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# Existence of the solutions to the Euler–Bernoulli plate model with semilinear boundary conditions

by

**M. BRADLEY**[1]

**I. LASIECKA**[2]

Department of Applied Mathematics, Thornton Hall
University of Virginia
Charlottesville, VA 22903

We consider a semilinear model of the Euler-Bernoulli plate where nonlinearities are allowed to appear both on the equation and on the boundary. Local existence of solutions is proven for initial conditions in the space of finite energy and global solutions are shown to exist for sufficiently small initial data. An appropriate fixed point argument is employed in the proof.

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# 1 Introduction

We consider the following semilinear model of the Euler-Bernoulli plate

$$\left.\begin{array}{lll} u_{tt}+\Delta^2 u = h_0(u,u_x,u_y) & \text{in } Q & (a) \\ u(t=0)=u_0;\, u_t(t=0)=u_1 & \text{in } \Omega & (b) \\ u=\frac{\partial}{\partial\nu}u=0 & \text{on } \Sigma_0 & (c) \\ \Delta u+(1-\mu)B_1u+k_1\frac{\partial}{\partial\nu}u_t = h_1(u,u_x,u_y) & \text{on } \Sigma_1 & (d) \\ \frac{\partial}{\partial\nu}\Delta u+(1-\mu)B_2u-k_2u_t = h_2(u,u_x,u_y) & \text{on } \Sigma_1 & (e) \end{array}\right\} \quad (1.1)$$

where the operators $B_1$ and $B_2$ are given by

$$\left.\begin{array}{rcl} B_1u & = & 2n_1n_2u_{xy}-n_1^2u_{yy}-n_2^2u_{xx} \\ B_2u & = & \frac{\partial}{\partial\tau}[(n_1^2-n_2^2)u_{xy}+n_1n_2(u_{yy}-u_{xx})] \end{array}\right\} (1.1)(f).$$

Here $\Omega\subset R^2$ is an open, bounded domain with sufficiently smooth boundary, $\Gamma=\Gamma_0\cup\Gamma_1$ with $\Gamma_0\cap\Gamma_1=\emptyset$, $Q=\Omega\times(0,\infty)$, and $\Sigma_j=\Gamma_j\times(0,\infty)$ for $j=0,1$. Appearing as parameters in our system, we have the constants $k_1\geq 0$ (equality holding only if $h_1\equiv 0$), $k_2>0$ and $\mu$ (Poisson's ratio) which we will assume satisfies $0\leq\mu\leq 1/2$. We take $\nu=[n_1,n_2]$ and $\tau=[-n_2,n_1]$ to be the unit normal and tangent, respectively, to the boundary.

The main goal of this paper is to establish the local and global existence of a unique solution, $(u,u_t)$, for (1.1) with initial data $(u_0,u_1)$ in the space of finite energy, $H^2(\Omega)\times L^2(\Omega)$.

In what follows, we shall make the following assumptions on the nonlinear functions $h_i(u, u_x, u_y)$

$$\text{(H-1)} \qquad \begin{cases} (i) \quad h_i : R^3 \to R^1 \text{ are continuously differentiable} \\ (ii) \quad |\frac{\partial}{\partial y_k} h_i(y_1, y_2, y_3)| \le |f_1(y_1)|\,|y_2|^r + |f_2(y_1)|\,|y_3|^s \end{cases}$$

where $r,\ s > 0$ are arbitrary and $f_1,\ f_2$ are continuous functions of $y_1$.

Our first result deals with local existence.

**Theorem 1.1 (Local Existence)** *Assume the hypothesis* (H-1). *Then for all initial data* $u_0 \in H^2(\Omega)$, $u_1 \in L^2(\Omega)$ *with* $u_0$ *satisfying (1.1)(c), there exists a unique solution* $(u(t), u_t(t)) \in C([0,T]; H^2(\Omega) \times L^2(\Omega))$ *for some* $T > 0$.

To state the global existence result, we need to impose more restrictive conditions on the $h_i$. In addition to (H-1), we will assume

$$\text{(H-2)} \qquad h_i(0,0,0) = 0 \quad \text{and} \quad \frac{\partial}{\partial y_k} h_i(y_1, y_2, y_3)\,|_{\bar{y}=0} = 0$$

where $\bar{y} = (y_1, y_2, y_3)$.

**Theorem 1.2 (Global Existence)** *Assume the hypotheses* (H-1) *and* (H-2). *Then there exists* $R > 0$ *such that for all initial data satisfying (1.1)(c) and*

$$\|(u_0, u_1)\|_{H^2(\Omega) \times L^2(\Omega)} \le R$$

*there exists a unique solution* $(u(t), u_t(t))$ *to (1.1). Furthermore, the solution pair* $(u(t), u_t(t))$ *satisfies the estimate*

$$\|(u(t), u_t(t))\|_{H^2(\Omega) \times L^2(\Omega)} \le M e^{-\alpha t} \qquad 0 < t < \infty$$

*for some* $M > 0$ *and some* $\alpha > 0$.

