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The existence of solutions to two-phase Stefan problems for nonlinear parabolic equations

by

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In this paper, we consider one-dimensional two-phase Stefan problems for a class of parabolic equations with nonlinear heat source terms and with nonlinear flux conditions on the fixed boundary. The flux conditions are described by time-dependent subdifferential operators in R and are interpreted as feedback boundary controls. The main purpose of this paper is to establish a local existence theorem for the Stefan problems, and our approach is based on the abstract theory of nonlinear evolution equations governed by time-dependent subdifferentials in Hilbert spaces.

Introduction

Let us consider the following two-phase Stefan problem:
Find a function $u = u(t,x)$ on $Q_0 = (0, T_0) \times (0,1), 0 < T_0 < \infty$, and a curve $x = l(t)$, $0 < l < 1$, on $[0, T_0]$ such that

$$\rho(u)_t - a(u_x)_x + h(t,x) = \begin{cases} f_0 & \text{in } Q_l^+, \\ f_1 & \text{in } Q_l^-, \end{cases} \tag{0.1}$$

$$h(t,x) \in g(t,x,u(t,x)) \text{ for a.e. } (t,x) \in Q_0,$$

$$Q_l^+ = \{(t,x)\,;\, 0 < t < T_0,\, 0 < x < l(t)\},$$

$$Q_l^- = \{(t,x)\,;\, 0 < t < T_0,\, l(t) < x < 1\},$$

$$\begin{cases} u(t,l(t)) = 0 \quad \text{for } 0 \leqslant t \leqslant T_0, \\ l'(t) = -a(u_x(t,l(t)-)) + a(u_x(t,l(t)+)) \text{ for a.e. } t \in [0,T_0], \\ l(0) = l_0, \end{cases} \tag{0.2}$$

$$u(0,x) = u_0(x) \quad \text{for } x \in (0,1), \tag{0.3}$$

$$\begin{cases} a(u_x(t,0+)) \in \partial b_0{}^t(u(t,0)) & \text{for a.e. } t \in [0,T_0], \\ -a(u_x(t,1-)) \in \partial b_1{}^t(u(t,1)) & \text{for a.e. } t \in [0,T_0], \end{cases} \tag{0.4}$$

where ρ: $R \to R$ and a: $R \to R$ are continuous increasing functions; for a.e. $(t,x) \in Q_0$, $r \to g(t,x,r)$ is a multivalued mapping in R; f_i $(i = 0,1)$ is a function on Q_0; l_0 is a number with $0 < l_0 < 1$ and u_0 is a function on $[0,1]$; b_i^t $(i = 0,1)$ is a proper l.s.c. convex function on R for each $t \in [0,T_0]$ and ∂b_i^t denotes its subdifferential in R.

In this paper, we treat a class of nonlinear parabolic equations of the form (0.1) that in particular reduce to

$$c_i u_t - (|u_x|^{p-2} u_x)_x + \sigma(u) \ni f_i, \qquad i = 0,1,$$

for positive constants c_0, c_1 and $2 \leqslant \rho < \infty$, where

$$\sigma(r) = \begin{cases} 1 & \text{for } r > 0 \\ [-1,1] & \text{for } r = 0 \\ -1 & \text{for } r < 0 \end{cases} \tag{0.5}$$

Also, it should be noticed that (0.4) represents various linear or nonlinear boundary conditions, such as Dirichlet, Neumann - or Signorini - type.

Many interesting results about existence and uniqueness of solutions and regularity of free boundaries are known for standard Stefan problems described by linear parabolic equations with Dirichlet or Neumann boundary conditions (cf. [2, 3, 4, 5, 8]). Also, Stefan problems for nonlinear parabolic equations have been studied, for instance, in [6, 7, 13], and those with non - standard boundary conditions of subdifferential type were earlier treated in [11, 15]. In particular, Stefan problems for nonlinear equation $cu_t - (|u_x|^{p-2} u_x)_x = f$ were formulated and an existence result was established in [10].

Our problem (0.1) - (0.4) is more general than those in the papers quoted above. We are especially interested in the heat source terms f_i, $i = 0,1$ and $g(t,x,u)$ in $(0,1)$, which cause that the set $\{(t,x); u(t,x) = 0\}$ may have the non - empty interior in Q_0. Also, we are interested in the boundary conditions (0.4) which are interpreted as a feedback flux control on the boundary $x = 0, 1$. The uniqueness of solution to the same type of problems as (0.1) - (0.4) was already discussed in [12]. The aim of this paper is to establish a general existence result for problem (0.1) - (0.4).

1. Main results

In what follows, let $0 < T < \infty$ and $Q = (0, T) \times (0, 1)$. We begin with introducing the precise assumptions (a1)-(a4) on ρ, a, g and b_i^t, $i = 0, 1$, under which Stefan problem (0.1)-(0.4) is discussed.

(a1) ρ: $R \to R$ is a bi-Lipschitz continuous and increasing function with $\rho(0) = 0$; denote by C_ρ a Lipschitz constant of ρ and φ^{-1}

(a2) a: $R \to R$ is a continuous function such that
$a_0 |r|^p \leqslant a(r) r \leqslant a_1 |r|^p$ for any $r \in R$,
$a_0 (r - r')^{p-1} \leqslant a(r) - a(r')$ for any $r, r' \in R$, $r \geqslant r'$,
where a_0 and a_1 are positive constants and $2 \leqslant p < \infty$.

(a3) For a.e. $(t, x) \in Q$, the mapping $r \to g(t, x, r)$ is set-valued in R such that $g(t, x, r)$ is a non-empty closed interval in R for any $r \in R$, $0 \in g(t, x, 0)$ and $g(t, x, r)$ is u.s.c. with respect to $r \in R$. Moreover suppose that for each $M > 0$, g has the following properties (i)-(iii):

(i) $r \to g(t, x, r) + C_M r$ is monotone in r with $|r| \leqslant M$ for a positive constant C_M depending only on M, that is,

$$(r_1^{'} + C_M r_1 - r_2^{'} - C_M r_2)(r_1 - r_2) \geqslant 0 \text{ if } |r_i| \leqslant M \text{ and } r_i^{'} \in g(t, x, r_i), i = 1, 2.$$

(ii) $|r^{'}| \leqslant g_{0,M}(t, x)$ for $r^{'} \in g(t, x, r)$, r with $|r| \leqslant M$ and a.e. $(t, x) \in Q$, where $g_{0,M}$ is a non-negative function in $L^2(Q)$;

(iii) for any λ with $0 < \lambda < 1/C_M$ and r with $|r| \leqslant M$, $[I + \lambda g(t, x, \cdot)]^{-1} r$ is measurable in $(t, x) \in Q$.

(a4) For $i = 0, 1$ and each $t \in [0, T]$, b_i^t is a proper l.s.c. convex function $\alpha_0 \in W^{1,2}(0, T)$, $\alpha_1 \in W^{1,1}(0, T)$:
(*) For any $0 \leqslant s \leqslant t \leqslant T$ and $r \in D(b_i^s) \equiv \{ r \in R; b_i^s(r) < \infty \}$ there exists $r' \in D(b_i^t)$ such that

$$|r' - r| \leqslant |\alpha_0(t) - \alpha_0(s)| (1 + |r| + |b_i^s(r)|^{1/p}),$$

$$b_i^t(r') - b_i^s(r) \leqslant |\alpha_1(t) - \alpha_1(s)| (1 + |r|^p + |b_i^s(r)|).$$

Furthermore, for f_i, $i = 0, 1$, u_0 and l_0 we suppose that

(a5) $f_0, f_1 \in L^2(Q)$;

(a6) $0 < l_0 < 1$, $u_0 \in W^{1,p}(0, 1)$, $u_0(l_0) = 0$, $u_0(i) \in D(b_i^0)$, $i = 0, 1$.

