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On topology and boundary variations in shape optimization

by

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This study is concerned with the development of an integrated procedure for the computation of the optimal topology as well as the optimal boundary shape of a two-dimensional, linear elastic body. The topology is computed by regarding the body as a domain of the plane with a high density of material and the objective is to maximize the overall stiffness, subject to a constraint on the material volume of the body. This optimal topology is then used as the basis for a shape optimal design method that regards the body as given by boundary curves. For this case the objective is to minimize the maximum value of the Von Mises equivalent stress in the body, subject to an isoperimetric constraint on the area as well as a constraint on the stiffness. Computational results are presented for the design of a beam and a portal frame.

1. Introduction

This study is concerned with the development of an integrated procedure for the computation of the optimal topology as well as the optimal boundary shape of a two-dimensional linear elastic body, using finite element discretizations.

The procedure is based on integrating the possibilities of two fundamentally different approaches to the optimization of the shape of a body. Traditionally, in shape design of mechanical bodies, a shape is defined by the orientated boundary curves of the body and in shape optimization the optimal form of these boundary curves is computed. This approach is very well established and the literature is extensive; the reader is referred to the excellent review papers [1] by Haftka and Gandhi and [2] by Ding. Alternatively, the mechanical body can be considered as a domain in space with a high density of material, that is, the body is described by a global density function that assigns material to points that are part of the body. Such a measure theoretical approach has been promoted by Kohn and Strang in a series of papers [3], [4]. Recent computational implementations of such an approach ([5] - [8]) using composites with micro-voids has shown that it is possible to predict a change in topology, a feature that cannot be achieved by the boundary variations technique. On the other hand, the material distribution formulation can only give a rough estimate of the boundary curves of the structure, and a reasonable prediction of the finer details of the boundaries requires very large FEM models.

It is thus natural to integrate the material distribution method and the boundary variations approach into one design tool. The possibility of generating the optimal topology for a body can be used by the designer to decide on the shape of the initial proposed form of the body for the boundary variations technique. This latter part is usually left entirely to the designer, but the material distribution method gives the designer a rational basis for his choice of initial form.

The nature of the topology optimization method is such that for problems in mechanics, the objectives used for the optimization should be global criteria, e.g. compliance, volume, average stress etc. In the present study the objective for the material distribution method is to minimize the compliance of the structure (i.e. maximize the global stiffness), subject to a constraint on the volume of the structure. On the other hand, the description of the body by boundary curves allows the finer details of the body to be controlled and this is utilized for the minimization of the maximum value of the Von Mises equivalent stress in the body, subject to a constraint on the compliance of the body as well as to an isoperimetric constraint on the area of the body. For both methods, finite element models are employed. For the topology optimization a virtual displacement-based finite element model is used while for the boundary variations technique a mixed finite element method provides for accurate computation of stresses and strains at the element nodes. The generation of meshes for both cases is performed by an elliptical automatic mesh generator that assures an orthogonal finite element mesh at the domain boundary. The mesh generator handles non-simply connected domains by dividing such domains into a number of simply connected domains. Its use gives added flexibility to the topology generation scheme and its use at each shape redesign in the boundary variations method eliminates the problem of mesh distortion. This in connection with the use of a mixed finite element method results in a stable boundary variations optimization method (see also [9], [10]).

Interfacing the topology optimization method with the boundary variations method is a problem of generating outlines of objects from grey level pictures. A procedure for an automatic computation of the proposed initial form for the boundary variations technique could thus be based on ideas and techniques from image analysis and pattern recognition. For the examples presented in this paper, the outlines for the initial proposed form were generated manually thus mimicking a design situation where the ingenuity of the designer is utilized to generate a 'good' initial form from the topology optimization results. The term 'good' in this context covers considerations such as ease of production, aesthetics etc. that may not have a quantified form. A reduction of the number of holes proposed by the topology optimization by ignoring relatively small holes exemplifies design decisions that could be taken before proceeding with the boundary variations technique.

2. Topology optimization

2.1. Problem formulation

The key for obtaining a topology generating shape design method is to regard a mechanical element as a domain in space with a high density of material, so that the optimal design problem in mathematical terms becomes a sizing problem.