Although the linear model of the Euler-Bernoulli plate is well-known (see [5,4,3,6]) very few existence results are known in the nonlinear case. Only a few special cases, such as monotone nonlinearities, have been considered. The main mathematical difficulty in studying the solvability of (1.1) with nonlinearities appearing in the boundary conditions is the intrinsic low regularity of solutions to the "uncontrolled" dynamics (i.e. system (1.1) with $k_1 = k_2 = 0$ and $L^2$-nonhomogeneous boundary data). Indeed, $L^2$ boundary data do not produce finite energy solutions (i.e. the solutions $(u(t), u_t(t))$ do not lie in $H^2(\Omega) \times L^2(\Omega)$unless dim $\Omega = 1$). Consequently, standard methods of nonlinear analysis which would lead to the well-posedness of the system do not apply. To cope with the problem, our idea is to introduce dynamic feedbacks $k_1 \frac{\partial}{\partial \nu} u_t$ and $k_2 u_t$ on the boundary. These feedbacks, on one hand, cause the dissipation of energy for the linear model. On the other hand, they induce, as we shall see, regularity properties of the linearized solution which are "better" than those provided for by standard trace theory. We shall then exploit this regularizing effect of the boundary feedbacks in order to control the nonlinearities in the system.

The outline of the paper is as follows. In section 2 we provide some background material on the semigroup representation of the solutions to (1.1). In section 3, we shall prove certain "trace" regularity properties for the solutions to the linearized problem. These properties will be critically used in section 4, where the proofs of both theorems are provided.

## 2 Preliminary Material

We find it convenient to represent the solution to (1.1) in the semigroup form. To accomplish this we introduce a few appropriate function spaces and several operators. Let $H^2_{\Gamma_0}(\Omega) = \{x \in H^2(\Omega) : x = \frac{\partial}{\partial\nu}x = 0 \text{ on } \Gamma_0\}$. Set $\mathcal{H} = H^2_{\Gamma_0}(\Omega) \times L^2(\Omega)$ and define $\mathcal{U} = L^2(\Gamma)$. We define $\mathcal{A}$ on $H^2_{\Gamma_0}(\Omega)$ by

$$\begin{aligned}\mathcal{A}u &\equiv \Delta^2 u \quad \text{with domain}\\ D(\mathcal{A}) = &\ \{u \in H^4 \cap H^2_{\Gamma_0}(\Omega) : \Delta u + (1-\mu)B_1 u = 0\\ &\quad \text{and } \tfrac{\partial}{\partial\nu}\Delta u + (1-\mu)B_2 u = 0 \text{ on } \Gamma_1\}\end{aligned}$$

which is well-defined, positive and self-adjoint. We will also introduce the Green maps, $G_1$ and $G_2$, defined by

$$G_1 h = v \iff \begin{cases} \Delta^2 v = 0 & \text{in } Q \\ v = \frac{\partial}{\partial\nu}v = 0 & \text{on } \Sigma_0 \\ \left.\begin{array}{l}\Delta v + (1-\mu)B_1 v = h \\ \frac{\partial}{\partial\nu}\Delta v + (1-\mu)B_2 v = 0\end{array}\right\} & \text{on } \Sigma_1 \end{cases} \tag{2.1}$$

and

$$G_2 h = v \iff \begin{cases} \Delta^2 v = 0 & \text{in } Q \\ v = \frac{\partial}{\partial\nu}v = 0 & \text{on } \Sigma_0 \\ \left.\begin{array}{l}\Delta v + (1-\mu)B_1 v = 0 \\ \frac{\partial}{\partial\nu}\Delta v + (1-\mu)B_2 v = h\end{array}\right\} & \text{on } \Sigma_1. \end{cases} \tag{2.2}$$

It can be shown that (2.1) and (2.2) are regular elliptic problems and hence (see [8]),

$$\left.\begin{array}{l} G_1 \in \mathcal{L}(H^s(\Gamma) \to H^{s+\frac{3}{2}}(\Omega)) \\ G_2 \in \mathcal{L}(H^s(\Gamma) \to H^{s+\frac{7}{2}}(\Omega)) \end{array}\right\} \text{ for } s \in R. \tag{2.3}$$

We are now in a position to define the operator $A : \mathcal{H} \to \mathcal{H}$ by

$$A = \begin{bmatrix} 0 & I \\ \mathcal{A} & 0 \end{bmatrix} \text{ with } \begin{array}{rl} D(A) = & D(\mathcal{A}) \times H^2_{\Gamma_0}(\Omega) \\ = & D(\mathcal{A}) \times D(\mathcal{A}^{1/2}). \end{array} \tag{2.4}$$

A direct computation shows that $A$ generates a $C_0$-semigroup of contractions on $\mathcal{H}$. We also define, for $i = 1, 2$, $\mathcal{B}_i : \mathcal{U} \to [D(A^*)]'$

$$\mathcal{B}_i g = \begin{bmatrix} 0 \\ \mathcal{A} G_i g \end{bmatrix} \ (\, g \in \mathcal{U} = L^2(\Gamma)) \tag{2.5}$$

and $F_i : \mathcal{H} \to \mathcal{U}$

$$F_i \tilde{u} = -k_i \mathcal{B}_i^* \tilde{u} \tag{2.6}$$

where $\tilde{u} = [u_1, u_2]$. We shall see that the $F_i$ defined by (2.6) coincide with the boundary feedbacks $-k_1 \frac{\partial}{\partial\nu} u_t$ and $k_2 u_t$. Indeed,

$$\mathcal{B}_i{}^* \tilde{u} = G_i^* \mathcal{A} u_2 \qquad \text{for } \tilde{u} \in D(A) \tag{2.7}$$

and

$$\left.\begin{array}{l} G_1^* \mathcal{A} u = \frac{\partial}{\partial\nu} u\big|_{\Gamma_1} \\ G_2^* \mathcal{A} u = -u\big|_{\Gamma_1} \end{array}\right\} \text{ for } u \in \mathcal{H}. \tag{2.8}$$

Since $\mathcal{B}_i \in \mathcal{L}(\mathcal{U} - [D(A^*)]')$ and $D(A) \subset \mathcal{H}$ is dense with $\mathcal{H}$ reflexive, we have that $\mathcal{B}_i F_i$ are well-defined in the topology of $[D(A^*)]' \ = [D(A)]'$. For functions $h_i$ satisfying (H-1) and (H-2), we define the operators $H_i : \mathcal{H} \to \mathcal{U}$

$$H_i\tilde{u} = h_i\left(\gamma_0(u^1), \gamma_0(u^1_x), \gamma_0(u^1_y)\right) \qquad i = 1, 2 \tag{2.9}$$

where $\gamma_0$ is the trace operator.