Now, we denote by $P = P(b_0^t, b_1^t; g; f_0, f_1; u_0; l_0)$ the system (0.1)-(0.4), and say that a pair $\{u, l\}$ is a solution of P on $[0, T_0]$, $0 < T_0 \leqslant T$, if the following properties (i)-(iii) are fulfilled:

(i) $u \in W^{1,2}(0, T_0; L^2(0, 1)) \cap L^\infty(0, T_0; W^{1,p}(0, 1))$ (hence $u \in C(\bar{Q}_0)$), and $l \in W^{1,2}(0, T_0) (\subset C([0, T_0]))$ with $0 < l < 1$ on $[0, T_0]$,

(ii) (0.1) holds in the sense of $D'(Q_l^+)$ and $D'(Q_l^-)$ for some $h \in L^2(Q_0)$ with $h(t,x) \in g(t,x,u(t,x))$ for a.e. $(t,x) \in Q_0$, and (0.2) and (0.3) are satisfied,

(iii) $b_i^{(\cdot)}(u(\cdot, i))$ is bounded on $[0, T_0]$, $u(t, i) \in D'(\partial b_i^t)$ for a.e. $t \in [0, T_0]$, $i = 0, 1$, and (0.4) holds.

THEOREM 1.1. *Suppose that assumptions (a1)-(a6) hold. Then there exists T_0 with $0 < T_0 \leqslant T$ such that problem P has at least one solution $\{u, l\}$ on $[0, T_0]$.*

The class of functions $g(t,x,r)$ satisfying (a3) includes any locally Lipschitz continuous function $k(r)$ on R. Therefore, in general, a solution u of $\rho(u)_t - a(u_x)_x + g(t,x,u) \ni f$ blows up at some finite time, so problem P might not be expected to have a solution on the whole time interval $[0, T]$. In this paper we devote ourselves to the proof of the above local existence theorem, and shall discuss global existence and behavior of solutions in the author's forthcoming paper [1].

The construction of a solution to problem (0.1)-(0.4) is done in the following way.

First, given a curve $x = l(t)$, $0 < l(t) < 1$, on $[0, T]$, $f_i \in L^2(Q)$, $i = 0, 1$, and $u_0 \in W^{1,p}(0, 1)$, we consider the initial boundary value problem $(\mathrm{IBP})_0 = (\mathrm{IBP})_0(l; f_0, f_1; u_0)$ which is formulated by

$$\rho(u)_t - a(u_x)_x = \begin{cases} f_0 & \text{in } Q_l^+ = \{(t,x);\, 0 < x < l(t),\, 0 < t < T\}, \\ f_1 & \text{in } Q_l^- = \{(t,x);\, l(t) < x < 1,\, 0 < t < T\}, \end{cases} \tag{1.1}$$

$$u(0, x) = u_0(x) \quad \text{for } 0 \leqslant x \leqslant 1, \tag{1.2}$$

$$u(t, l(t)) = 0 \quad \text{for } 0 \leqslant t \leqslant T, \tag{1.3}$$

$$a(u_x(t, 0+)) \in \partial b_0^t(u(t, 0)) \quad \text{for a.e. } t \in [0, T], \tag{1.4}$$

$$-a(u_x(t, 1-)) \in \partial_1^t(u(t, 1)) \quad \text{for a.e. } t \in [0, T]. \tag{1.5}$$

This problem can be uniquely solved as a direct application of the theory of nonlinear evolution equations generated by time-dependent subdifferentials.

Next, we consider the problem $(\mathrm{IBP})_0$ with (1.1) replaced by

$$\rho(u)_t - a(u_x)_x + h = \begin{cases} f_0 & \text{in } Q_l^+, \\ f_1 & \text{in } Q_l^-, \end{cases} \tag{1.6}$$

$$h(t, x) \in g_M(t, x, u(t, x)) \quad \text{for a.e. } (t, x) \in Q,$$

where f_0 and f_1 are given functions in $L^2(Q)$, M is a positive number so that $|u_0| < M$ on $[0, 1]$ and

$$g_M(t, x, r) = \begin{cases} \sup\{r'; r' \in g(t,x,M)\} & \text{for } r > M, \\ g(t,x,r) & \text{for } r \text{ with } |r| \leqslant M, \\ \inf\{r'; r' \in g(t,x,-M)\} & \text{for } r < -M. \end{cases} \tag{1.7}$$

This problem is denoted by $(\mathrm{IBP})_M = (\mathrm{IBP})_M(l; f_0, f_1; u_0)$ and can be solved by using the uniform estimates for solutions of $(\mathrm{IBP})_0$ with respect to l, f_0 and f_1.

Finally, by the standard fixed point theorem we seek for a curve $x = l(t)$ on $[0, T]$ and a time T', $0 < T' \leqslant T$, satisfying

$$\begin{cases} l(0) = l_0, \ 0 < l < 1 \text{ on } [0, T'], \\ l'(t) = -a(u_x(t, l(t)-)) + a(u_x(t, l(t)+)) \quad \text{for a.e. } t \in [0, T'], \end{cases}$$

where u is a solution of $(\mathrm{IBP})_M$ on Q. It is proved that this pair $\{u, l\}$ gives a solution of problem (0.1)-(0.4) on a time-interval $[0, T_0]$ with $0 < T_0 \leqslant T'$.

2. Initial boundary value problem $(\mathrm{IBP})_0$

In the sequel, for simplicity we put

$$H = L^2(0, 1), \qquad X = W^{1,p}(0, 1) \ (\subset C([0, 1]))$$

and

$$(f, g)_H = \int_0^1 f(x) g(x)\, dx, \qquad \text{for } f, g \in H.$$

Throughout this section, we assume (a1)-(a6) to be satisfied.
We fix a number δ with $0 < \delta < 1 - \delta$ and put

$$\Lambda_\delta = \{l \in C([0, T]); \delta \leqslant l(t) \leqslant 1 - \delta \text{ on } [0, T]\}.$$

In order to solve $(\mathrm{IBP})_0(l; f_0, f_1; u_0)$ for each $l \in \Lambda_\delta$ by an application of the subdifferential operator theory, we introduce the family $\{\varphi_l^t\}_{t\in[0,T]}$ of functions φ_l^t on H formulated by

$$\varphi_l^t(z) = \begin{cases} \int_0^1 A(z_x(x))dx + b_0^t(z(0)) + b_1^t(z(1)) & \text{if } z \in K^l(t), \\ \infty & \text{otherwise,} \end{cases} \tag{2.1}$$

where

$$K^l(t) = \{z \in X; z(l(t)) = 0, z(i) \in \mathrm{D}(b_i^t), i = 0, 1\}, 0 \leqslant t \leqslant T,$$

and

$$A(r) = \int_0^r a(s)ds \quad \text{for any } r \in R.$$

LEMMA 2.1. (*cf., Kenmochi* [11; *Lemmas* 2.1, 3.1]). (1) *There are positive constants* R_1, R_2 *depending only on* α_0, α_1 *and* T *such that*

$$b_i^t(r) + R_1|r| + R_1 \geqslant 0 \quad \textit{for any } t \in [0, T], \textit{any } r \in R \textit{ and } i = 0, 1,$$

and

$$|b_i^t(r)| \leqslant b_i^t(r) + R_2|r| + R_2 \textit{ for any } t \in [0, T], \textit{any } r \in R \textit{ and } i = 0, 1.$$

(2) *For each* $l \in \Lambda_\partial$ *and* $t \in [0, T]$, $\varphi_l^t(\cdot)$ *is a proper l.s.c. convex function on* H *and* $\mathrm{D}(\varphi_l^t) = K^l(t)$.