Consider a mechanical element as a body occupying a domain, which is part of a larger reference domain. The body is subject to body forces, and boundary tractions and boundary conditions etc. are defined on the boundary of the reference domain. Refering to the reference domain we can define the optimal shape design problem as the problem of finding the optimal choice of material distribution over the domain.

The optimization goal could be to minimize the compliance, subject to a constraint on the volume.

Notice that by defining the shape design problem in this way, for each point in the reference domain there is a dicrete choice as to whether that point is a material point or not. That is, we have formulated a distributed parameter optimization problem with a discrete valued design function. A direct approach to such an optimization problem by discretization of the analysis problem (by FEM) would thus require the use of discrete optimization algoritms. However, in general the distributed problem does not have any solution ([3], [4]), so such an approach would be unstable with respect to choice of element type and mesh size. The design problem should be regularized and composites introduced into the formulation. The use of composites moves the on-off nature of the problem from the macroscopic scale to a microscopic scale and the design variable becomes a density function that can take on all values between 0 and 1 . The introduction of composites thus removes the discrete nature of the problem.

2.2. Approach by homogenization

For a material with a periodic microstructure of known geometry, the method of homogenization provides a readily available recipe for the computation of the effective moduli of such a material. This allows for the use of composites in the formulation of shape design problems, with voids at a microlevel introduced in the base material employed for the mechanical element. The design variables will then be the geometric quantities defining the local dimensions of the voids, with the volume given through a density function which also depend on these geometric quantities.

Several alternative cell geometries could be considered, as long as the chosen geometry can describe a complete removal of material as well as a solid cell. This excludes the use of circular holes in square cells, while the use of a composite with square cells with square holes is a simple possible choice for transforming the discrete valued shape design problem into a standard sizing problem. For the latter case the effective properties of the material are given entirely by the density of the material and an angle of rotation of the cell, with respect to some fixed reference frame. As we are rotating the cell, the dependence on this angle follow from the well known frame rotation formulas, while the dependence on density for this case has to be computed numerically. The rigidity is a convex function in the density and as the volume is linear in density it is to be expected that the optimized design will have density values 0 or 1 in large parts of the reference domains, as required for prediction of topology.

More complicated cell-geometries than square holes in square cells typically implies more geometric variables and thus more design variables for the shape optimization. Experience with other micro-geometries indicate that this is not of a significent parameter for generating the topology of a structure, unless the cellrotation angle is fixed in the optimization process ([5]).

The most important microstructure for comparison and for flexibility is the so-called layering of second rank. For our purposes this consists of the base material and of a very weak material taking the place of voids. The composite is then a layering of the base material with another material, which is a lower scale layering of the base material and the weak material. The effective material propertics can be computed analytically, by recursive use of the homogenization formulas ([6]). For such materials the optimal shape design problem, in the form of the minimum compliance problem, is assured to have a solution ([12], [13]). This feature and the fact that there exist analytical expressions for the effective moduli are properties that favour the use of such layered materials. However, compared to square cells with square holes the number of variables needed to describe the cell geometry are doubled. Examples of topology optimized structures are shown in Fig. 1.

3. Shape optimization by a boundary variations method

3.1. Problem formulation

Once the optimal topology and initial boundary shape is defined, the objective is to refine this initial shape, such that the von - Mises equivalent stress in the body is minimized.

For the two-dimensional linear elastic body described as shown in Fig. 2 the objective is to find, by means of the boundary variation, the shape of the domain $\Omega \in D$ (set of local geometric design constraints) such that the maximum value of the von-Mises equivalent stress is minimized, i.e. to achieve,

$$\min_{\Omega \in D} \max_{x \in \Omega} \quad \bar{\sigma}_{eq} \tag{3.1}$$

subject to the resource constraint

$$\int_{\Omega} d\Omega - \bar{A} \leq 0. \tag{3.2}$$

to the compliance constraint

$$\int_{\Omega} f \bullet u d\Omega + \int_{\Gamma} t \bullet u d\Gamma - \bar{\Phi} \leq 0. \tag{3.3}$$

and to the condition of equilibrium. Here D denotes the set of admissible shapes, defined through local geometric constraints. The equilibrium is defined via the stationarity condition for the Hu -Washizu variational principle ([14]), and by employing the well known speed method for boundary shape variations we can derive the set of necessary conditions, to be satisfied at the optimal domain, Ω^*, (Ref. [10] contains the complete derivation of these results).