Next, we define

$$\begin{aligned} A_F \;&\equiv\; A + \mathcal{B}_1F_1 + \mathcal{B}_2F_2 \quad \text{with domain} \\ D(A_F) \;&=\; \{[u_1, u_2] \in \mathcal{H} : u_2 \in D(\mathcal{A}^{1/2}) \text{ and} \\ &\qquad u_1 + k_1G_1G_1^*\mathcal{A}u_2 + k_2G_2G_2^*\mathcal{A}u_2 \in D(\mathcal{A})\}. \end{aligned} \tag{2.10}$$

It was proved in [4] that $A_F$ is a generator of a $C_0$-semigroup on $\mathcal{H}$ and

$$\left\|e^{A_F t}\right\|_{\mathcal{H}\to\mathcal{H}} \le Me^{-\omega t} \qquad t > 0 \tag{2.11}$$

where $M, \omega > 0$.

Now we are in a position to formulate an abstract semigroup model for the original problem (1.1) (see [1]). With $\tilde{u} \equiv (u, u_t)$

$$\begin{aligned} \frac{d}{dt}\tilde{u}(t) &= A\tilde{u}(t) + \sum_{i=1}^{2}(\tilde{\mathcal{B}}_iF_i\tilde{u}(t) + \mathcal{B}_iH_i(\tilde{u}(t))) + H_0(\tilde{u}(t)) \\ &\equiv A_F\tilde{u} + \sum_{i=1}^{2}\mathcal{B}_iH_i(\tilde{u}) + H_0(\tilde{u}) \\ \tilde{u}(0) &= \tilde{u}_0 \in \mathcal{H} \end{aligned} \tag{2.12}$$

where this system is considered in the topology of $[D(A^*)]'$.

Our main goal is to establish the existence of the solutions $\tilde{u}$ to (2.12). Since we are interested in mild (or weak) solutions, we shall represent the sought after $\tilde{u}$ in integral form. To this end, let us (formally) define the operators $\mathcal{L}_i : C([0,\infty);\mathcal{U}) \to C([0,\infty);\mathcal{H})$ for $i = 1,2$ by

$$(\mathcal{L}_i u)(t) \equiv \int_0^t e^{A_F(t-\tau)}\mathcal{B}_i u(\tau)d\tau. \tag{2.13}$$

Later, we shall prove that the above definition is meaningful. We also define the operator $\mathcal{L}_0 : \mathcal{L}(C([0,\infty);\mathcal{H}))$ by

$$(\mathcal{L}_0 f)(t) \equiv \int_0^t e^{A_F(t-\tau)} f(\tau)d\tau. \tag{2.14}$$

With the above definitions, the "mild" solution to (2.12) may be written as

$$\tilde{u}(t) = e^{A_F t}\tilde{u}_0 + \sum_{i=1}^{2} \mathcal{L}_i\left(H_i(\tilde{u}(\cdot))\right)(t) + \mathcal{L}_0\left(H_0(\tilde{u}(\cdot))\right)(t) \tag{2.15}$$

The main idea behind the proofs of the theorems 1.1 and 1.2 is to seek a fixed point for the integral equation (2.15) under appropriate assumptions on the operators $H_i$. Notice that, due to the unboundedness of the operators $\mathcal{B}_i$, the expressions defining the operators $\mathcal{L}_i$ are only formal. Consequently, we must prove that the $\mathcal{L}_i$ i=1,2, are well-defined and, moreover, possess an additional regularity property which will allow us to apply an appropriate fixed point argument. This regularity requirement will be the subject of the next section.

## 3 Regularity of the maps $\mathcal{L}_1$ and $\mathcal{L}_2$

Notice first, that from the definitions of $A_F$, (2.10), and $\mathcal{B}_i$, (2.5), it follows that

$$A_F^{-1}\mathcal{B}_i \in \mathcal{L}(\mathcal{U};\mathcal{H}). \tag{3.1}$$

Indeed,

$$A_F^{-1}\mathcal{B}_i u = \begin{pmatrix} -k_1 G_1 G_1^* \mathcal{A} - k_2 G_2 G_2^* \mathcal{A} & -\mathcal{A}^{-1} \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \mathcal{A} G_i u \end{pmatrix} = \begin{pmatrix} -G_i u \\ 0 \end{pmatrix}$$

and the conclusion (3.1) follows from (2.3).

In order to give meaning to the formula defining $\mathcal{L}_i$ for i = 1,2, we shall prove

**Proposition 3.1** *For any $T > 0$ and $i = 1, 2$,*

$$\mathcal{L}_i : C([0,T];\mathcal{U}) \to C([0,T];\mathcal{H})$$

*are closed and densely defined.*

**Proof:** By (3.1) and (2.12)

$$A_F^{-1}\mathcal{L}_i \in \mathcal{L}(C([0,T];\mathcal{U}), C([0,T];\mathcal{H})). \tag{3.2}$$

Hence (see [2]) the $\mathcal{L}_i$ are closed.