(3) *There are positive constants* R_3, R_4 *and* R_5 *depending only on* α_0, α_1, a_0, a_1, T *and* ∂ *such that*

$$|z_x|^p L_p(0,1) \leqslant R_3\varphi_l^t(z) + R_4 \textit{ for any } t \in [0, T], z \in K^l(t) \textit{ and } l \in \Lambda_\partial,$$

and

$$|b_i^t(z(i))| \leqslant \varphi_l^t(z) + R_5 \textit{ for any } t \in [0, T], z \in K^l(t), l \in \Lambda_\partial \textit{ and } i = 0, 1.$$

(4) *There is a positive constant* R_6 *depending only on* α_0, α_1, a_0, a_1, T *and* ∂ *such that for any* $l \in \Lambda_\partial$, $\varphi_l^t(\cdot)$ *has the following property* (**):

(**) *For any* $0 \leqslant s \leqslant t \leqslant T$ *and* $z \in \mathrm{D}(\varphi_l^s)$ *there is* $\tilde{z} \in \mathrm{D}(\varphi_l^t)$ *such that*
$$|\tilde{z} - z|_H \leqslant R_6\{|l(t) - l(s)| + |\alpha_0(t) - \alpha_0(s)|\}(1 + |\varphi_l^s(z)|^{1/2})$$
and
$$\varphi_l^t(\tilde{z}) - \varphi_l^s(z) \leqslant R_6\{|l(t) - l(s)| + |\alpha_0(t) - \alpha_0(s)| + |\alpha_1(t) - \alpha_1(s)|\} \cdot \cdot (1 + |\varphi_l^s(z)|)$$

(5) *For each* $l \in \Lambda_\partial$ *and* $t \in [0, T]$, *the subdifferential* $\partial\varphi_l^t$ *of* φ_l^t *in* H *is single-valued and characterized as follows*: $z' = \partial\varphi_l^t(z)$ *if and only if* $z' \in H$, $z \in X$ *and*

$$\begin{cases} -a(z_x)_x = z' & \text{in } D'(0, l(t)) \text{ and } D'(l(t), 1), \\ z(l(t)) = 0, \\ a(z_x(0^+)) \in \partial b_0{}^t(z(0)), \\ -a(z_x(1^-)) \in \partial b_1{}^t(z(1)). \end{cases}$$

By γ_0 we denote the following function on $L^2(0,1)$:

$$\gamma_0(z) = \int_0^1 z^+(x)\,dx \quad \text{for } z \in L^2(0, 1).$$

This function is continuous, non-negative and convex on $L^2(0, 1)$. Also, by B we denote the operator from $D(B) = H$ into itself given by

$$[Bz](x) = \rho^{-1}(z(x)) \quad \text{for } z \in H,\ x \in (0, 1),$$

which is bi-Lipschitz continuous in H, and is the subdifferential of the convex function j on H given by

$$j(v) = \int_0^1 \int_0^{v(x)} \rho^{-1}(r)\,drdx \quad \text{for } v \in H.$$

LEMMA 2.2 (*cf., Kenmochi* [9; *Lemma* 3.4.3]).

(1) $\gamma_0(z) + \gamma_0(-z) = |z|\, L^1(0,1)$ *for* $z \in L^2(0, 1)$.

(2) *For each* $l \in \Lambda_\delta$ *and* $t \in [0, T]$, $\partial\varphi_l^t \circ B$ *is* γ_0*-accretive in* H, i.e. *if* z *and* z_1 *are any points with* $Bz, Bz_1 \in D(\partial\varphi_l^t)$, *then* $(\partial\varphi_l^t(Bz) - \partial\varphi_l^t(Bz_1), w)_H \geqslant 0$ *for some* $w \in \partial\gamma_0(z - z_1)$, *where* $\partial\gamma_0$ *is the subdifferential of* γ_0 *in* H.

(3) *For each* $l \in \Lambda_\delta$, $t \in [0, T]$ *and* $r \geqslant 0$, *the set* $\{z \in H;\ |z|_H \leqslant r,\ \varphi_l^t(z) \leqslant r\}$ *is compact in* H.

For $L > 0$, we put

$$\Lambda_\partial(L) = \{l \in \Lambda_\partial;\ |l'|\,L_2(0, T) \leqslant L\}.$$

The above Lemmas 2.1 and 2.2 allow us to apply the abstract results from [9; Theorems 2.8.1.-2.8.3], and we see that for $l \in \Lambda_\delta(L)$ and $\bar f \in L^2(0, T; H)$ the Cauchy problem

$$\begin{cases} v'(t) + \partial\varphi_l^t(B(v(t))) = \bar f(t), & \text{for a.e. } t \in [0, T], \\ v(0) = \rho(u_0) \end{cases} \tag{2.2}$$

has a unique solution v in $W^{1,2}(0, T; H)$ such that the function $t \to \varphi_l^t(Bv(t))$ is bounded on $[0, T]$. Let v be a solution of (2.2) for $\bar f = X_l^+ \bar f_0 + X_l^- \bar f_1$, where X_l^+, X_l^- are the characteristic functions of Q_l^+, Q_l^-, respectively. Then, by Lemma 2.1 it is easy to see that $u = Bv$ is a unique solution of $(\mathrm{IBP})_0(l; \bar f_0, \bar f_1; u_0)$ and $u \in W^{1,2}(0, T; H) \cap L^\infty(0, T; X)$.

Next we mention a lemma on some uniform estimates for solutions to $(\mathrm{IBP})_0$ with respect to the data.

LEMMA 2.3. *Let k_1 be any positive number. Then there is a constant $R_7 > 0$, depending only on the quantities in conditions* (a1), (a2), (a4), *the class $\Lambda_\delta(L)$ and k_1, such that*

$$|u|L^\infty(0, T; X) + |u|W^{1,2}(0, T; H) \leqslant R_7, \tag{2.3}$$

and

$$\sup_{t \in [0, T]} |b_i'(u(t, i))| \leqslant R_7 \quad \text{for } i = 0, 1, \tag{2.4}$$

whenever u is the solution of $(\mathrm{IBP})_0(l; \bar{f}_0, \bar{f}_1; u_0)$ and $l \in \Lambda_\delta(L), \bar{f}_i \in L^2(Q)$ with $|\bar{f}_i|L^2(Q) \leqslant k_1$, $i = 0, 1$, and $u_0 \in X$ satisfying $u_0(l(0)) = 0$ and $|b_i^0(u_0(0))| + |u_0|_X \leqslant k_1$, $i = 0, 1$.

This lemma is a direct consequence of [9; Theorem 2.8.4], too.

The following lemma is concerned with convergence of solutions of $(\mathrm{IBP})_0$.

LEMMA 2.4. *Let $l \in \Lambda_\delta(L)$, $l_n \in \Lambda_\delta(L)$, $n = 1, 2, \ldots$ and $\bar{f}_i \in L^2(Q)$, $i = 0, 1$, $\bar{f}_{i,n} \in L^2(Q)$, $i = 0, 1$, $n = 1, 2, \ldots$. Suppose that*

$$l_n \to l \text{ uniformly on } [0, T],$$

and for $i = 0, 1$,

$$\bar{f}_{i,n} \to \bar{f}_i \text{ weakly in } L^2(Q).$$

Let u and u_n be the solutions of $(\mathrm{IBP})_0(l; \bar{f}_0, \bar{f}_1; u_0)$ and $(\mathrm{IBP})_0(l_n; \bar{f}_{0,n}, \bar{f}_{1,n}; u_{0,n})$ on $[0, T]$. Then,

$$u_n \to u \text{ in } C([0, T]; H), \text{ in } L^p(0, T; X), \text{ weakly in } W^{1,2}(0, T; H) \text{ and} \tag{2.5}$$

$$\text{weakly * in } L^\infty(0, T; X), \text{ (hence in } C(\bar{Q})),$$

$$a(u_{n,x}(\cdot, 0+)) \to a(u_x(\cdot, 0+)) \quad \text{in } L^{p'}(0, T), \tag{2.6}$$

$$a(u_{n,x}(\cdot, 1-)) \to a(u_x(\cdot, 1-)) \quad \text{in } L^{p'}(0, T), \tag{2.7}$$

$$a(u_{n,x}(\cdot, l_n(\cdot) \pm)) \to a(u_x(\cdot, l(\cdot) \pm)) \quad \text{in } L^{p'}(0, T), \tag{2.8}$$

where $1/p + 1/p' = 1$.

