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Optimality conditions and numerical approximations for some optimal design problems [1]

by

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In structural optimization one often encounters the problem of minimizing the weight of a plate or a beam under some geometric constraints and considering some bounds on the deflection or stress. Sometimes these problems have been studied as state constrained optimal control problems governed by elliptic differential equations, the control being a parameter that appears in the coefficients of the corresponding differential operator. In this paper we consider three aspects of these problems. Firstly we do the sensitivity analysis and prove the existence of a solution. Next we derive the optimality system from an abstract theorem of existence of a Lagrange multiplier. And finally we perform the numerical discretization of the control problem and prove the convergence of approximate solutions. In order to derive the optimality conditions and to prove the convergence of the numerical approximations we make a stability hypothesis of

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the Slater type, which avoids the necessity of enlarging the set of admissible states in the discretization. This approach is interesting since it is well known that this increasing of the admissible state set diminishes the order of convergence.

1. Setting of the problems

We consider two cases of optimal design of structures.

1.1. Beam

Let us consider a clamped beam, subject to a vertical load f and let $u(x)$ be the variable cross-sectional area of this beam. The weight of the beam is to be minimized. It is required that the deflection remains within certain limits: $|y(x)| \leq \delta$. So the problem can be formulated as follows:

$$\min_{u \in U_{ad}} \quad J(u) = \int_0^L u(x)dx$$

subject to

$$A_u y = \frac{d^2}{dx^2}\left[E\sigma u^2(x)\frac{d^2y}{dx^2}(x)\right] = f(x) \text{ in } \Omega = (0, L)$$

$$y(0) = y(L) = y'(0) = y'(L) = 0$$

$$|y(x)| \leq \delta \text{ in } \overline{\Omega} = [0, L]$$

where

$$U_{ad} = \{u \in C^{0,1}(\overline{\Omega}) : a \leq u(x) \leq b, |u'(x)| \leq c\},$$

δ, a, b and c are positive constants, L is the length of the beam, $E > 0$ is the Young's modulus of the material used and σ is a constant depending on the shape of cross-section of the beam. $C^{0,1}(\overline{\Omega})$ denotes the space of Lipschitz functions in $\overline{\Omega}$.

For every $u \in U_{ad}$ and every $f \in H^{-2}(\Omega)$ there exists a unique solution y_u of the above Dirichlet problem belonging to the space $H_0^2(\Omega)$. Furthermore there exists a constant $C_0 > 0$ such that

$$\|y_u\|_{H_0^2(\Omega)} \leq C_0\|f\|_{H^{-2}(\Omega)} \quad \forall u \in U_{ad} \tag{1.1}$$

It is important to note that while we are considering clamped-clamped beam in this paper, we could carry out the study assuming other boundary conditions such as

- Cantilevered: $y(0) = y'(0) = y''(L) = \frac{d}{dx}[E\sigma u^2(x)y''(x)]_{x=L} = 0.$
- Simply supported: $y(0) = y''(0) = y(L) = y''(L) = 0.$

This problem has been studied by E. Casas [4], J. Haslinger and P. Neittaanmäki [9], E.J. Haug and J.S. Arora [10] and E.J. Haug and B. Rousselet [11]. See also I. Hlaváček, I. Bock and J. Lovíšek [12].

1.2. Plate

Let Ω be a convex and bounded subset of R^2 and let Γ be its boundary. Let us consider now a clamped plate transversaly loaded, f being the distribution of load. Suppose that Ω is the mean plane of the plate and $u(x)$ is the thickness. The structural optimization problem is:

$$\min_{u \in U_{ad}} \quad J(u) = \int_\Omega u(x)dx$$

subject to

$$A_u y = \partial^2_{x_1}\left[D(u)(\partial^2_{x_1}y + \nu\partial^2_{x_2}y)\right] + \partial^2_{x_2}\left[D(u)(\partial^2_{x_2}y + \nu\partial^2_{x_1}y)\right] +$$

$$2(1-\nu)\partial^2_{x_1x_2}\left[D(u)\partial^2_{x_1x_2}y\right] = f \text{ in } \Omega$$

$$y = \partial_n y = 0 \quad \text{on } \Gamma$$

$$|y(x)| \le \delta \quad \text{in } \overline{\Omega}$$

where

$$U_{ad} = \{u \in C^{0,1}(\overline{\Omega}) : a \le u(x) \le b, \|\nabla u(x)\| \le c\},$$

$$D(u) = \frac{Eu^3}{12(1-\nu^2)},$$

$E > 0$ is the Young's modulus and $\nu \in (0, 1/2)$ is the Poisson's coefficient.

