T^*) \cap [T^* - \varepsilon, T^*] \times [0, 1]$ for each $0 < \varepsilon < T^*$. Similarly u is bounded on $Q_l^-(T^*) \cap [T^* - \varepsilon, T^*] \times [0, 1]$. ■

Proof of Theorem 2.1. We first take a sequence $\{u_{o,n}, l_o\}$ of initial data satisfying the strong compatibility condition and $u_{o,n} \to u_o$ in H (as $n \to \infty$). Also, put

$$z_{o,n} \text{ (resp. } z_o) = \begin{cases} u_{o,n} \text{ (resp. } u_o) & \text{on } [0, l_o], \\ 0 & \text{on } [l_o, \infty), \end{cases}$$

and

$$z_{1,n}\ (\text{resp. } z_1) = \begin{cases} 0 & \text{on } [-\infty, l_o], \\ u_{o,n}\ (\text{resp. } u_o) & \text{on } [l_o, 1]. \end{cases}$$

Now, denote by $\{u_n, l_n\}$ the solution of SP $(\rho; \{b_o^t\}, \{b_1^t\}, u_{o,n}, l_o)$ on the maximal interval $[0, T_n^*)$ of existence, and by $\{u_i^n, l_n^i\}$, $i = 0, 1$, the solution of one-phase Stefan problem $\mathrm{SP}_i(\rho; \{b_i^t\}; z_{i,n}, l_o)$ on R_+. Then, by Theorem 2.2 we have

$$u_n^1 \leq u_n \leq u_n^o \text{ on } (0, T_n^*) \times [0, 1],\ l_n^1 \leq l_n \leq l_n^o \text{ on } (0, T_n^*), \tag{7.4}$$

because $\{u_n^o, l_n^o\}$ (resp. $\{u_n^1, l_n^1\}$) is a solution of SP $(\rho; \{b_o^t\}, \{\hat{b}_1^t\}, z_{o,n}, l_o)$ (resp. SP $(\rho;$ $\{\hat{b}_o^t\}, \{b_1^t\}; z_{1,n}, l_o))$ on $[0, T_n')$ with $T_n' = \sup\{t > 0; l_n^o(t) < 1 \text{ (resp. } l_n^1(t) > 0)\}$, where

$$\hat{b}_1^t(r) = \begin{cases} 0 & \text{for } r \geq 0, \\ \infty & \text{for } r < 0, \end{cases} \qquad \hat{b}_o^t(r) = \begin{cases} 0 & \text{for } r \leq 0, \\ \infty & \text{for } r > 0. \end{cases}$$

We observe from Proposition 6.2 that

$$\begin{cases} l_n^i \to l^i \text{ in } C([0, T]),\ i = 0, 1, \\ u_n^o \to u^o \text{ in } C([0, T]; L^2(0, \infty)), \\ u_n^1 \to u^1 \text{ in } C([0, T]; L^2(-\infty, 1)) \end{cases} \tag{7.5}$$

for every finite $T > 0$, where $\{u^i, l^i\}$ is the solution of $\mathrm{SP}_i(\rho; \{b_i^t\}; z_i, l_o)$ on R_+, $i = 0, 1$. We note that there are positive constants δ, T_o such that

$$\delta \leq l_n^i \leq 1 - \delta \text{ on } [0, T_o] \text{ for } i = 0, 1 \text{ and large } n,$$

and by (7.4)

$$\delta \leq l_n(t) \leq 1 - \delta \text{ for } t \in [0, T_n^*) \cap [0, T_o] \text{ and large } n.$$

Hence, Lemma 7.1 implies that $T^*_n > T_o$ for large n, and just as in Lemmas 5.2 and 5.3 we see from (2.12) for $\{u_n, l_n\}$ on $[0, T_o]$ that $\{u_n\}$ is bounded in $W^{1,2}(T_o-\varepsilon, T_o; H) \cap L^\infty(T_o-\varepsilon, T_o; X)$, $\{l_n\}$ is bounded in $W^{1,3}(T_o-\varepsilon, T_o)$, and $\{b_i^{(\cdot)}(u_n(\cdot, i))\}$, $i = 0, 1$, are bounded in $L^\infty(T_o-\varepsilon, T_o)$ for every $0 < \varepsilon < T_o$. Using these facts together with (7.4) and (7.5), we can extract a subsequence of $\{n\}$, denote again by $\{n\}$, such that $u_n \to u$ in $C([0, T_o]; H)$, weakly in $W^{1,2}_{\mathrm{loc}}((0, T_o]; H)$ an weakly* in $L^\infty_{\mathrm{loc}}((0, T_o]; X)$, and $l_n \to l$ in $C([0, T_o])$ and weakly in $W^{1,3}_{\mathrm{loc}}((0, T_o])$. Besides, it is not difficult to see that the limit $\{u, l\}$ is the solution of SP $(\rho; \{b_o^t\}, \{b_1^t\}; u_o, l_o)$ on $[0, T_o]$, having the required properties. ■