P r o o f. The lemma can be proved in a way similar to that of [9; Lemma 2.8.8] and [11; Proposition 4.1]. For simplicity we write $\varphi_n{}^t$, φ^t for $\varphi_{l_n}^t$, φ_l^t and $\bar{f}_n$, $\bar{f}$ for $X_{l_n}^+ \bar{f}_{0,n} + X_{l_n}^- \bar{f}_{1,n}$, $X_l^+ \bar{f}_0 + X_l^- \bar{f}_1$, respectively, where $X_{l_n}^+$, $X_{l_n}^-$, X_l^+ and X_l^- are the characteristic functions of $Q_{l_n}^+$, $Q_{l_n}^-$, Q_l^+, and Q_l^-, respectively. Clearly

$$\bar{f}_n \to \bar{f} \text{ weakly in } L^2(0, T; H) \text{ as } n \to \infty . \tag{2.9}$$

By virtue of a convergence result concerning convex functions [11; Lemma 4.1], we have

$$\varphi_n{}^t \to \varphi^t \text{ on } H \text{ in the sense of Mosco for each } t \in [0, T] \text{ as } n \to \infty . \tag{2.10}$$

By Lemma 2.3,

$$|u_n|_{L^\infty(0,T;X)} + |u_n|_{W^{1,2}(0,T;H)} \leqslant R_7 \qquad \text{for } n = 1, 2, \dots .$$

Therefore we can select a subsequence of $\{u_n\}$, denoted by $\{u_n\}$ again, such that u_n converges to some v in $C([0, T]; H)$, and evidently $v \in W^{1,2}(0, T; H)$ and

$$\rho(u_n)_t \to \rho(v)_t \quad \text{weakly in } L^2(0, T; H), \tag{2.11}$$

$$\varphi^t(v(t)) \leqslant \lim_{n\to\infty} \inf \varphi_n{}^t(u_n(t)) \quad \text{for any } t \in [0, T]. \tag{2.12}$$

Next we show that

$$\int_0^T \varphi_n{}^t(u_n(t))dt \to \int_0^T \varphi^t(v(t))dt. \tag{2.13}$$

In fact, since $\bar{f}_n(t) - \rho(u_n)_t(t) = \partial\varphi_n{}^t(u_n(t))$, it follows that

$$\begin{aligned} &\int_0^T (\bar{f}_n(t) - \rho(u_n)_t(t), w(t) - u_n(t))_H dt \\ &\leqslant \int_0^T \varphi_n{}^t(w(t))\, dt - \int_0^T \varphi_n{}^t(u_n(t))\, dt \end{aligned} \tag{2.14}$$

for any $w \in L^2(0, T; H)$ with $\varphi_n{}^{(\cdot)}(w(\cdot)) \in L^1(0, T)$. Corresponding to the function v, we take a sequence $\{z_n\} \subset L^2(0, T; H)$ such that

$$z_n \to v \text{ in } L^2(0, T; H), \quad \int_0^T \varphi_n{}^t(z_n(t))\, dt \to \int_0^T \varphi^t(v(t))\, dt :$$

indeed such a sequence $\{z_n\}$ exists by (2.10) (cf., [9; Proposition 2.7.1]). Substituting z_n as w in (2.14) and letting $n \to \infty$, we get by (2.9) and (2.14)

$$\lim_{n\to\infty} \sup \int_0^T \varphi_n{}^t(u_n(t))dt \leqslant \int_0^T \varphi^t(v(t))dt. \tag{2.15}$$

Hence from (2.12) and (2.15) we see (2.13). Next, given $\bar{w} \in L^2(0, T; H)$ with $\varphi^{(\cdot)}(w(\cdot)) \in L^1(0, T)$, we choose a sequence $\{w_n\}$ in $L^2(0, T; H)$ such that

$$w_n \to \bar{w} \text{ in } L^2(0, T; H), \quad \int_0^T \varphi_n{}^t(w_n(t))\, dt \to \int_0^T \varphi^t(v(t))\, dt ,$$

and substitute w_n as w of (2.14). Then by (2.13), letting $n \to \infty$ yields

$$\int_0^T (\rho(v)_t(t) - \bar{f}(t), v(t) - \bar{w}(t))_H dt \leqslant \int_0^T \varphi^t(\bar{w}(t))\, dt - \int_0^T \varphi^t(v(t))\, dt ,$$

which shows by the definition of subdifferential $\partial\varphi^t$ that

$$\bar{f}(t) - \rho(v)_t(t) \in \partial\varphi^t(v(t)) \quad \text{for a.e. } t \in [0, T].$$

Since the solution of Cauchy problem (2.2) is unique, we see that $u = v$ and that $u_n \to u$ in $C([0, T]; H)$, weakly* in $L^\infty(0, T; X)$ and weakly in $W^{1,2}(0, T; H)$ without subtracting any subsequence of $\{u_n\}$. Furthermore, by the uniform convexity of $L^p(0, T; X)$, (2.13) implies

$$u_n \to u \quad \text{in} \quad L^p(0, T; X).$$

Thus (2.5) has been proved.

Finally, convergences (2.6)-(2.8) are obtained by using (2.5) in the following way. By virtue of (a2) and (2.5) we have

$$a(u_{n,x}) \to a(u_x) \quad \text{in} \quad L^{p'}(0, T; L^{p'}(0, 1)). \tag{2.16}$$

Given $\varepsilon > 0$, we choose a smooth function $\bar{l}$ in $\Lambda_\delta(L)$ so that

$$0 \leqslant l_n - \bar{l} \leqslant \varepsilon \quad \text{on } [0, T] \quad \text{for all large } n.$$

Observe that for a.e. $t \in [0, T]$,

$$\begin{aligned} &|a(u_{n,x}(t, l_n(t)-)) - a(u_x(t, l(t)-))|^{p'} \\ &\leqslant 9|a(u_{n,x}(t, l_n(t)-)) - a(u_{n,x}(t, \bar{l}(t)))|^{p'} + \\ &+ 9|a(u_{n,x}(t, \bar{l}(t))) - a(u_x(t, \bar{l}(t)))|^{p'} + \\ &+ 9|a(u_x(t, \bar{l}(t))) - a(u_x(t, l(t)-))|^{p'}. \end{aligned}$$

Also, for a.e. $t \in [0, T]$,

$$\begin{aligned} &|a(u_x(t, \bar{l}(t))) - a(u_x(t, l(t)-))|^{p'} \\ &= \Big|\int_{\bar{l}(t)}^{l(t)} a(u_x(t, x))_x dx\Big|^{p'} \end{aligned}$$

$$\leqslant \{ \int_{\bar{l}(t)}^{l(t)} (|\rho(u)_t(t,x)| + |\bar{f}_0(t,x)|)\, dx \}^{p'}$$

$$\leqslant 4|l(t) - \bar{l}(t)|^{p'/2} \{ |\rho(u)_t(t)|_H^{p'} + |\bar{f}_0(t)|_H^{p'} \}$$

$$\leqslant 4\varepsilon^{p'/2} \{ 2 + |\rho(u)_t(t)|_H^2 + |\bar{f}_0(t)|_H^2 \},$$

and similarly

$$|a(u_{n,x}(t, l_n(t)^-)) - a(u_{n,x}(t, \bar{l}(t)))|^{p'}$$

$\leqslant 4\varepsilon^{p'/2} \{ 2 + |\rho(u_n)_t(t)|_H^2 + |\bar{f}_{0,n}(t)|_H^2 \}$ for all large n. We choose a smooth function $\eta \in C^1([0,T])$ such that $0 \leqslant \eta \leqslant 1$ on $[0,T]$, $\eta(0) = 0$, $\eta = 1$ on $[\delta, 1]$, and set $C(\eta) = \sup\{|\eta_x(x)|; x \in [0,1]\}$. Then we have

$$\int_0^T |a(u_{n,x}(t, \bar{l}(t))) - a(u_x(t, \bar{l}(t)))|^{p'} dt$$

$$= \int_0^T | \int_0^{\bar{l}(t)} \frac{\partial}{\partial x} \{ \eta(x)(a(u_{n,x}(t,x)) - a(u_x(t,x))) \}^{p'} dx |\, dt$$

$$\leqslant \int_0^T \int_0^{\bar{l}(t)} C(\eta) p' |(a(u_{n,x}(t,x)) - a(u_x(t,x))|^{p'} dx dt +$$

$$+ \int_0^T \int_0^{\bar{l}(t)} p' |a(u_{n,x}(t,x))_x - a(u_x(t,x))_x| \, |a(u_{n,x}(t,x)) - a(u_x(t,x))|^{p'-1} dx dt$$

$$\leqslant C(\eta) p' |(a(u_{n,x}) - a(u_x)|^{p'}{}_{L^{p'}(0,T;L^{p'}(0,1))} +$$