For every $u \in U_{ad}$ and every $f \in H^{-2}(\Omega)$ there exists a unique solution $y_u \in H^2_0(\Omega)$ of previous boundary value problem. Moreover this solution satisfies the inequality 1.1 and it is a continuous function in $\overline{\Omega}$ because $H_0^2(\Omega) \subset C_0(\Omega)$ since $n = 2$ (Adams [1]), where $C_0(\Omega)$ denotes the space of the continuous functions in $\overline{\Omega}$ vanishing on Γ. Let us remark that Γ is a Lipschitz boundary because Ω is bounded and convex, Grisvard [8]. The convexity assumption on Ω is useful only for the numerical approximation, the study of the continuous problem can be made simply supposing Γ to be Lipschitz.

This problem has been considered by E. Casas [4], E.J. Haug and J.S. Arora [10] and E.J. Haug and B. Rousselet [11]. See also I. Hlaváček, I. Bock and J. Lovíšek [13].

In the next section we will prove that the previous optimal design problems have at least one solution, supposed the existence of a function $u \in U_{ad}$ such that $|y_u(x)| \leq \delta$ for all $x \in \overline{\Omega}$, and we derive the optimality conditions. The numerical approximation and the proofs of convergence are considered in Section 3.

2. Sensivity analysis and existence of a solution

In the sequel (P_δ) will denote the optimal design problem corresponding to the beam or to the plate, δ being the limit imposed to the displacements $y(x)$. In these problems the constraints $a \leq u(x) \leq b$ are motivated by technological reasons, but it is known that they are not enough to assure the existence of a solution, see F. Murat [14] and W. Velte and P. Villaggio [15]. In order to overcome this difficulty we have added the constraint on the gradient. This constraint is justified because the previous state equations describe the physical situation correctly only if the variation of the plate thickness or the cross-sectional area of the beam is smooth. Before proving that (P_δ) has a solution we are going to state some continuity and differentiability properties of the function $u \longrightarrow y_u$.

THEOREM 1 *Let $A(\Omega)$ be the open subset of $C(\overline{\Omega})$ formed by the strictly positive functions in $\overline{\Omega}$. Then the mapping $F : A(\Omega) \longrightarrow H_0^2(\Omega)$ defined by $F(u) = y_u$ is infinitely differentiable. Moreover for every $u \in A(\Omega)$ and every $v \in C(\overline{\Omega})$, $z = DF(u) \cdot v$ is the unique solution of the following boundary problem:*

$$\begin{cases} A_u z + A_{u,v} y_u = 0 & \text{in } \Omega \\ z = 0 & \text{on } \Gamma \end{cases}$$

where $A_{u,v}$ is given by

BEAM: $A_{u,v}y = \frac{d^2}{dx^2}\left[2E\sigma u(x)v(x)\frac{d^2y}{dx^2}(x)\right]$

PLATE: $A_{u,v}y = \partial^2_{x_1}\left[D(u,v)(\partial^2_{x_1}y + \nu\partial^2_{x_2}y)\right] + \partial^2_{x_2}\left[D(u,v)(\partial^2_{x_2}y + \nu\partial^2_{x_1}y)\right]$

$+2(1-\nu)\partial^2_{x_1x_2}\left[D(u,v)\partial^2_{x_1x_2}y\right]$

with $D(u,v) = \frac{Eu^2v\cdot}{4(1-\nu^2)}$.

Proof. In order to prove this theorem we are going to consider the mapping $G : A(\Omega) \times H^2_0(\Omega) \longrightarrow H^{-2}(\Omega)$ defined by $G(u,y) = A_u y - f$. It is obvious that G is of class C^∞, $\partial_y G(u,y) = A_u$ is an isomorphism of $H^2_0(\Omega)$ onto $H^{-2}(\Omega)$ and $G(u, y_u) = 0$ for every $u \in A(\Omega)$, then from the implicit function theorem we deduce that the mapping F is of class C^∞ □.