4. Automatic mesh generation

4.1. Introduction

To solve numerically the necessary conditions associated with the shape optimal design problem, there is a critical need for and automatic grid generator for the finite element model used to estimate the state and adjoint state variables, at each new shape design. Note that estimates for stress and adjoint strains fields

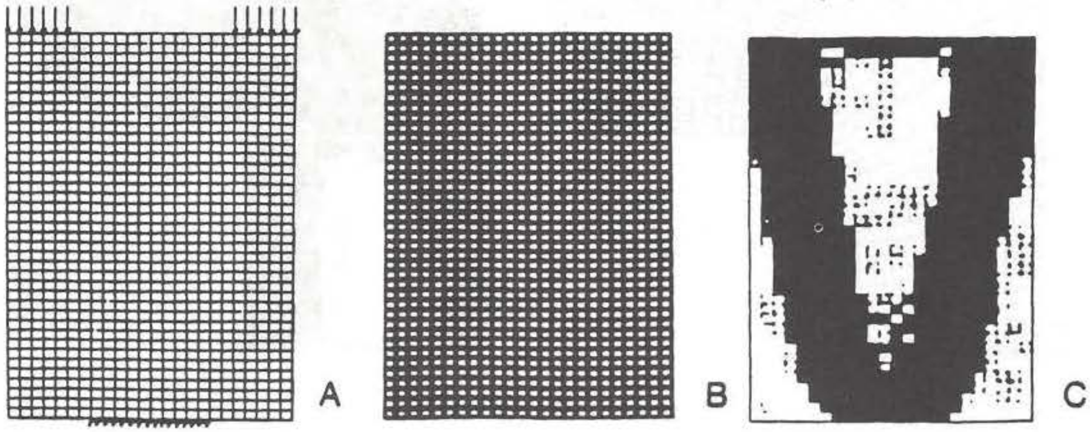

Fig. 1.1 A design case (a support), topology optimization. A: The meshed reference domain with loads and boundary conditions. B: The initial design. The density of material is shown by drawing, for each element, a (rotated) square, white hole with area $1-\mu$ inside the black (material) element. In reality these holes are at a microlevel. C: The optimal design.

Fig. 1.2 Modified support problem, based on results shown in Fig. 2. A: The reference domain, with mesh, loads ets. B: The initial design, with reduced design area. C: The optimal topology design.

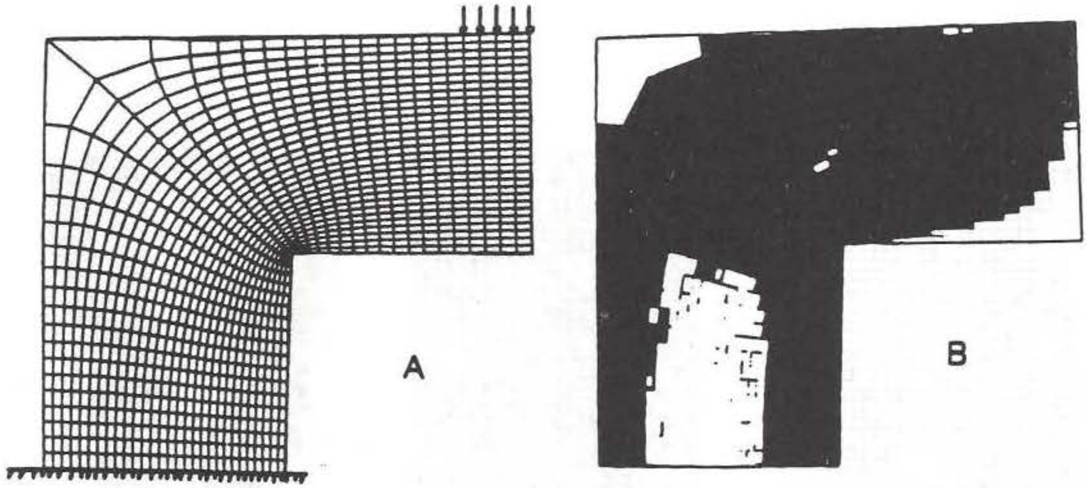

Fig. 1.3 A L-shaped support, optimized with use of layered materials. A: The reference domain showing mesh, loadings etc. B: The optimal design.