$$+ p' |(a(u_{n,x}) - a(u_x)|^{p'-1}{}_{L^{p'}(0,T;L^{p'}(0,1))} \times$$

$$\times |(a(u_{n,x})_x - a(u_x)_x|_{L^2(0,T;H)} T^{(2-p')/2p'}$$

$$\leqslant C(\eta) p' |(a(u_{n,x}) - a(u_x)|^{p'}{}_{L^{p'}(0,T;L^{p'}(0,1))} +$$

$$+ p' T^{(2-p')/2p'} |(a(u_{n,x}) - a(u_x)|_{L^{p'}(0,T;L^{p'}(0,1))} \{ |\rho(u_n)_t|_{L^2(0,T;H)} +$$

$$+ |\rho(u)_t|_{L^2(0,T;H)} + |\bar{f}_{0,n}|_{L^2(0,T;H)} + |\bar{f}_0|_{L^2(0,T;H)} +$$

$$+ |\bar{f}_{1,n}|_{L^2(0,T;H)} + |\bar{f}_1|_{L^2(0,T;H)}\}.$$

Consequently, using (2.16), we have

$$\limsup_{n\to\infty} |a(u_{n,x}(\cdot, l_n(\cdot)-)) - a(u_x(\cdot, l(\cdot)-))|^{p'}_{L^{p'}(0,T)} \leqslant \text{const. } \varepsilon^{p'/2}.$$

Thus

$$a(u_{n,x}(\cdot, l_n(\cdot)-)) \to a(u_x(\cdot, l(\cdot)-)) \quad \text{in } L^{p'}(0,T).$$

The other convergences of (2.6), (2.7) and (2.8) can be shown in a similar way

3. Initial boundary value problem $(\mathrm{IBP})_\mathrm{M}$

In this section we solve $(\mathrm{IBP})_\mathrm{M}$.

Let $l \in \Lambda_\delta(L)$, $u_0 \in X$ and $f_i \in L^2(Q)$, $i = 0, 1$, and suppose that

$$u_0(l(0)) = 0,\ u_0(i) \in D(b_i^0),\ i = 0,1. \tag{3.1}$$

First of all we take a number $M > 0$ so that

$$|u_0(x)| < M \qquad \text{for all} \quad x \in [0,1], \tag{3.2}$$

and consider the multivalued function g_M on $[0,T] \times [0,1] \times R$ given by (1.7). Clearly, by (a3) the mapping $r \to g_M(t,x,r) + C_M r$ is maximal monotone in $r \in R$ for a.e. $(t,x) \in Q$, and

$$|r'| \leqslant g_{0,M}(t,x) \text{ for any } r' \in g_M(t,x,r),\ r \in R \text{ and a.e. } (t,x) \in Q, \tag{3.3}$$

where C_M and $g_{0,M}$ are correspondingly the non-negative constant and the function in condition (a3).

Now, for each $0 < \varepsilon \leqslant C_M^{-1}$ consider the following approximate problem $(\mathrm{IBP})_{M,\varepsilon}$:

$$\rho(u)_t - a(u_x)_x + g_{M,\varepsilon}(t,x,u) = \begin{cases} f_0 & \text{in } Q_l^+, \\ f_1 & \text{in } Q_l^-, \end{cases} \tag{3.4}$$

and initial-boundary conditions (1.2)-(1.5). Here, for a.e. $(t,x) \in Q$, $g_{M,\varepsilon}(t,x,\cdot)$ is the Yosida approximation of $g_M(t,x,\cdot)$, i.e.

$$g_{M,\varepsilon}(t,x,r) = \frac{1}{\varepsilon}\{r - [I + \varepsilon g_M(t,x,\cdot)]^{-1} r\},\ r \in R.$$

From (3.3) it is easy to see that

$$|g_{M,\varepsilon}(t,x,r)| \leqslant g_{0,M}(t,x) \quad \text{for any} \quad r \in R \text{ and a.e. } (t,x) \in Q. \tag{3.5}$$

LEMMA 3.1. *For each* $0 < \varepsilon \leqslant C_M^{-1}$, $(\mathrm{IBP})_{M,\varepsilon}$ *has a solution* u_ε *in* $W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$. *Moreover, there is a positive constant* R_M, *independent of* ε, *such that*

$$|u_\varepsilon|_{L^\infty(0,T;X)} + |u_\varepsilon|_{W^{1,2}(0,T;H)} \leqslant R_M. \tag{3.6}$$

P r o o f. For simplicity we write $g_{M,\varepsilon}(w)$ for $g_{M,\varepsilon}(t,x,w)$. Consider the mapping Γ_ε with assigns to each $w \in C([0,T];H)$ the solution u of $(\mathrm{IBP})_0(l;\bar{f}_0,\bar{f}_1;u_0)$ with $\bar{f}_0 = f_0 - g_{M,\varepsilon}(w)$ and $\bar{f}_1 = f_1 - g_{M,\varepsilon}(w)$. Since

$$|f_i - g_{M,\varepsilon}(w)|_{L^2(Q)} \leqslant |f_i|_{L^2(Q)} + |g_{0,M}|_{L^2(Q)}$$

for any $w \in C([0,T];H)$, it follows from Lemma 2.3 that

$$|\Gamma_\varepsilon w|_{L^\infty(0,T;X)} + |\Gamma_\varepsilon w|_{W^{1,2}(0,T;H)} \leqslant R_M \text{ for all } w \in C([0,T];H),$$

where R_M is a positive constant independent of ε.

Therefore Γ_ε is a compact operator from

$\{w \in C([0,T];H); w(0,x) = u_0(x)$ for $x \in [0,1]$,

$|w|_{C([0,T];H)} \leqslant T^{1/2} R_M + |u_0|_H\}$ into itself, so that Γ_ε has a fixed point u_ε, i.e. $\Gamma_\varepsilon u_\varepsilon = u_\varepsilon$. Clearly, u_ε is a solution of $(\mathrm{IBP})_{M,\varepsilon}$ satisfying (3.6).

LEMMA 3.2. *Let* u_ε *be the solution of* $(\mathrm{IBP})_{M,\varepsilon}$, *obtained by Lemma* 3.1, *for each* $0 < \varepsilon \leqslant C_M^{-1}$. *Then, there is a sequence* $\{\varepsilon_n\}$ *with* $\varepsilon_n \downarrow 0$ (as $n \to \infty$) *such that* $u_n := u_{\varepsilon_n}$ *converges to the solution* u *of* $(\mathrm{IBP})_M$ *in the following*

$$u_n \to u \text{ in } L^p(0,T;X), \text{ weakly in } W^{1,2}(0,T;H) \text{ and weakly * in } L^\infty(0,T;X), (\text{hence in } C(\bar{Q})), \tag{3.7}$$

$$a(u_{n,x}(\cdot,0+)) \to a(u_x(\cdot,0+)) \quad \text{in } L^{p'}(0,T), \tag{3.8}$$

$$a(u_{n,x}(\cdot,1-)) \to a(u_x(\cdot,1-)) \quad \text{in } L^{p'}(0,T), \tag{3.9}$$

$$a(u_{n,x}(\cdot,l(\cdot)\pm)) \to a(u_x(\cdot,l(\cdot)\pm)) \quad \text{in } L^{p'}(0,T), \tag{3.10}$$

where $1/p + 1/p' = 1$.

P r o o f. From (3.5)

$$|g_{M,\varepsilon}(t,x,u_\varepsilon(t,x))| \leqslant g_{0,M}(t,x) \quad \text{for a.e. } (t,x) \in Q.$$

Then there exists a subsequence $\{\varepsilon_n\}$ of $\{\varepsilon\}$ with $\varepsilon_n \downarrow 0$ and $h \in L^2(Q)$ such that $g_{M,\varepsilon_n}(t,x,u_{\varepsilon_n}(t,x)) \to h(t,x)$ weakly in $L^2(Q)$. By virtue of Lemma 2.4, $u_n := u_{\varepsilon_n}$ converges to the solution u of $(\mathrm{IBP})_0(l;f_0,f_1;u_0)$ with $f_0 = f_0 - h$ and $f_1 = f_1 - h$ in the of (3.7)-(3.10). Using basic properties of Yosida-approximation, we see that

$$h(t,x) \in g_M(t,x,u(t,x)) \quad \text{for a.e. } (t,x) \in Q.$$

Finally it is not difficult to check that u is the solution of $(\mathrm{IBP})_M$.