THEOREM 2 *Let us assume that there exists an element $u \in U_{ad}$ such that the associated state satisfies the constraint $|y_u(x)| \leq \delta\ \forall x \in \overline{\Omega}$. Then (P_δ) has at least one solution.*

Proof. First let us note that U_{ad} is a closed and bounded subset of $C^{0,1}(\overline{\Omega})$ and that the inclusion $C^{0,1}(\overline{\Omega}) \subset C(\overline{\Omega})$ is compact. Moreover $U_{ad} \subset A(\Omega)$, the mapping $F : A(\Omega) \longrightarrow H^2_0(\Omega)$ is continuous and the inclusion $H^2_0(\Omega) \subset C_0(\Omega)$ is continuous, hence the set

$$\{u \in U_{ad} : |y_u(x)| \leq \delta\ \forall x \in \overline{\Omega}\}$$

is compact and nonempty in $C(\overline{\Omega})$. Thus the problem (P_δ) consists in minimizing a continuous function J on a nonempty and compact set, so it has at least one solution. □

Let us remark that (P_δ) has admissible points if b or δ are sufficiently large.

3. Optimality conditions

We are going to derive the optimality conditions for problem (P_δ). This is done by using the following result proved by J.F. Bonnans and E. Casas [2] and [3].

THEOREM 3 *Let U and Z be two Banach spaces, U being separable, and $K \subset U$ and $C \subset Z$ two convex subsets, C having a nonempty interior. Given $z_0 \in \overset{o}{C}$ and $\delta > 0$, let $C_\delta = (1-\delta)z_0 + \delta C$ and $\overline{u} \in K$ be a solution of problem*

$$(P_\delta)\left\{\begin{array}{l} Min\ J(u) \\ u \in K \ and\ G(u) \in C_\delta \end{array}\right.$$

where $J : U \longrightarrow R$ *and* $G : U \longrightarrow Z$ *are of class* C^1. *Then there exist a real number* $\overline{\lambda} \geq 0$ *and an element* $\overline{\mu} \in Z'$ *such that*

$$\overline{\lambda} + \|\overline{\mu}\|_{Z'} > 0$$

$$< \overline{\mu}, z - G(\overline{u}) > \leq 0 \quad \forall z \in C_\delta$$

$$< \overline{\lambda} J'(\overline{u}) + [DG(\overline{u})]^* \overline{\mu}, u - \overline{u} > \geq 0 \quad \forall u \in K.$$

Moreover $\overline{\lambda}$ *can be taken equal to one in the following cases:*

1. *If the following condition of Slater type is satisfied:*

$$\exists u_0 \in K \ such\ that\ G(\overline{u}) + DG(\overline{u}) \cdot (u_0 - \overline{u}) \in \overset{o}{C}_\delta$$

2. *For almost every* δ *belonging to the interval* $[\delta_0, \delta_1]$, *supposed that* (P_δ) *has a solution for each* δ *of this interval.*

In the sequel we will denote by $\overline{B}_\delta$ the closed ball of $C_0(\Omega)$ of center 0 and radio δ, and by $M(\Omega)$ the space of real and regular Borel measures on Ω, that is to say $M(\Omega) = C_0(\Omega)'$. Now we prove the next result.

THEOREM 4 *Let* $\overline{u}$ *be a solution of problem* (P_δ), *then there exist a real number* $\overline{\lambda} \geq 0$ *and elements* $\overline{y}, \overline{p} \in H_0^2(\Omega)$ *and* $\overline{\mu} \in M(\Omega)$ *satisfying*

$$\overline{\lambda} + \|\overline{\mu}\|_{M(\Omega)} > 0 \tag{3.1}$$

$$\left\{\begin{array}{ll} A_{\overline{u}}\overline{y} = f & in\ \Omega \\ \overline{y} = \partial_n \overline{y} = 0 & on\ \Gamma \end{array}\right. \tag{3.2}$$

$$\left\{\begin{array}{ll} A_{\overline{u}}\overline{p} = \overline{\mu} & in\ \Omega \\ \overline{p} = \partial_n \overline{p} = 0 & on\ \Gamma \end{array}\right. \tag{3.3}$$

$$\int_\Omega (z(x) - \overline{y}(x)) d\overline{\mu}(x) \leq 0 \quad \forall z \in \overline{B}_\delta \tag{3.4}$$

and one of the following inequalities:
BEAM:

$$\int_0^L (\overline{\lambda} - 2E\sigma\overline{u}\,\overline{y}''\overline{p}'')(u - \overline{u})dx \geq 0 \ \forall u \in U_{ad} \tag{3.5}$$