In order to prove that the $\mathcal{L}_i$ are densely defined, it is enough to show that

$$\mathcal{L}_i \in \mathcal{L}(C^1([0,T];\mathcal{U}), C([0,T];\mathcal{H})) \text{ for } i = 1, 2. \tag{3.3}$$

On the other hand, using (2.12) and integrating by parts we obtain (for $u \in C^1([0,T];\mathcal{U})$)

$$(\mathcal{L}_i u) = -A_F^{-1}\mathcal{B}_i u(t) + e^{A_F t} A_F^{-1}\mathcal{B}_i u(0) + A_F^{-1}\mathcal{L}_i u'(t).$$

The conclusion (3.3) now follows from (3.1) and (3.2).□

In the sequel we shall need stronger regularity results for the operators $\mathcal{L}_i$. In fact, the main result of this section is

**Lemma 3.1** *For $i = 1,2$:*

$$(i) \qquad \mathcal{L}_i \in \mathcal{L}(C([0,\infty);\mathcal{U}), C([0,\infty);\mathcal{H}))$$

$$(ii) \quad \mathcal{L}_i \in \mathcal{L}\left(L^2([0,T];\mathcal{U}), C([0,T];\mathcal{H})\right) \qquad \textit{for any } T > 0.$$

We will prove Lemma 3.1 through a sequence of propositions.

**Proposition 3.2** *For $i = 1,2$ and any $T > 0$*

$$\int_0^T \left\| \mathcal{B}_i^* e^{A_F^* t}\tilde{x} \right\|_{\mathcal{U}}^2 dt \le C_T \|\tilde{x}\|_{\mathcal{H}}^2 \qquad \textit{for } \tilde{x} \in D(A_F^*). \tag{3.4}$$

**Proof:** Notice first that by (3.1) we have $\mathcal{B}_i^*(A_F^*)^{-1} \in \mathcal{L}(\mathcal{H};\mathcal{U})$. Hence, (3.4) is well-defined for $\tilde{x} \in D(A_F^*)$. In order to prove (3.4), we shall invoke its p.d.e. interpretation.

From (2.7) and (2.8), we see that (3.4) is equivalent to proving

$$\begin{aligned} \int_0^T \left\{ \left\| \frac{\partial}{\partial \nu}(e^{A_F^* t}\tilde{x})_2\Big|_\Gamma \right\|_{L^2(\Gamma)} + \left\| (e^{A_F^* t}\tilde{x})_2\Big|_\Gamma \right\|_{L^2(\Gamma)} \right\} dt \\ \le C_T \cdot \left\{ \|x_1\|_{D(\mathcal{A}^{1/2})} + \|x_2\|_{L^2(\Omega)} \right\} \end{aligned} \tag{3.5}$$

for some $0 < T < \infty$ and all $\tilde{x} = [x_1, x_2] \in D(A_F^*)$.

*Remark:* Notice that the regularity in (3.5) does not follow from the standard regularity of the underlying dynamics. It is an additional trace regularity result. Indeed, with $x_1 \in D(\mathcal{A}^{1/2})$ and $x_2 \in L^2(\Omega)$ one has by standard results that $e^{A_F^* t}\tilde{x} \in C([0,T];\mathcal{H})$. Hence, $(e^{A_F^* t}\tilde{x})_2 \in C([0,T];L^2(\Omega))$. The above regularity does not allow us to define the traces $(e^{A_F^* t}\tilde{x})_2\Big|_\Gamma$ and $\frac{\partial}{\partial \nu}(e^{A_F^* t}\tilde{x})_2\Big|_\Gamma$.

We now define a new variable $\tilde{v}(t) = [v_1(t), v_2(t)] \equiv e^{A_F^* t}\tilde{x}$.Then by virtue of semigroup properties of $e^{A_F^* t}$, we know that $\tilde{v}$ satisfies the abstract o.d.e.

$$\tilde{v}_t(t) \;=\; A_F^* \tilde{v}(t)$$

$$\tilde{v}(0) \;=\; \tilde{x}. \tag{3.6}$$

We then observe that (3.5) is equivalent to

$$\int_0^T \left\{ \left\| \left(\tfrac{\partial}{\partial \nu} v_2\right)\big|_\Gamma \right\|_{L^2(\Gamma)} + \left\| (v_2)\big|_\Gamma \right\|_{L^2(\Gamma)} \right\} dt \;\le\; C_T \cdot \|\tilde{v}(0)\|_{D(\mathcal{A}^{1/2}) \times L^2(\Omega)}. \tag{3.7}$$

To prove (3.7), we note that

$$A^*_F \tilde{v} \;=\; \begin{bmatrix} 0 & -I \\ \mathcal{A} & -k_1 \mathcal{A} G_1 G_1^* \mathcal{A} - k_2 \mathcal{A} G_2 G_2^* \mathcal{A} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$=\; \begin{bmatrix} -v_2 \\ \mathcal{A}(v_1 - k_1 G_1 G_1^* \mathcal{A} v_2 - k_2 G_2 G_2^* \mathcal{A} v_2) \end{bmatrix}.$$

Hence (3.6) can be written as the second order o.d.e.

$$\begin{aligned} v_{tt}(t) \;&=\; \mathcal{A}(v + k_1 G_1 G_1^* \mathcal{A} v_t + k_2 G_2 G_2^* \mathcal{A} v_t) \\ v(0) = x_1 \quad &; \quad v_t(0) = x_2 \end{aligned} \tag{3.8}$$

(where we have set $v_1 = v$ and consequently, $v_2 = -v_t$). The system (3.8) is equivalent to the p.d.e.