4. Proof of the Theorem 1.1

Let M be a positive number satisfying (3.2), L be any positive number and fix them in this section. Next, taking a positive number δ with

$$2\delta < l_0 < 1 - 2\delta,$$

we consider the subclass $\Lambda_\delta(L,l_0)$ of $\Lambda_\delta(L)$:

$$\Lambda_\delta(L,l_0) = \{l \in \Lambda_\delta(L); l(0) = l_0\}.$$

LEMMA 4.1. *Let u^l be a solution of $(\mathrm{IBP})_M(l;f_0,f_1;u_0)$ for each $l \in \Lambda_\delta(L,l_0)$. Then there is a number T', independent of l, with $0 < T' \leqslant T$ such that for all $l \in \Lambda_\delta(L,l_0)$*

$$|a(u_x^l(\cdot,l(\cdot)-))|_{L^2(0,\,T),} + |a(u_x^l(\cdot,l(\cdot)+))|_{L^2(0,\,T)} \leqslant L. \tag{4.1}$$

P r o o f. By Lemma 2.3 and (3.3) we have

$$|u^l|_{L^\infty(0,T;X)} + |u^l|_{W^{1,2}(0,T;H)} \leqslant R_8 \quad \text{for any } l \in \Lambda_\delta(L,l_0). \tag{4.2}$$

where R_8 is some positive constant. From (0.1) it follows that

$$\begin{aligned} |a(u_x^l)_x|_{L^2(Q_l^+)} &= |\rho(u^l)_t - h^l + f_0|_{L^2(Q_l^+)} \\ &\leqslant |\rho(u^l)_t|_{L^2(Q_l^+)} + |h^l|_{L^2(Q_l^+)} + |f_0|_{L^2(Q_l^+)} \\ &\leqslant C_\rho R_8 + |g_{0,M}|_{L^2(Q)} + |f_0|_{L^2(Q)} \equiv R_9. \end{aligned} \tag{4.3}$$

where $h^l = f_0 - \rho(u^l)_t + a(u_x^l)_x$ in Q_l^+ and R_9 is some positive constant independent of l.

For simplicity, we put $u \equiv u^l$. To prove the lemma, we use the following inequalities:

$$\int_0^y |v(x)|^2 dx \leqslant K_0(y)\,(|v|_{L^{p'}(0,y)}^{4p'/(p'+2)} |v_x|_{L^2(0,y)}^{(4-2p')/(p'+2)} + |v|_{L^{p'}(0,y)}^2)$$

$$\int_0^y |v(x)||v_x(x)|dx \leqslant K_0(y)(|v|_{L^{p'}(0,y)}^{p'/(p'+2)} |v_x|_{L^2(0,y)}^{(p'+4)/(p'+2)} +$$

$$+ |v|_{L^{p'}(0,y)} |v_x|_{L^2(0,y)}),$$

for $v \in \{v \in L^{p'}(0,y);\ v_x \in L^2(0,y)\}$ where $1/p + 1/p' = 1$ and $K_0(y)$ is a positive decreasing function of $y \geqslant 0$ independent of v (cf., O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Ural'ceva [14; Chap.2, Theorem 2.2]). $K_0 = K_0(\delta)$, using the same function η as in the proof of Lemma 2.4, we have for any $t > 0$,

$$\int_0^t |a(u_x(\tau, l(\tau)-))|^2 d\tau$$

$$= \int_0^t \int_0^{l(t)} \frac{\partial}{\partial x}(\eta(x)a(u_x(\tau,x)))^2 dxd\tau$$

$$= \int_0^t \int_0^{l(t)} 2\eta_x(x)\eta(x)a(u_x(\tau,x))^2 dxd\tau +$$

$$+ \int_0^t \int_0^{l(t)} 2\eta(x)^2 a(u_x(\tau,x))_x a(u_x(\tau,x))\,dxd\tau$$

$$\leqslant 2C(\eta)K_0 \int_0^t (|a(u_x(\tau,\cdot))|_{L^{p'}(0,l(\tau))}^{4p'/(p'+2)} |a(u_x(\tau,\cdot))_x|_{L^2(0,l(\tau))}^{(4-2p')/(p'+2)} +$$

$$+ |a(u_x(\tau,\cdot))|_{L^{p'}(0,l(\tau))}^2)\,d\tau +$$

$$+ 2K_0 \int_0^t (|a(u_x(\tau,\cdot))|_{L^{p'}(0,l(1))}^{p'/(p'+2)} |a(u_x(\tau,\cdot))_x|_{L^2(0,l(\tau))}^{(p'+4)/(p'+2)} +$$

$$+ |a(u_x(\tau,\cdot))|_{L^{p'}(0,l(\tau))} |a(u_x(\tau,\cdot))_x|_{L^2(0,l(\tau))})\,d\tau$$

$$\leqslant 2C(\eta)K_0 a_1^{4p'/(p'+2)} \int_0^t |u_x(\tau,\cdot)|_{L^p(0,l(\tau))}^{4p/(p'+2)} |a(u_x(\tau,\cdot))_x|_{L^2(0,l(\tau))}^{(4-2p')/(p'+2)} d\tau +$$

$$+ 2C(\eta)K_0 a_1^2 \int_0^t |u_x(\tau)|_{L^p(0,1)}^{2(p-1)} d\tau +$$

$$+ 2K_0 a_1^{p'/(p'+2)} \int_0^t |u_x(\tau,\cdot)|_{L^p(0,1)}^{p/(p'+2)} |a(u_x(\tau,\cdot))_x|_{L^2(0,l(\tau))}^{(5p-4)/(3p-2)} d\tau +$$

$$+ 2K_0 a_1 \int_0^t |u_x(\tau)|_{L^p(0,1)}^{(p-1)} |a(u_x(\tau))_x|_{L^2(0,l(\tau))} d\tau$$

$$\leqslant 2C(\eta)K_0 a_1^{4p'/(p'+2)} R_8^{4p/(p'+2)} R_9^{(2p-4)/(3p-2)} t^{2p/(3p-2)} +$$

$$+ 2C(\eta)K_0 a_1^2 R_8^{2(p-1)} t +$$

$$+ 2K_0 a_1^{p'/(p'+2)} R_8^{p/(p'+2)} R_9^{(5p-4)/(6p-4)} t^{p/(6p-4)} + 2K_0 a_1 R_8^{p-1} R_9^{1/2} t^{1/2}.$$

We have the same type of inequality for $a(u_x(\cdot, l(\cdot)+))$ as above. Hence the required inequality is inferred for a certain T' with $0 < T' \leqslant T$.

Now we define an operator $N: \Lambda_\delta(L, l_0) \to C([0,T])$ by putting

$$[Nl](t) = l_0 - \int_0^t a(u_x^l(\tau, l(\tau)-))d\tau + \int_0^t a(u_x^l(\tau, l(\tau)+))d\tau,$$

for each $l \in \Lambda_\delta(L, l_0)$ and $t \in [0,T]$. Moreover, let us consider the operator $N^*: \Lambda_\delta(L, l_0) \to C([0,T])$ given by

$$[N^* l](t) = \begin{cases} [Nl](t) & \text{for } 0 \leqslant t \leqslant T_1, \\ [Nl](T_1) & \text{for } T_1 \leqslant t \leqslant T, \end{cases}$$

for each $l \in \Lambda_\delta(L, l_0)$, where $T_1 = \min\{T', (\delta/L)^2\}$, T' being as in Lemma 4.1.

LEMMA 4.2. (1) *N^* maps $\Lambda_\delta(L, l_0)$ into itself.*
(2) *N^* is continuous in the topology of $C([0,T])$.*

Proof. We observe that

$$\left|\frac{d}{dt}[Nl](t)\right| \leqslant |a(u_x^l(t, l(t)-))| + |a(u_x^l(t, l(t)+))| \text{ for a.e. } t \in [0,T],$$

so that

$$\left|\frac{d}{dt}[N^* l]\right|_{L^2(0,T)} \leqslant \left|\frac{d}{dt}[Nl]\right|_{L^2(0,T_1)} \leqslant L \quad \text{for any } l \in \Lambda_\delta(L, l_0).$$

Also, by elementary calculation and the definition of T_1,

$|[N^* l](t) - l_0| \leqslant \delta$ for any $l \in \Lambda_\delta(L, l_0)$ and $t \in [0, T_1]$.