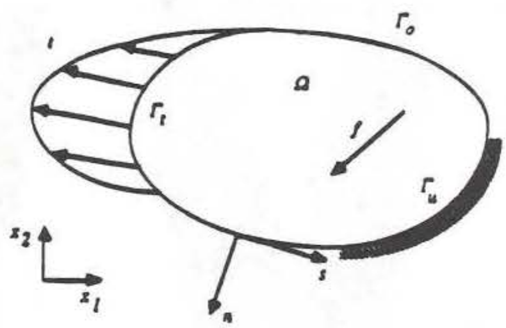

Fig. 2 Notation for mixed variational formulation.

are critical for the evaluation of the cost function and the optimality condition defining the optimal domain.

If grid adaptation and optimization is used, then additional finite element analyses are needed to define the new adapted (optimized) grid. This approach was considered in other research works in shape optimal design. However to avoid such a high computational cost, derived from the additional finite element analysis needed to define the optimal grid and augmented by the fact that a mixed finite element model is used, which already implies a very expensive analysis, this approach was not considered in this work.

However we are interested in maintaining 'good' mesh properties, that might otherwise be destroyed during the shape modification at each new design iteration. Those properties that affect the shape redesign process are the main concern.

The choice of an automatic grid generator should not be arbitrary and it should relate sensibly to the problem type to be solved.

In the case of shape optimal design and based on the form of the necessary conditions for the optimal domain, some requirements should be taken into account in the selection of an automatic grid generation system:

A) The solution of the optimality condition requires very accurate stress and adjoint strain estimates along the design boundary.

B) During the domain shape variation, geometric singulatities can develop along the design boundaries. The grid generator should minimize the propagation into the domain of mesh non-uniformities, due to these singularities.

C) To minimize the interpolation error of the finite element solution there is a need for grid smootheness and orthogonality.

D) Initial shapes based on topology optimization, as described in section 2, can be quite arbitrary. The grid generator should be able to operate on quite general shapes and permit interior boundaries.

To cater for these requirements it was chosen to employ an elliptical mesh generator ([15]), based on a subdivision of the domain by blocks. In this method the grid is obtained via the solution of a system of elliptical partial differential equations. Defining appropriate boundary conditions, mesh orthogonality can be obtained along the domain boundary.

5. Implementation and integration

5.1. Numerical model for topology optimization

The discrete version of the analysis problem for topology optimization is accomplished through a discretization using four node isoparametric finite elements, with the element mesh discretization of the reference domain given via the mesh generator described in Section 4.

The design variables (density and rotation angle for square holes in square cells) are discretized as elementwise constant and the iterative update of the design variables is based on a recursive scheme with an elementwise evaluation of the optimality criteria for the optimization problem, ([5], [6]). Element strain levels are taken as an average of nodal values in each element. The angle of rotation can be updated through a Newton method, or by rotating the cells of the microstructure to align with the principal stress directions ([16]). For the topology optimization, with holes in square cells, the dependence of the effective rigidity tensor on the density is computed for a number of values of the density and the complete functional dependence is approximated by interpolation with Legendre polynomials; the sensitivity, as needed for the update scheme, can then be given in closed form. If layered materials are used, as described in section 2, analytical expression for the effective rigidity and its derivatives can be derived ([6]).

5.2. Topology optimization in practice

The introduction of the material density as a design variable in shape design, as described above, results in a flexible and reliable tool for predicting topology and boundary shapes. The problem should be formulated on a reference domain, which should be chosen as simple as possible so as to reduce the size of the analysis problem. The domain should allow for definition of loads and tractions and of boundary condition. The use of the automatic elliptical mesh generator simplifies the treatment of problems with complicated geometry such as non-simply connected reference domains. Complicated reference domains are needed for cases where design requirements implies the exclusion of certain parts of space as parts of the structure. If the precise shapes of inner holes in a non-simply connected reference domain are of minor importance, it is advisable to cater for such holes by fixing the density as zero for elements defining the hole (or parts of it). Likewise, it is often required that certain parts of a structure

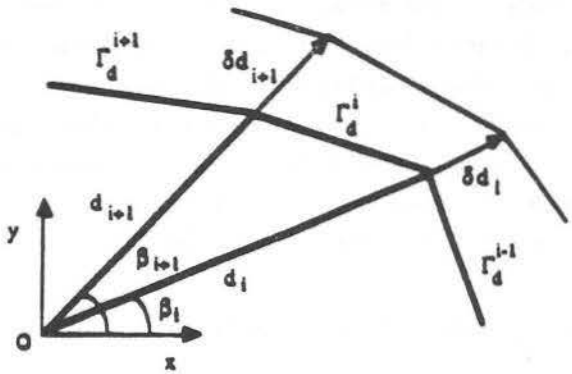

Fig. 3 Definition of boundary design element.

cannot be removed, and this can be handled by setting the density equal to 1 for elements defining such subdomains of the structure.