$$\begin{aligned} & v_{tt} + \Delta^2 v = 0 && \text{in } Q \\ & v = \tfrac{\partial}{\partial \nu} v = 0 && \text{on } \Sigma_0 \\ & \Delta v + (1-\mu) B_1 v = -k_1 \cdot \tfrac{\partial}{\partial \nu} v_t && \text{on } \Sigma_1 \\ & \tfrac{\partial}{\partial \nu} \Delta v + (1-\mu) B_2 v = k_2 \cdot v_t && \text{on } \Sigma_1. \end{aligned} \tag{3.9}$$

Using the "method of multipliers" with multiplier $v_t(t)$, we have via (3.9)

$$0 \;=\; \int_\Omega v_{tt} \cdot v_t \, d\Omega + \int_\Omega \Delta^2 v \cdot v_t \, d\Omega$$

$$\begin{aligned}
&= \int_\Omega \frac{1}{2}\frac{d}{dt}\{(v_t)^2 + (\Delta v)^2\}\, d\Omega \\
&\qquad + \int_\Gamma \left\{(1-\mu)[B_1 v \tfrac{\partial}{\partial\nu} v_t - B_2 v \cdot v_t] + k_1(\tfrac{\partial}{\partial\nu} v_t)^2 + k_2(v_t)^2\right\}\, d\Gamma \\
&= \int_\Omega \frac{1}{2}\frac{d}{dt}\{(v_t)^2 + (\Delta v)^2\}\, d\Omega \\
&\qquad + (1-\mu)\int_\Omega \{2v_{xy}v_{xyt} - v_{yy}v_{xxt} - v_{xx}v_{yyt}\}\, d\Omega \\
&\qquad + k_1 \cdot \left\| (\tfrac{\partial}{\partial\nu} v_t)|_\Gamma \right\|^2_{L^2(\Gamma)} + k_2 \cdot \|(v_t)|_\Gamma\|^2_{L^2(\Gamma)}.
\end{aligned}$$

(Here we have denoted the two spacial variables as $x$ and $y$). The last inequality follows from a Lemma proven in [4]:

**Lemma 3.2** *For sufficiently smooth $u$ and $v$ in $\Omega$ we have*

$$\int_\Omega (2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})\,d\Omega = \int_\Gamma \{(B_1 u)(\tfrac{\partial}{\partial\nu} v) - (B_2 u)\cdot v\}\,d\Gamma$$

*where $B_1$ and $B_2$ are given by (1.1)(f).*

By combining terms, we have

$$\begin{aligned}
0 &= \frac{1}{2}\int_\Omega \frac{d}{dt}\left[(v_t)^2 + (1-\mu)(v_{xx}^2 + v_{yy}^2) + \mu(\Delta v)^2 + 2(1-\mu)(v_{xy})^2\right] d\Omega \\
&\qquad + k_1 \cdot \left\| (\tfrac{\partial}{\partial\nu} v_t)|_\Gamma \right\|^2_{L^2(\Gamma)} + k_2 \cdot \|(v_t)|_\Gamma\|^2_{L^2(\Gamma)}.
\end{aligned}$$

Integrating in time, we arrive at

$$\begin{aligned}
& k_1 \int_0^T \left\| \tfrac{\partial}{\partial\nu} v_t|_\Gamma \right\|^2_{L^2(\Gamma)}\, dt + k_2 \int_0^T \|v_t|_\Gamma\|^2_{L^2(\Gamma)}\, dt \\
&\le C\left\{\|v_t(0)\|_{L^2(\Omega)} + \|v(0)\|_{H^2_{\Gamma_0}(\Omega)}\right\} \\
&= C\,\|\bar{v}(0)\|_{D(A^{1/2}) \times L^2(\Omega)}
\end{aligned}$$

for $\tilde{x} = \tilde{v}(0) \in D(A^*_F)$. Since $T < \infty$, we conclude that (3.7), and consequently (3.5), hold.□

Next, let us introduce the operator

$$\hat{\mathcal{L}}_i x(t) \equiv \int_t^T \mathcal{B}_i^* e^{A^*_F(\tau - t)} x(\tau) d\tau \qquad \text{with } \alpha < \alpha_0, \;\; i = 1, 2. \tag{3.10}$$

Since $\mathcal{B}_i^*(A^*_F)^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{U})$, we have

$$\hat{\mathcal{L}}_i \in \mathcal{L}\left(L^1([0,T]; D(A^*_F)), C([0,T]; \mathcal{H})\right). \tag{3.11}$$

We shall need stronger regularity than (3.11) to prove Lemma 3.1.

**Proposition 3.3** *The* $\hat{\mathcal{L}}_i : L^1([0,T]; \mathcal{H}) \to L^1([0,T]; \mathcal{U})$ *are continuous.*

**Proof:** Let $x \in L^1([0,T]; D(A^*_F))$ . Then we have

$$\begin{aligned} \left\| \hat{\mathcal{L}}_i x \right\|_{L^1([0,T];\mathcal{U})} &= \int_0^T \left\| \int_t^T \mathcal{B}_i^* e^{A^*_F(\tau - t)} x(\tau) d\tau \right\|_{\mathcal{U}} dt \\ &\leq \int_0^T \int_0^\tau \|\mathcal{B}_i^* e^{A^*_F \bar{t}} x(\tau)\|_{\mathcal{U}} d\bar{t}\, d\tau, \end{aligned}$$