PLATE:

$$\int_\Omega \left(\overline{\lambda} - D(\overline{u},1)[\nu\Delta\overline{y}\Delta\overline{p} + (1-\nu)(\partial^2_{x_1}\overline{y}\partial^2_{x_1}\overline{p} + \partial^2_{x_2}\overline{y}\partial^2_{x_2}\overline{p} +\right.$$

$$\left. 2\partial^2_{x_1x_2}\overline{y}\partial^2_{x_1x_2}\overline{p})]\right)(u - \overline{u})dx \geq 0 \ \forall u \in U_{ad} \tag{3.6}$$

where $D(\overline{u},1) = \dfrac{E\sigma\overline{u}^2}{4(1-\nu^2)}$. *Moreover* $\overline{\lambda}$ *can be chosen equal to one in the following cases:*

1. *If the Slater condition is satisfied:*

$$\begin{array}{l} \exists (u_0, z_0) \in U_{ad} \times H_0^2(\Omega) \text{ such that} \\ A_{\overline{u}}z_0 + A_{\overline{u},u_0-\overline{u}}\overline{y} = 0 \text{ and } \overline{y} + z_0 \in B_\delta. \end{array} \tag{3.7}$$

2. *For almost every* $\delta \in [\delta_0, \infty)$ *supposed that* (P_{δ_0}) *has a solution.*

Proof. It is enough to apply the theorem 3 taking $U = C(\overline{\Omega})$, $Z = C_0(\Omega)$, $K = U_{ad}$, $C_\delta = \overline{B}_\delta$, $J(u) = \int_\Omega u(x)dx$ and $G(u) = y_u$. Now taking $\overline{y} = G(\overline{u})$ and $\overline{p}$ as the solution of 3.3, we deduce that 3.1–3.4 is verified. In order to prove the inequalities 3.5 and 3.6 we use the theorem 1 and so we can deduce for every $u \in U_{ad}$ that $z = DG(\overline{u}) \cdot (u - \overline{u})$ belongs to $H_0^2(\Omega)$ and verifies the equation $A_{\overline{u}}z + A_{\overline{u},u-\overline{u}}\overline{y} = 0$. Now from theorem 3, integrating by parts and using 3.3, we get

$$0 \leq < \overline{\lambda}J'(\overline{u}) + [DG(\overline{u})]^*\overline{\mu}, u - \overline{u} > = \overline{\lambda}J'(\overline{u}) \cdot (u - \overline{u}) + < \overline{\mu}, z > =$$

$$\int_0^L \overline{\lambda}(u - \overline{u})dx + < A_{\overline{u}}\overline{p}, z > = \int_0^L \overline{\lambda}(u - \overline{u})dx - < A_{\overline{u},u-\overline{u}}\overline{y}, \overline{p} > =$$

$$\int_0^L (\overline{\lambda} - 2E\sigma\overline{u}\,\overline{y}''\overline{p}'')(u - \overline{u})dx.$$

Inequality 3.6 is got in the same way. The rest of the proof follows easily from theorem 3. □

REMARKS 1 *1. Following F.H. Clarke [7] we will say the problem* (P_δ) *is normal if there exist* $\overline{\mu} \in M(\Omega)$ *and* $\overline{y}, \overline{p} \in H_0^2(\Omega)$ *such that the optimality system 3.1–3.6 is verified with* $\overline{\lambda} = 1$. *The previous theorem states* *that almost every problem (P_δ) is normal. Also it affirms that the Slater constraint qualification implies (P_δ) is normal. In the next Section we will show that under hypothesis 3.7 it is possible to prove the convergence of the numerical discretization.*