For the density approach using homogenized materials, the resulting topologies are almost insensitive to the form of the FEM mesh that is employed. It is, of course, always advisable to test the results by changing the mesh, and the use of an automatic mesh generator simplifies this process significantly.

5.3. Numerical model for boundary variations

The discrete version of the mixed variational formulation used for the boundary shape design is achieved through a discretization of stress, strain and displacement fields using four node isoparametric finite elements. This leads to an indefinite system of equation in the nodal values of the stress, strain and displacement fields.

The discretization of the design pertubation field is obtained through interpolation using linear boundary elements. The shape of the body is then given through the design variables d , which are the lengths of the position i vectors of the respective interpolation nodes, expressed with respect to a pre-defined origin (see Fig. 3).

Since this discretization is needed in order to express the discrete version of the optimality condition the design boundary and finite element meshes should coincide at this portion of the domain boundary. This implies that the design variables are directly related to the finite element nodes at the design boundary.

With the discretizations described above, the design variables d for the optimal boundary shape can be computed iteratively, either based on a direct solution of the discrete optimality condition or by employing well-known gradient type algorithms. The results shown in this paper were obtained by use of the Pshenichny version of the linearization method ([9]).

5.4. Boundary variations in practice

The boundary variations technique employed assures great freedom in the definition of boundary shapes. Oscillations of the boundary, as reported in the literature (cf. [1], [2]), are avoided by use of a precise FEM model which can detect local non-smootheness of the boundary. Boundary variations techniques based on more global representations of the boundaries (splines etc.) achieves smootheness of the optimal design boundaries at the expense of reduced design freedom. However, the freedom inherent in the present model implies that great care needs to be exhibited when defining the FEM mesh, the design boundaries and the design reference points. Preferably, several runs of the optimization method should be made in order to understand the nature of the problem being solved and it is especially important to test the effect of different FEM discretizations. The use of the elliptic mech generator naturally alleviates this task.

5.5. Integration of the methods

The topology optimization scheme and the boundary shape optimization method arise due to two fundamentally different definitions of shape. The optimization of topology results in a prediction of the overall lay-out of the structure, and gives a rough description of the shape of outer as well as inner boundaries of the structure.

The boundary shape optimization controls the finer details of the boundary, but cannot change the topology of the initial chosen reference domain. It is thus clear that the two methods will benefit from an integration, with the topology optimization method playing the role of a pre-processor for the boundary shape optimization.

For the integration, it was decided to employ the graphics facilities of modern workstation, thus mimicking a CAD-type environment ([17]). Input of all data for the two optimization methods is done interactively, with maximum possible use of graphics input devices. The integration of the topology optimization results into the boundary shape optimization is done by drawing the shape of the initial form directly on the screen on top of a presentation of the topology optimization results. In this way complicated image processing techniques are not needed, and it is up to the user to decide on what information from the topology optimization that should be used. Typically, it will be decided to ignore small holes etc.

For both methods, the elliptical mesh generator described in Section 4 is used for the construction of the FEM meshes that are needed, and again all data is inputted in an interactive computer enviroment based on extensive use of graphics. The assignment of mesh data is the most complicated part of the integrated procedure as considerable care is needed for the correct definition of boundary and block numbering.

5.6. The integrated method in practice

The use of the topology optimization method as a pre-processor for the boundary shape optimization method results in very good initial forms being obtained for the latter method. Generally, only small and localized design changes occur in the boundary optimization. Typically, the minimization of the stress level during the boundary optimization also results in some decrease in the compliance, but this is not unexpected as the drawing of the initial form from the topology data constitutes a not insignificant pertubation of the minimum compliance design.