by Fubini's theorem and a change of variables,

$$\begin{aligned} &\leq \int_0^T \int_0^T \|\mathcal{B}_i^* e^{A^*_F \bar{t}} x(\tau)\|_{\mathcal{U}} d\bar{t}\, d\tau \\ &= \widehat{C_T}\, \|x\|_{L^1([0,T];\mathcal{H})} \end{aligned}$$

by Proposition 3.2. Now since $D(A^*_F) \subset \mathcal{H}$ is dense, we have (by a standard density argument)

$$\hat{\mathcal{L}}_i : L^1([0,T]; \mathcal{H}) \to L^1([0,T]; \mathcal{U}) \qquad i = 1, 2$$

are bounded, as desired. □

Taking the adjoint of the operator $\hat{\mathcal{L}}_i$, we notice that

$$\left(\hat{\mathcal{L}}_i^{*}\tilde{u}\right)(t) = \left(\mathcal{L}_i\tilde{u}\right)(t) \text{ on } [D(A^*)]'.$$

Thus, from the result of Proposition 3.3 and from the usual density argument we obtain

**Proposition 3.4** $\mathcal{L}_i \in \mathcal{L}\left(C([0,T];\mathcal{U});C([0,T];\mathcal{H})\right)$.□

Our final step is to extend the regularity of the $\mathcal{L}_i$ from $(0,T)$ to $(0,\infty)$, and thus obtain the proof of Lemma 3.1.

**Proof of Lemma 3.1:** From Proposition 3.4,

$$\sup_{0\le t\le T}\left\|\int_0^t e^{A_F(t-\tau)}\mathcal{B}_i u(\tau)d\tau\right\|_{\mathcal{H}} \le C_T\|u\|_{C([0,T];\mathcal{U})}. \tag{3.12}$$

Now suppose $u \in C([0,2T];\mathcal{U})$. Then for $T \le t \le 2T$ (with $\delta < \omega$), we have

$$\begin{aligned}
&\left\|\int_0^t e^{A_F(t-\tau)}\mathcal{B}_i u(\tau)d\tau\right\|_{\mathcal{H}} \\
&\le\ Me^{-\delta(t-T)}C_T\|u\|_{C([0,T];\mathcal{U})} + \left\|\int_0^t e^{A_F(t-\tau)}\mathcal{B}_i u(\tau+T)d\tau\right\|_{\mathcal{H}} \\
&\le\ Me^{-\delta(t-T)}C_T\|u\|_{C([0,T];\mathcal{U})} + C_T\|u(\cdot+T)\|_{C([0,T];\mathcal{U})} \\
&\le\ MC_T\{e^{-\delta(t-T)}+1\}\|u\|_{C([0,2T];\mathcal{U})}
\end{aligned}$$

since $M \ge 1$. Furthermore, we have the estimate

$$\sup_{0\le t\le 2T}\left\|\int_0^t e^{A_F(t-\tau)}\mathcal{B}_i u(\tau)d\tau\right\|_{\mathcal{H}} \le 2M^2C_T\|u\|_{C([0,2T];\mathcal{U})}$$

In general, we have for $nT \le t \le (n+1)T$

$$\left\|\int_0^t e^{A_F(t-\tau)}\mathcal{B}_i u(\tau)d\tau\right\|_{\mathcal{H}}$$

$$\begin{aligned}
&\leq \quad M e^{-\delta(t-nT)} \sum_{j=0}^{n-1} \left\| \int_{jT}^{(j+1)T} e^{(A_F)(nT-\tau)} \mathcal{B}_i u(\tau) d\tau \right\|_{\mathcal{H}} \\
&\qquad + \left\| \int_{nT}^{t} e^{A_F(t-\tau)} \mathcal{B}_i u(\tau) d\tau \right\|_{\mathcal{H}} \qquad (3.13)\\
&\leq \quad \left\{ M e^{-\delta(t-nT)} \cdot M \left( \sum_{j=0}^{n-1} e^{-\delta jT} \right) + 1 \right\} C_T \, \|u\|_{C([0,(n+1)T];\, \mathcal{U})} \\
&\leq \quad M^2 C_T \left\{ e^{-\delta(t-nT)} \left( \sum_{j=0}^{n-1} e^{-\delta jT} \right) + 1 \right\} \|u\|_{C([0,(n+1)T];\, \mathcal{U})}.
\end{aligned}$$

We claim that the estimates (3.13) and (3.12) provide us with

$$\begin{aligned}
&\sup_{0 \leq t \leq nT} \left\| \int_0^t e^{A_F(t-\tau)} \mathcal{B}_i u(\tau) d\tau \right\|_{\mathcal{H}} \\
&\qquad \leq M^2 C_T \left[ \left( \sum_{j=0}^{n-2} e^{-\delta jT} \right) + 1 \right] \|u\|_{C([0,\, nT];\, \mathcal{U}))}. \qquad (3.14)
\end{aligned}$$

The proof is by induction. Clearly, for $n = 2$, (3.14) holds. We then have

$$\begin{aligned}
&\sup_{0 \leq t \leq nT} \left\| \int_0^t e^{A_F(t-\tau)} \mathcal{B}_i u(\tau)\, d\tau \right\|_{\mathcal{H}} \\
&\leq \quad max \left\{ \sup_{0 \leq t \leq (n-1)T} \left\| \int \cdots \right\|_{\mathcal{H}}, \sup_{(n-1)T \leq t \leq nT} \left\| \int \cdots \right\|_{\mathcal{H}} \right\} \\
&\leq \quad M^2 C_T \cdot max \left\{ \begin{array}{c} \left[ \left( \sum_{j=0}^{n-3} e^{(-\delta jT)} \right) + 1 \right] \|u\|_{C([0,(n-1)T],\mathcal{U})}, \\ \sup_{(n-1)T \leq t \leq nT} \left\| \int \cdots \right\|_{\mathcal{H}} \end{array} \right\}.
\end{aligned}$$

But then

$$\begin{aligned}
&\sup_{(n-1)T \leq t \leq nT} \left\| \int_0^t e^{A_F(t-\tau)} \mathcal{B}_i u(\tau) d\tau \right\|_{\mathcal{H}} \\
&\qquad \leq \quad M^2 C_T \left\{ \left( \sum_{j=0}^{n-2} e^{-\delta jT} \right) + 1 \right\} \|u\|_{C([0,nT];\mathcal{U})},
\end{aligned}$$

which proves Lemma 3.1(i).