2. *The Slater condition is satisfied if the constraints $\|\nabla\overline{u}(x)\| \leq c$ and $\overline{u}(x) \leq b$ are not active. Indeed it is enough to take $u_0 = (1+\epsilon)\overline{u}$, with $\epsilon > 0$ small enough, so $u_0 \in U_{ad}$ and*

$$A_{\overline{u}} z_0 = -A_{\overline{u},u_0-\overline{u}}\overline{y} = -n\epsilon A_{\overline{u}}\overline{y} = -n\epsilon f, \ n = 2 \text{ or } 3,$$

thus $z_0 = -n\epsilon\overline{y}$ and then $\overline{y} + z_0 = (1-n\epsilon)\overline{y} \in B_\delta$.

In optimal design it is frequent to have $f \leq 0$ in such a way that the deflection is negative $\overline{y}(x) \leq 0$. In this case the Slater condition is satisfied if we can find an element $u_0 \in U_{ad}$ such that z_0 is positive and smaller than δ. I suspect that $u_0 = \overline{u} + \epsilon$ could be a correct element in many realistic cases.

3. *From 3.4 it is easy to deduce that the Lagrange multiplier $\overline{\mu}$ associated to the state constraint is concentrated on the set of points where the constraint is active. In particular if this set is finite, let us say $\{x_k\}_{k=1}^m$, then*

$$\overline{\mu} = \sum_{k=1}^{m} \lambda_k \delta_{[x_k]}, \quad \text{with} \quad \begin{cases} \lambda_k \geq 0 & \text{if } \overline{y}(x_k) = \delta \text{ and} \\ \lambda_k \leq 0 & \text{if } \overline{y}(x_k) = -\delta, \end{cases}$$

where $\delta_{[x_k]}$ is the Dirac measure concentrated at the point x_k. See Casas [5] for this question.

4. Numerical approximation

In order to carry out the numerical approximation we must distinguish the one dimensional and two dimensional cases. So we are going to consider both cases separately, studying the discretization properties, and then we will prove some convergence results.

4.1. Beam

Let $\mathcal{T}_h : 0 = x_0 < x_1 < \cdots < x_{n(h)} = L$ be a partition of $[0, L]$ such that

$$h = \max_{1 \le j \le n(h)} (x_j - x_{j-1}).$$

Let us consider the following function spaces

$$U_h = \{u_h \in C([0, L]) : u_h|_{[x_{j-1}, x_j]} \in \mathcal{P}_1,\ 1 \le j \le n(h)\}$$

$$V_h = \{y_h \in C^1([0, L]) : y_h|_{[x_{j-1}, x_j]} \in \mathcal{P}_3,\ 1 \le j \le n(h)\}$$

where $\mathcal{P}_1$ and $\mathcal{P}_3$ are the spaces of the polynomials of degree less than or equal to 1 and 3 respectively. It is well known that $V_h \subset H^2(0, L)$, Ciarlet [6]. Now we take $V_{0h} = V_h \cap H_0^2(0, L)$ and $U_{adh} = U_h \cap U_{ad}$. If we denote by $\{e_j\}_{j=0}^{n(h)}$ the functions of U_h such that $e_j(x_i) = \delta_{ij}$, then this set of functions constitutes a basis of U_h and each element $u_h \in U_h$ can be written in the following way

$$u_h = \sum_{j=0}^{n(h)} u_j e_j, \quad \text{with } u_j = u_h(x_j),\ 0 \le j \le n(h).$$

So we have

$$U_{adh} = \{u_h \in U_h : a \le u_j \le b\ (0 \le j \le n(h)) \text{ and } \max_{1 \le j \le n(h)} \frac{|u_j - u_{j-1}|}{x_j - x_{j-1}} \le c\}.$$