Thus (1) holds. In order to prove (2), let $l_n, l \in \Lambda_\delta(L, l_0)$ and assume $l_n \to l$ in $C([0, T])$. For simplicity we write u_n (resp. u) for the solution of $(\mathrm{IBP})_M(l; f_0, f_1; u_0)$ (resp. $(\mathrm{IBP})_M(l_n; f_0, f_1; u_0)$). By (3.3),

$$|h_n|_{L^2(Q)} \leqslant |g_{0,M}|_{L^2(Q)} \qquad \text{for } n = 1, 2, \dots,$$

where

$$h_n = \begin{cases} f_0 - \rho(u_n)_t + a(u_{n,x})_x & \text{in } Q_{l_n}^+, \\ f_1 - \rho(u_n)_t + a(u_{n,x})_x & \text{in } Q_{l_n}^-. \end{cases}$$

Therefore we can choose a subsequence $\{h_{n'}\}$ of $\{h_n\}$ with a function $\bar{h} \in L^2(Q)$ such that $h_{n'} \to \bar{h}$ weakly in $L^2(Q)$. We put $\bar{f}_{i,n'} = f_i + h_{n'}$ for $i = 0, 1$. Now, applying Lemma 2.4, we see that

$$u_{n'} \to v \text{ in } C([0, T]; H),$$

$$a(u_{n'x}(\cdot, l_{n'}(\cdot) \pm)) \to a(v_x(\cdot, l(\cdot) \pm)) \qquad \text{in } L^{p'}(0, T),$$

where $1/p + 1/p' = 1$ and v is the solution of $(\mathrm{IBP})_0(l; f_0, f_1; u_0)$ with $f_i = f_i + \bar{h}$ for $i = 0, 1$. Besides, by (a3),

$$\bar{h}(t, x) \in g_M(t, x, v(t, x)) \qquad \text{for a.e. } (t, x) \in Q.$$

This shows that v is the solution $(\mathrm{IBP})_M(l; f_0, f_1; u_0)$, and the uniqueness of a solution of $(\mathrm{IBP})_M(l; f_0, f_1; u_0))$ shows that $u = v$. Hence, without taking any subsequence of $\{n\}$, we see that

$$a(u_{n,x}(\cdot, l_n(\cdot) \pm)) \to a(u_x(\cdot, l(\cdot) \pm)) \qquad \text{in } L^{p'}(0, T).$$

Therefore $N^* l_n \to N^* l$ in $C([0, T])$.

We now accomplish the proof of Theorem 1.1. Since $\Lambda_\delta(L, l_0)$ is a compact convex set in $C([0, T])$, by the fixed point theorem there exists $l \in \Lambda_\delta(L, l_0)$ such that $N^* l = l$, and the pair $\{u^l, l\}$ is a solution of $P(b_0{}^l, b_1{}^l; g_M; f_0, f_1; u_0)$ on $[0, T_1]$. Noting that $u \in C(\bar{Q})$ and $|u_0| < M$, we can choose a number T_0 with $0 < T_0 \leqslant T_1$ such that $|u(t, x)| < M$ for $(t, x) \in [0, T_0] \times [0, 1]$. From the definition of g_M, it is follows that $g(t, x, u) = g_M(t, x, u)$ a.e. on $(0, T_0) \times (0, 1)$ and hence $\{u^l, l\}$ is a solution of $P(b_0{}^l, b_1{}^l; g; f_0, f_1; u_0)$ on $[0, T_0]$.

5. A comparison result and some examples

In this section we give a result on the comparison of solutions to Stefan problems and some examples of flux condition (0.4) and source term $g(t,x,u)$.

A comparison result is proved under the following sign conditions on the data f_0, f_1, b_0^t, b_1^t and u_0:

$$\left[\begin{array}{ll} \partial b_0^t(r) \subset (-\infty, 0] & \text{for any } r<0 \text{ and } t \in [0,T], \text{ and} \\ \partial b_1^t(r) \subset [0,\infty) & \text{for any } r>0 \text{ and } t \in [0,T]; \end{array}\right. \tag{5.1}$$

$$(-1)^i f_i \geqslant 0 \text{ a.e. on } Q, \text{ and } f_i \in L^1([0,T]; L^\infty(0,1)),\ i=0,1; \tag{5.2}$$

$$u_0 \geqslant 0 \text{ on } [0, l_0],\ u_0 \leqslant 0 \text{ on } [l_0, 1]. \tag{5.3}$$

THEOREM 5.1. (*cf., Kenmochi* [12]). *Let ρ and a be functions satisfying* (a1) *and* (a2), *respectively, a.d consider the Stefan problems $P = P(b_0^t, b_1^t; g; f_0, f_1; u_0, l_0)$ and $P = P(b_0^t, b_1^t; g; f_0, f_1; u_0\, l_0)$, where the set of data $(b_0^t, b_1^t; g; f_0, f_1; u_0, l_0)$ as well as $(b_0^t, b_1^t; g; f_0, f_1; u_0\, l_0)$ satisfies* (a3)-(a6) *and* (5.1)-(5.3).
Further suppose that

$$f_0 \leqslant f_0,\ f_1 \leqslant f_1 \text{ a.e. on } Q,$$

$$\left[\begin{array}{l} (r' - r')(r - r)^+ \geq 0 \text{ for any } r \in D(\delta b_i^t),\ r \in D(\delta b_i^t), \\ r' \in \delta b_i^t(r),\ \bar{r}' \in \delta \bar{b}_i^t(\bar{r}),\ i = 0, 1, \text{ and } t \in [0, T]; \end{array}\right.$$

for each $M > 0$ there is a positive constant C_M^* such that

$$\left[\begin{array}{ll} (r' - \bar{r}')(r - r)^+ + C_M^* |(r-r)^+|^2 \geqslant 0 & \text{for any } r \text{ with } |r| \leqslant M, \\ r' \in g(t,x,r),\ r \text{ with } |r| \leqslant M,\ r' \in \bar{g}(t,x,r) \text{ and a.e. } (t,x) \in Q; & \end{array}\right.$$

Let $\{u, l\}$ and $\{u, l\}$ be solutions of P and P on $[0, T_0]$, $0 < T_0 \leqslant T$, respectively. Then, we have

$$\begin{aligned} & |[\rho(u(t)) - \rho(u(t))]^+|_{L^1(0,1)} + [l(t) - l(t)]^+ \\ \leqslant & \{|[\rho(u(s)) - \rho(u(s))]^+|_{L^1(0,1)} + [l(s) - l(s)]^+\} \times \\ \times & \exp\{CC_\rho(t-s) + \int_s^t [|f_0(\tau)|_{L^\infty(0,1)} + |f_1(\tau)|_{L^\infty(0,1)}] d\tau\}, \end{aligned} \tag{5.4}$$

for any $0 \leqslant s \leqslant t \leqslant T_0$, *where M is any constant with* $|u| < M$ *and* $|\bar{u}| < M$ *a.e. on* $Q_0 = (0, T_0) \times (0, 1)$ *and* $C = \max\{C_M^*, C_M, \bar{C}_M\}$ *with the constants* $C_M, \bar{C}_M$ *in condition* (a3)-(i) *corresponding to the data of* P, $\bar{P}$, *respectively.*

P r o o f. Let M be any positive constant such that $|u| < M$ and $|\bar{u}| < M$ on $\bar{Q}_0$, and consider the multivalued functions g_M and $\bar{g}_M$, defined in the similar way as (1.7), corresponding to g and $\bar{g}$, respectively.

Now, denote by P_M (resp. $\bar{P}_M$) the problem P (resp. $\bar{P}$) with g (resp. $\bar{g}$) replaced by g_M (resp. $\bar{g}_M$). Then, since $g_M(t,x,u) = g(t,x,u)$ and $\bar{g}_M(t,x,\bar{u}) = \bar{g}(t,x,\bar{u})$, we see that $\{u, l\}$ and $\{\bar{u}, \bar{l}\}$ are solutions of P_M and $\bar{P}_M$, respectively. According to the comparison result of [12; Theorem], the inequality (5.4) holds for the solutions $\{u, l\}$ and $\{\bar{u}, \bar{l}\}$ of P_M and $\bar{P}_M$, respectively.