As for part (ii) of Lemma 3.1, a proof similar to that of Proposition 3.3 gives us that

$$\hat{\mathcal{L}}_i \in \mathcal{L}\left(L^2([0,T];\mathcal{U}), L^2([0,T];\mathcal{H})\right).$$

An application of the lifting theorem proven in [7] yields the desired result.□

## 4 Proofs of Theorems 1.1 and 1.2

To prove theorems 1.1 and 1.2, we return to the integral equation, (2.15). In both proofs, we seek to employ the Contraction Mapping Theorem (CMT) to prove that a unique solution to (2.12), and consequently to system (1.1), exists.

### 4.1 Proof of Theorem 1.1

We define

$$(\mathcal{F}v)(t) = e^{A_F t}\tilde{u}_0 + \sum_{i=0}^{2} \mathcal{L}_i\left(H_i(v)\right)(t). \tag{4.1}$$

To establish the result of Theorem 1.1, it suffices to prove that $\mathcal{F}$ has the unique fixed point in the space $\mathcal{Z}$ defined by

$$\mathcal{Z} \equiv \{z \in C([0,T];\mathcal{H}) : \|z\|_{C([0,T];\mathcal{H})} \leq 2M\|\tilde{u}_0\|_{\mathcal{H}} \equiv R_0\}.$$

By using hypothesis (H-1) together with the Sobolev imbeddings (dim $\Omega$=2)

$$\left.\begin{aligned} &H^1(\Omega) \subset L^p(\Omega) \quad \text{for any } p \geq 1 \\ \text{and} \quad &H^2(\Omega) \subset C(\bar{\Omega}) \end{aligned}\right\} \tag{4.2}$$

one can easily show that for $i = 1, 2$, the operators $H_i : \mathcal{H} \to \mathcal{U}$ and $H_0 : \mathcal{H} \to \mathcal{H}$ are bounded and locally Lipschitz continuous. In particular,

$$\|H_i(v) - H_i(u)\|_{\mathcal{U}} \leq |L_i(\|u\|_{\mathcal{H}}, \|v\|_{\mathcal{H}})| \; \|v - u\|_{\mathcal{H}} \tag{4.3}$$

where each $L_i(x, y)$ is a continuous function. Thus, by (2.11) and by Lemma 3.1(ii) we have for $t < T$ and $v \in \mathcal{Z}$

$$\begin{aligned} \|(\mathcal{F}v)(t)\|_{\mathcal{H}} &\leq \frac{R_0}{2} + \sum_{i=0}^{2} \|\mathcal{L}_i\|_{op} \cdot \left[ \int_0^T \|(H_i v)(t)\|_{\mathcal{V}}^2 dt \right]^{1/2} \\ &\leq \frac{R_0}{2} + T^{\frac{1}{2}} \sum_{i=0}^{2} \|\mathcal{L}_i\|_{op} \cdot C(R_0) \end{aligned}$$

where $C(R_0)$ is a continuous function of $R_0$ and $\|\cdot\|_{op} \equiv \|\cdot\|_{L^2([0,T];\mathcal{U}) \to C([0,T];\mathcal{H})}$ with $\mathcal{V} = \mathcal{U}$ for $i = 1, 2$ and $\|\cdot\|_{op} \equiv \|\cdot\|_{L^2([0,T];\mathcal{H}) \to C([0,T];\mathcal{H})}$ with $\mathcal{V} = \mathcal{H}$ for $i = 0$. Thus, given $R_0 > 0$ and $\|\tilde{u}_0\|_{\mathcal{H}} \leq R_0$, we select $T$ such that

$$T^{1/2} \sum_{i=0}^{2} \|\mathcal{L}_i\|_{op} C(R_0) < \frac{R_0}{2}. \tag{4.4}$$

This gives us that $\mathcal{F}(\mathcal{Z}) \subset \mathcal{Z}$.

Similarly, we prove that $\mathcal{F}$ is a contraction. Indeed, by (4.3)

$$\begin{aligned} &\|(\mathcal{F}v_1)(t) - (\mathcal{F}v_2)(t)\|_{\mathcal{H}} \\ &\quad \leq T^{\frac{1}{2}} \sum_{i=0}^{2} \|\mathcal{L}_i\|_{op} \cdot \sup_{0 \leq t \leq T} |L_i(\|v_1(t)\|_{\mathcal{H}}, \|v_2(t)\|_{\mathcal{H}})| \, \|v_1 - v_2\|_{\mathcal{H}}. \end{aligned}$$