For problems with little available material ('thin' structures), the topology optimization will predict optimal topologies with, usually, many holes and often truss-like lay-outs. Such designs are very difficult to handle with the boundary optimization method. Firstly, the number of boundary design variables will be excessive and, secondly, the meshes needed for reasonable prediction of stresses will result in huge FEM models. The latter problem is, of course, exaggerated by the mixed variational model employed in the present work. Thus for thin structures, alternative mechanical models are probably needed, with the boundary shape variations method as described in this paper only used for e.g. fillets connecting beams and bars in the truss-like lay-outs that are predicted from topology optimization.

In cases of more 'solid' structures, the integration of the two methods gives very nice results, as the examples in fig. 4 and 5 shows.

6. Conclusions

The integration of the material distribution method for topology optimization with a boundary shape optimization method results in an efficient and flexible design tool. The use of an automatic mesh generator is central for the performance of the boundary variations scheme and increases the flexibility of the overall system. The topology optimization can not only predict topology, but is also an efficient tool for generating initial designs for the boundary variations method even for cases where the correct basic topology can be chosen intuitively. The use of the boundary shape optimization is crucial in order to finalize a design, even though only small design changes occur.

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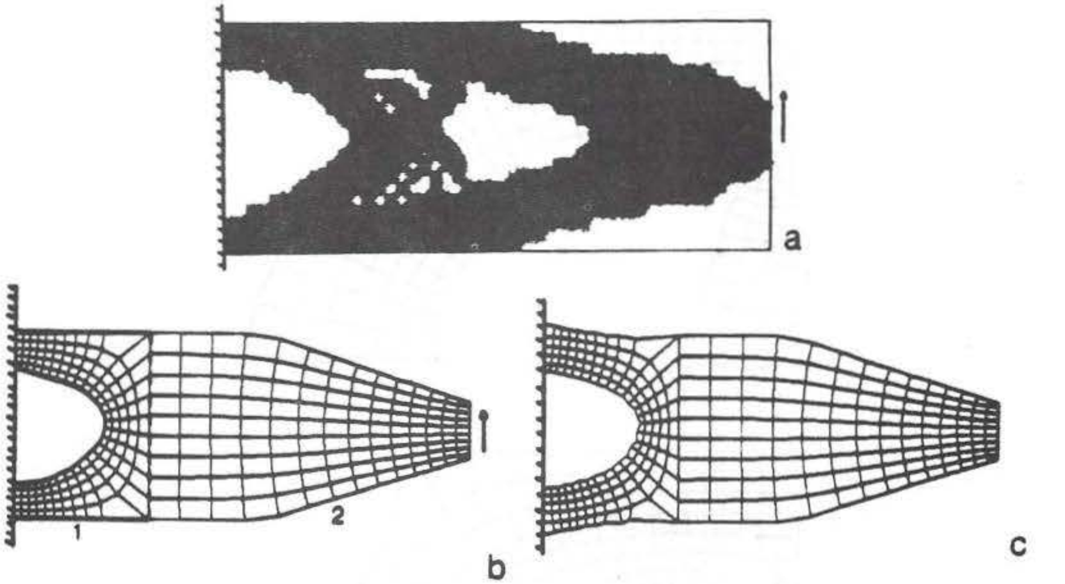
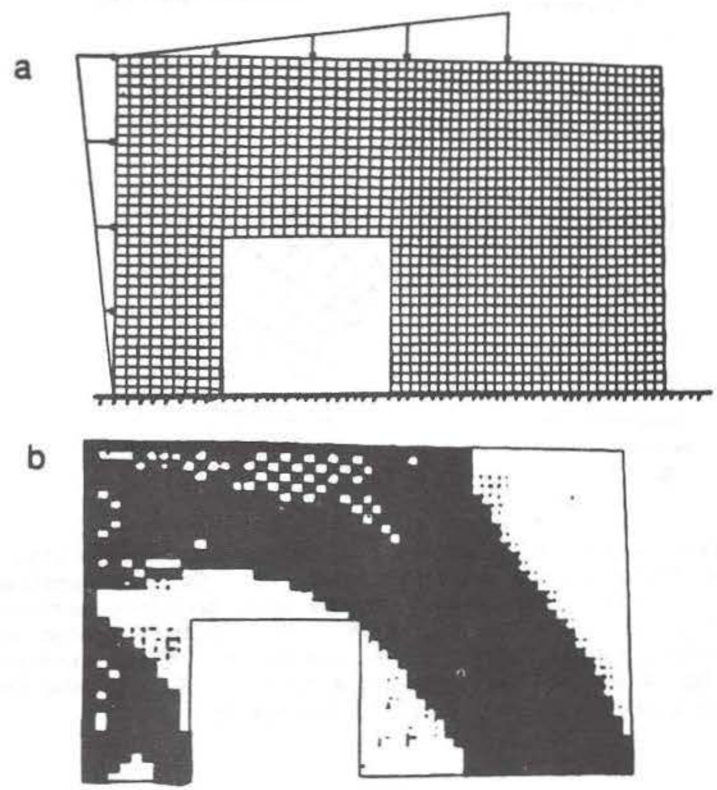