P r o o f o f T h e o r e m 2 . 4 . For each n, w denote by $\{u_n, l_n\}$ the solution of SP_n on $[0, T_n^*)$, where $[0, T_n^*)$ is the maximal interval of existence. Just as Lemma 5.4, it can be shown that $T_n^* > T$ for sufficiently large n. Moreover, making use of the inequalities (2.12) and (2.13) for $\{u_n, l_n\}$ and employing a standard argument on the convergence of subdifferential operators (cf. [9, 13]) we obtain the required convergences. ■

P r o o f o f T h e o r e m 2 . 5 . Suppose $T^* < \infty$ and either (b) or (c) does not holds. Then, there would exist a sequence $\{t_n\}$ with $t_n \uparrow T^*$ (as $n \to \infty$) and two numbers $0 < x_1 < x_o < 1$ such that

$$x_1 < l(t_n) < x_o \text{ for any } n.$$

Now, let $\{V^i, L^i\}$ be the solution of one-phase problem $SP_i(\rho; \{b_{i,M}\}; V^i_o, x_i)$ on R_+ for $i = 0, 1$, where for a positive constant M

$$V^o_o(x) \text{ (resp. } V^1_o(x)) = \begin{cases} M \text{ (resp. } -M) \text{ for } 0 \le x \le x_o \text{ (resp. } x_1 \le x \le 1), \\ 0 \text{ for } x > x_o \text{ (resp. } x < x_1), \end{cases}$$

and

$$b_{o,M}(r) \text{ (resp. } b_{1,M}(r)) = \begin{cases} 0 \text{ for } r = M \text{ (resp. } r = -M), \\ \infty \text{ otherwise.} \end{cases}$$

We then note that these are one-phase Stefan problems with the usual Dirichlet boundary conditions $V^o = M$ on $x = 0$ and $V^1 = -M$ on $x = 1$, respectively. Here, the constant M is chosen so as to satisfy

$$|u| \le M \qquad \text{on } \left[\frac{T^*}{2}, T^*\right) \times [0, 1]; \tag{7.6}$$

this choice of M is possible by Lemma 7.2. Moreover, take positive number δ and T_o $(\le T^*/2)$ so that

$$\delta \le L^1(t) < L^o(t) \le 0 - \delta \qquad \text{for } t \in [0, T_o].$$

In this case, on account of (7.6), it follows from the usual comparison result for Stefan problems with Dirichlet boundary conditions that

$$L^1(t - t_n) \le l(t) \le L^o(t - t_n) \text{ for any } t \in [t_n, T^*) \text{ with } T^* - t_n \le T_o.$$

Therefore,

$$0 < \inf_{t \in [0,T*)} l(t) \leq \sup_{t \in [0,T*)} l(t) < 1,$$

so that by Lemma 7.1, $\{u, l\}$ is extendable beyond the time T^*. This contradicts the definition of T^*. Thus the case (b) or (c) holds true, if $T^* < \infty$ ∎

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Globalne istnienie rozwiązań dwufazowego zadania Stefana z warunkami nieliniowymi

W pracy wyprowadzono wyniki dotyczące istnienia rozwiązań dwufazowych zadań Stefana z nieliniowymi warunkami przepływu. W tym celu dowodzi się własności dotyczących specjalnych ocen energii dla szerokiej klasy danych początkowych.

Глобальное существование решений двухфазной задачи Стефана снелинейными условиями

В работе представлены результаты существования решений двухфазных задач Стефана с нелинейными условиями потока. Для этой цели доказываются свойства касающиеся специальных оценок энергии для широкого класса начальных данных.