Selecting $T$ sufficiently small so that both (4.4) and

$$T^{1/2} \sum_{i=0}^{2} \|\mathcal{L}_i\|_{op} \cdot \sup_{0 \leq t \leq T} |L_i(\|v_1(t)\|_{\mathcal{H}}, \|v_2(t)\|_{\mathcal{H}})| < 1 \tag{4.5}$$

are satisfied, we may apply the (CMT) to give us the desired result.□

## 4.2 Proof of Theorem 1.2

To prove Theorem 1.2, we set $v(t) = e^{\alpha t}\tilde{u}(t)$ for $\alpha < \omega$ and observe that

$$v(t) \;=\; e^{(A_F+\alpha)(t)}\,\tilde{u}_0 \;+\; \sum_{i=0}^{2}\left(\mathcal{L}_i\,(e^{\alpha\cdot} H_i(e^{-\alpha\cdot} v(\cdot)))(t)\right). \tag{4.6}$$

Defining

$$(\mathcal{F}v)(t) \;=\; e^{(A_F+\alpha)(t)}\,\tilde{u}_0 \;+\; \sum_{i=0}^{2}\left(\mathcal{L}_i\,(e^{\alpha\cdot} H_i(e^{-\alpha\cdot} v(\cdot)))(t)\right)$$

we see that proving Theorem 1.2 is equivalent to proving

**Lemma 4.1** *$\mathcal{F}$ has a unique fixed point in the space $\mathcal{Z}$ defined by*

$$\mathcal{Z} = \{z \in C([0,\infty);\mathcal{H}) : \|z\|_{\,C([0,\infty);\mathcal{H})} \;\le\; R_0\}$$

*for some $R_0 > 0$.*

**Proof:** We will show that $\mathcal{F}\mathcal{Z} \subset \mathcal{Z}$ for $\|\tilde{u}_0\|_{\,\mathcal{H}} \le 2R_0/3M$. We have

$$\begin{aligned}
&\|\mathcal{F}v\|_{\,C([0,\infty);\mathcal{H})} \\
&\quad\le\; \|\,e^{(A_F+\alpha)(t)}\tilde{u}_0\,\|_{\,C([0,\infty);\mathcal{H})} \;+\; \sum_{i=0}^{2}\|\,\mathcal{L}_i(e^{\alpha\cdot} H_i(e^{-\alpha\cdot} v(\cdot)))\,\|_{\,C([0,\infty);\mathcal{H})} \\
&\quad\le\; \sup_{0\le t<\infty} Me^{-\delta t}\,\|\,\tilde{u}_0\,\|_{\,\mathcal{H}} \;+\; \sum_{i=0}^{2}\|\mathcal{L}_i\|\cdot\|\,e^{\alpha\cdot} H_i(e^{-\alpha\cdot} v(\cdot))\,\|_{\,C([0,\infty);\mathcal{V})}
\end{aligned}$$

where $\mathcal{V} = \mathcal{U}$ and $\|\mathcal{L}_i\| = \|\mathcal{L}_i\|_{\,C([0,\infty);\mathcal{U})\to C([0,\infty);\mathcal{H})} < \infty$ for $i = 1,2$ by Lemma 3.1(i) and $\mathcal{V} = \mathcal{H}$ and $\|\mathcal{L}_i\| = \|\mathcal{L}_i\|_{\mathcal{L}(C([0,\infty);\mathcal{H}))} < \infty$ for $i = 0$ by (2.14).

Notice that the assumptions (H-1) and (H-2) along with the Sobolev imbeddings (4.2) imply (see [1])

$$\left.\begin{aligned} &H_i(0) = 0 \\ &\text{and} \\ &\sum_{i=1}^{2} \|DH_i(v)\|_{\mathcal{L}(\mathcal{H};\mathcal{U})} + \|DH_0(v)\|_{\mathcal{L}(\mathcal{H})} \to 0 \quad \text{as } \|v\|_{\mathcal{H}} \to 0 \end{aligned}\right\} \tag{4.7}$$

where $DH$ represents the Frechét derivative of $H$.

Using (4.7) and a "mean value theorem" for Banach spaces (see [9]), we have for $i = 0, 1, 2$

$$\begin{aligned} &\|e^{\alpha\cdot} H_i(e^{-\alpha\cdot} v(\cdot))\|_{C([0,\infty);\mathcal{H})} \\ &\leq \|v\|_{C([0,\infty);\mathcal{H})} \cdot \sup_{0<\tau<1;\, r\geq 0} \|H_i'(\tau e^{-\alpha r} v(r))\|_{\mathcal{H}-\mathcal{V}}. \end{aligned}$$

Here, $\mathcal{V} = \mathcal{H}$ for $i = 0$ and $\mathcal{V} = \mathcal{U}$ for $i = 1, 2$. Then taking $R_0$ sufficiently small, we have, again by (4.7),

$$\sup_{0<\tau<1;\, r\geq 0} \|DH_i(\tau e^{-\alpha r} v(r)\|_{\mathcal{H}-\mathcal{V}} \leq \frac{1}{6\|\mathcal{L}_i\|}$$

for $v \in \mathcal{Z}\,(= \mathcal{Z}(R_0))$. Consequently,

$$\|\mathcal{F}v\|_{C([0,\infty);\mathcal{H})} \leq M\,\|\tilde{u}_0\|_{\mathcal{H}} + \frac{\|v\|}{3} \leq M\,\|\tilde{u}_0\|_{\mathcal{H}} + \frac{R_0}{3}.$$

Chosing $\|\tilde{u}_0\| \leq 2R_0/3M$, we have $\mathcal{F}\mathcal{Z} \subset \mathcal{Z}$.

Similarly, we can show that $\mathcal{F}$ is a contraction on $\mathcal{Z}$ (for details see [1]). Thus, by (CMT), we obtain the proof of Theorem 1.2.□

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