For each $u_h \in U_{adh}$ we define the bilinear form

$$a_{u_h} : V_h \times V_h \longrightarrow R$$

$$a_{u_h}(y_h, z_h) = < A_{u_h} y_h, z_h > = < y_h, A_{u_h} z_h > =$$

$$E\sigma \int_0^L u_h^2(x) y_h''(x) z_h''(x) dx$$

where $< \cdot, \cdot >$ denotes the duality product in $H_0^2(0, L)$. Now we denote by $y_h(u_h)$ the unique element belonging to V_{0h} that satisfies

$$a_{u_h}(y_h(u_h), z_h) = < f, z_h > \quad \forall z_h \in V_{0h}.$$

Finally the discretized optimal design problem is stated as follows:

$$(P_h) \begin{cases} \min J_h(u_h) = J(u_h) = \dfrac{1}{2} \displaystyle\sum_{j=1}^{n(h)} (u_j - u_{j-1})(x_j - x_{j-1}) \\ \\ u_h \in U_{adh} \text{ and } |y_h(u_h)(x_j)| \le \delta,\ 1 \le j \le n(h) - 1 \end{cases}$$

The following result states that $\{U_{adh}\}_h$ constitutes an internal approximation of U_{ad}.

LEMMA 1 *1. For every $u \in U_{ad}$ there exists a sequence $\{u_h\}$, with $u_h \in U_{adh}$, such that*

$$\lim_{h\to 0} \|u - u_h\|_{C([0,L])} = 0.$$

2. If $\{u_h\}$ is a sequence that converges towards u and if $u_h \in U_{adh}$ for every $h > 0$, then $u \in U_{ad}$.

Proof. In order to prove the first part is enough to take u_h as the element of U_h which interpolates u at the nodes $\{x_j\}_{j=0}^{n(h)}$, that is to say

$$u_h = \sum_{j=0}^{n(h)} u(x_j)e_j.$$

So we approximate u by a polygonal line and then $u_h \to u$ uniformly in $[0, L]$. Furthermore we have

$$a \leq u_j = u_h(x_j) = u(x_j) \leq b, \quad j = 0, \cdots, n(h)$$

$$\frac{|u_j - u_{j-1}|}{x_j - x_{j-1}} = \frac{1}{x_j - x_{j-1}} \left| \int_{x_{j-1}}^{x_j} u'(t)dt \right| \leq c$$

hence $u_h \in U_{adh}$. The second part follows from the inclusion $U_{adh} \subset U_{ad}$. □.

4.2. Plate

Let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}$ satisfying

1. $h_T = diam(T) = \max_{x,y\in T} \|x - y\|$ and $h = \max_{T\in\mathcal{T}_h} h_T$.
2. $T \subset \overline{\Omega}\ \forall T \in \mathcal{T}_h$.
3. If $T, T' \in \mathcal{T}_h, T \neq T'$ then we have $\overset{o}{T} \cap \overset{o}{T'} = \emptyset$ and either $T \cap T' = \emptyset$ or T and T' have in common one whole edge or only one vertex.
4. Let us take $\overline{\Omega}_h = \cup_{T\in\mathcal{T}_h} T$, Ω_h its interior and Γ_h its boundary. Then we assume that $\overline{\Omega}_h$ is convex and that the vertices of $\mathcal{T}_h$ placed on the boundary Γ_h are points of Γ.

5. The angles of all $T \in \mathcal{T}_h$ are bounded from below by a constant θ independently of h.

As consequence of these hypotheses we get that $\Omega_h \to \Omega$ in the following sense: for every compact set $E \subset \Omega$ there exists $h_0 > 0$ such that $E \subset \Omega_h$ for all $h < h_0$. Now to every boundary triangle T of $\mathcal{T}_h$ we associate another triangle $\tilde{T} \subset \overline{\Omega}$ with two interior sides to Ω coincident with two sides of T and the third side is the curvilinear arc of Γ limited by the other two sides. We denote by $\tilde{\mathcal{T}}_h$ the family formed by these boundary triangles with a cuvilinear side and the interior triangles to Ω of $\mathcal{T}_h$, so $\overline{\Omega} = \cup_{T \in \tilde{\mathcal{T}}_h} T$.

As in the one dimensional case, let us consider the spaces:

$$U_h = \{u_h \in C(\overline{\Omega}) : u_h|_T \in \mathcal{P}_1 \ \forall T \in \tilde{\mathcal{T}}_h\}$$

$$V_h = \{y_h \in C^1(\overline{\Omega}_h) : y_h|_T \in \mathcal{P}_5 \ \forall T \in \mathcal{T}_h\}.$$

So we have thought of the Argyris triangle to get the function space V_h, but obviously we could have considered other finite elements of class C^1, for instance the Bell triangle. In these conditions it is well known that $V_h \subset H^2(\Omega_h)$, then we define $V_{0h} = V_h \cap H_0^2(\Omega_h)$. In fact the elements of $H_0^2(\Omega_h)$ can be extended by zero to Ω and so we can consider $H_0^2(\Omega_h)$ as a subspace of $H_0^2(\Omega)$.