Fig. 4 . Optimal design of a beam. A: Optimal topology with outline showing reference domain. B and C: Initial and final design using two blocks in a boundary variations method. Only the boundaries of block 1 can move. The maximum stress is decreased by 55.7% and the compliance by 7.3%.



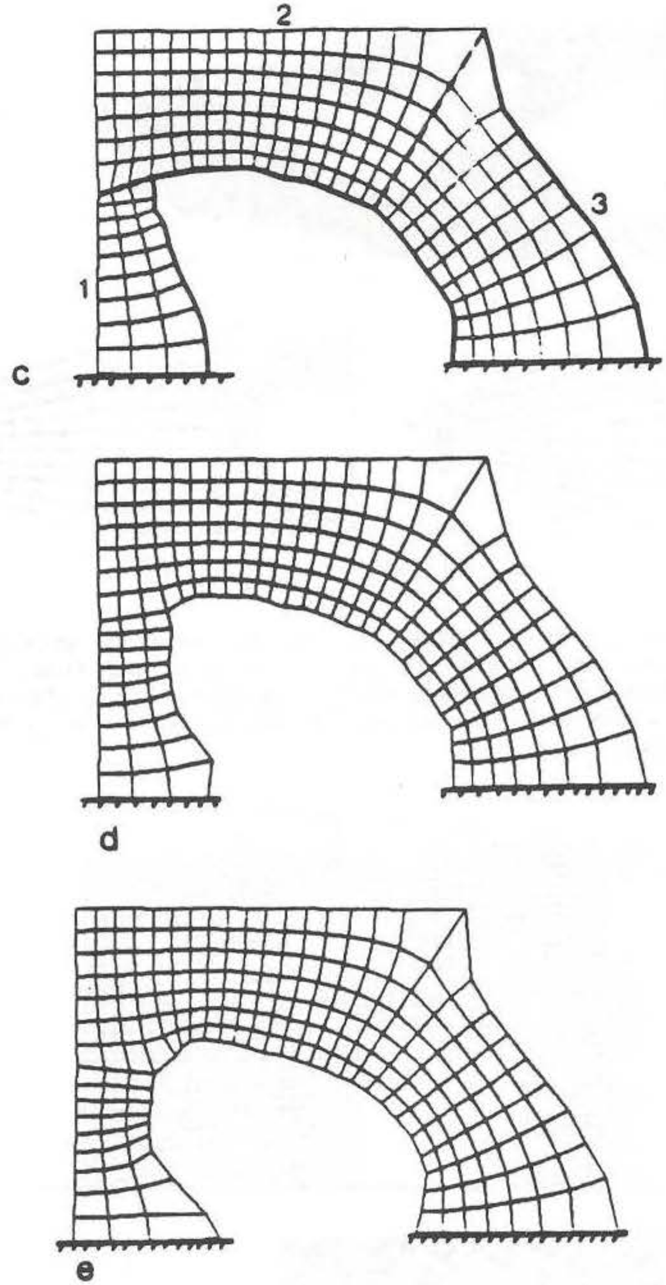


Fig. 5 Optimal design of a portal. A: Reference domain with mesh, loads ets. Three blocks used for mesh. B: Optimal topology. C and D: Initial and optimal design with boundary variations. Three blocks are used for the mesh. E: Optimal design when support boundary is not fixed. In D, maximal Von Mises stress is decreased by 14.5% and the compliance by 9.1%. In E the corresponding values are 23.8% and 17.8%, respectively.

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