COROLLARY. *Assume* (a1)-(a6) *and* (5.1)-(5.3) *hold. Then problem* $P = P(b_0^t, b_1^t; g; f_0, f_1; u_0, l_0)$ *has at most one solution on any interval* $[0, T_0]$, $0 < T_0 \leqslant T$.

P r o o f. Apply Theorem 5.1 in case $P = \bar{P}$. Then from (5.4) we infer that if $\{u, l\}$ and $\{\bar{u}, \bar{l}\}$ are any two solutions of P, then $u = \bar{u}$ and $l = \bar{l}$.

As mentioned in the Introduction, expression (0.4) includes various boundary conditions. We now give some examples.

EXAMPLE 5.1. (Dirichlet type).

$$u(t, i) = k_i(t), \quad 0 \leqslant t \leqslant T, \quad i = 0, 1;$$

this is written in the form (0.4), if $b_i^t(\cdot)$ is defined by

$$b_i^t(r) = \begin{cases} 0 & \text{if } r = k_i(t), \\ \infty & \text{if } r \neq k_i(t); \end{cases}$$

where $k_i \in W^{1,2}(0, T)$, $(-1)^i k_i \geqslant 0$ on $[0, T]$. Then for $i = 0, 1$, b_i^t satisfies (a4) and (5.1). In fact, put

$$\alpha_0(t) = \int_0^t (|k'_0(\tau)| + |k'_1(\tau)|)\, d\tau, \quad \alpha_1(t) \equiv 0 \quad \text{for any } t \in [0, T].$$

Then for any $s, t \in [0, T]$ with $s \leqslant t$, for $i = 0, 1$ and $r \in D(b_i^s)$ (i.e. $r = k_i(s)$), we can take $k_i(t)$ as $r' \in D(b_i^t)$, because

$$\begin{aligned} |r - r'| &= |k_i(s) - k_i(t)| \\ &\leqslant \int_s^t |k_i'(\tau)|\, d\tau \end{aligned}$$

$$\leqslant |\alpha_0(t) - \alpha_1(s)|(1 + |r|).$$

Hence we have (a4). Clearly, (5.1) holds.

EXAMPLE 5.2. (Neumann type).

$$u_x(t, 0+) = k_0(t) \text{ and } u_x(t, 1-) = k_1(t) \text{ for a.e. } t \in [0, T];$$

in this case we may take as $b_i^t(\cdot)$, $i = 0, 1$,

$$b_i^t(r) = k_i(t) r \quad \text{for } r \in R.$$

Assume that $k_i \in W^{1,1}(0, T)$ and $k_i \geqslant 0$ on $[0, T]$ for $i = 0, 1$, and put

$$\alpha_0(t) \equiv 0, \quad \alpha_1(t) = \int_0^t (|k'_0(\tau)| + |k'_1(\tau)|) d\tau, \quad \text{for any } t \in [0, T].$$

Then (a4) is satisfied. In fact, for any $s, t \in [0, T]$ with $s \leqslant t$, $i = 0, 1$ and $r \in D(b_i^s)$, we can take r as $r' \in D(b_i^t)$, because

$$\begin{aligned} b_i^t(r) - b_i^s(r) &= k_i(t) r - k_i(s) r \\ &\leqslant \left(\int_s^t |k_i'(\tau)| d\tau\right) |r| \\ &\leqslant |\alpha_1(t) - \alpha_1(s)|(1 + |r|^p + |b_i^s(r)|). \end{aligned}$$

Since $\partial b_i^t(r) = k_i(t)$ for $i = 0, 1$ and any $t \in [0, T]$, b_i^t satisfies (5.1).

EXAMPLE 5.3. (Signorini type).

$$\begin{cases} u(\cdot, 0) \geqslant k_0(\cdot) & \text{on } [0, T], \\ u_x(\cdot, 0+) = 0 & \text{a.e. on } \{u(\cdot, 0) > k_0(\cdot)\}, \\ u_x(\cdot, 0+) \leqslant 0 & \text{a.e. on } \{u(\cdot, 0) = k_0(\cdot)\}, \end{cases}$$

$$\begin{cases} u(\cdot, 1) \leqslant k_1(\cdot) & \text{on } [0, T], \\ u_x(\cdot, 1-) = 0 & \text{a.e. on } \{u(\cdot, 1) < k_1(\cdot)\}, \\ u_x(\cdot, 1-) \leqslant 0 & \text{a.e. on } \{u(\cdot, 1) = k_1(\cdot)\}; \end{cases}$$

these conditions are represented in the form (0.4) for $b_i^t(\cdot)$, $i = 0, 1$, given by

$$b_0^t(r) \text{ (resp. } b_1^t(r)) = \begin{cases} 0 & \text{if } r \geqslant k_0(t) \text{ (resp. } r \leqslant k_1(t)), \\ \infty & \text{otherwise.} \end{cases}$$

where for $i = 0, 1, k_i \in W^{1,2}(0, T), (-1)^i k_i \geqslant 0$ on $[0, T]$. Also, condition (a4) is satisfied. In fact, let

$$\alpha_0(t) = \int_0^t (|k'_0(\tau)| + |k'_1(\tau)|)\,d\tau, \quad \alpha_1(t) \equiv 0 \quad \text{for any } t \in [0, T].$$

Then for any $s, t \in [0, T]$ with $s \leqslant t$, and $r_0 \in D(b_0^s)$ (resp. $r_1 \in D(b_1^s)$), we can take $k_0(t) + (r_0 - k_0(t))^+$ (resp. $k_1(t) - (r_1 - k_1(t))^-$) as $r'_0 \in D(b_0^t)$ (resp. $r'_1 \in D(b_1^t)$). By elementary calculation, we obtain $(*)$ of (a4). It is clear that (5.1) holds.

Finally we give a typical example of $g(t, x, r)$.

EXAMPLE 5.4. Let $g_0(t, x)$ be a nonnegative function in $L^2(Q)$ and $g_1(t, x)$ be a bounded measurable function on Q. Also, let $k(r)$ be a locally Lipschitz continuous function in $r \in R$, and assume $k(0) = 0$. Then we define

$$g(t, x, r) = g_0(t, x)\sigma(u) + g_1(t, x)k(r),$$

where $\sigma(\cdot)$ is given by (0.5). For this function g, condition (a3) is easily verified; in fact, given any number $M > 0$, we can take

$$C_M = |g_1|_{L^\infty(Q)} \times (\text{Lipschitz constant of } k \text{ on } [-M, M])$$

and

$$g_{0,M}(t, x) = g_0(t, x) + |g_1|_{L^\infty(Q)} \times \max\{|k(r)|; -M \leqslant r \leqslant M\}.$$

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Istnienie rozwiązań dwufazowego zagadnienia Stefana dla nieliniowych równań parabolicznych

W pracy rozważane jest jednowymiarowe dwufazowe zagadnienie Stefana dla równań typu parabolicznego z nieliniowymi warunkami brzegowymi. Warunki brzegowe są opisywane przez zależne od czasu operatory subgradientu i interpretowane jako realizacja sterowania w układzie zamkniętym. Głównym wynikiem pracy jest twierdzenie o istnieniu lokalnego w czasie rozwiązania zagadnienia. Stosowana metoda dowodu oparta jest na abstrakcyjnej teorii nieliniowych równań ewolucyjnych z zależnymi od czasu operatorami subgradientu w przestrzeni Hilberta.

Существование решений двухфазной задачи Стефана для нелинейных параболических уравнений

В статье рассматривается одномерная двухфазная задача Стефана для параболических уравнений с нелинейными правыми частями и нелинейными граничными условиями. Граничные условия описаны изменяющимися во времени операторами субградиента и интерпретируются в качестве реализации управления в замкнутой связи. Главным результатом работы является теорема о существовании локального во времени решения задачи. Применяемый метод доказательства основан на абстрактной теории нелинейных эволюционных уравнений с изменяющимся во времени операторами субградиента в гильбертовом пространстве.