Let $\{x_j\}_{j=0}^{n(h)}$ be the set of vertices of $\mathcal{T}_h$. As in the beam case we denote by $\{e_j\}_{j=0}^{n(h)}$ the basis of U_h defined by the equalities $e_j(x_i) = \delta_{ij}$ and $\{u_j\}_{j=0}^{n(h)}$ are the coordinates of u_h in this basis: $u_j = u_h(x_j)$.

Now we define

$$\begin{aligned} U_{adh} = \{&u_h \in U_h : a \le u_j \le b \ (0 \le j \le n(h)) \\ &\text{and } \|\nabla u_h|_T\| \le c \ \forall T \in \mathcal{T}_h\}. \end{aligned}$$

We note that $\nabla u_h|_T$ is a constant of R^2 for each $T \in \mathcal{T}_h$, so the constraints on u_h are easy to handle. We must remark too that $U_{adh} \not\subset U_{ad}$ in general because an element $u_h \in U_{adh}$ can take values lower to a or upper to b in the domain $\tilde{T} \setminus T$, supposed $T \subset \tilde{T}$.

As in the one dimensional case we denote by $y_h(u_h)$ the unique element of V_{0h} that satisfies

$$a_{u_h}(y_h(u_h), z_h) = < f, z_h > \quad \forall z_h \in V_{0h}$$

where

$$a_{u_h}(y_h, z_h) = < A_{u_h} y_h, z_h > = \int_{\Omega_h} D(u_h)[\nu \Delta y_h \Delta z_h +$$

$$(1-\nu)(\partial^2_{x_1} y_h \partial^2_{x_1} z_h + \partial^2_{x_2} y_h \partial^2_{x_2} z_h + 2\partial^2_{x_1 x_2} y_h \partial^2_{x_1 x_2} z_h)]dx.$$

Now we state the discretized optimal design problem:

$$(P_h)\begin{cases} \min J_h(u_h) = \int_{\Omega_h} u_h(x)dx = \sum_{T\in\mathcal{T}_h} \text{meas}(T) u_h(x_T) \\ \\ u_h \in U_{adh} \text{ and } |y_h(u_h)(x_j)| \le \delta, j \in I_h \end{cases}$$

where x_T is the barycenter of the triangle T and I_h is the set of indices corresponding to interior vertices of $\mathcal{T}_h$. So the discrete state does not satisfy the constraint on the deflection in every point of Ω, but in order to solve numerically the problem it is necessary to take a system of constraints easy to handle.

An important question to note here is that if $u \in U_{ad}$ and u_h is the function of U_h which interpolates u in the points x_j, then it is not true in general that $u_h \in U_{adh}$ because of the constraint on the gradient. To verify this consider the triangle T with vertices $(0,0), (1,1)$ and $(-1,1)$, the function $u(x) = \|x\|$ and $c = 1$. Then we have that $u_h(x) = \sqrt{2}x_2$ is the element of $\mathcal{P}_1$ which interpolates u and it does not satisfy the constraint on the gradient. This makes more difficult to prove that U_{adh} constitutes an internal approximation of U_{ad}, in fact we need to prove the following previous lemma:

LEMMA 2 *Let U_0 be the set*

$$U_0 = \{u \in C^{0,1}(\overline{\Omega}) : \exists \epsilon = \epsilon(u) > 0 \;\; \textit{such that} \;\; a + \epsilon \le u(x) \le b - \epsilon$$

$$\textit{and} \;\; \|\nabla u(x)\| \le c - \epsilon\}.$$

Then for every $u \in U_{ad}$ there exists a sequence $\{u_k\} \subset C^\infty(\overline{\Omega}) \cap U_0$ converging uniformly to u.

Proof. Let $x_0 \in \Omega$ be a fixed element. For each $\lambda \in (0,1)$ we define

$$\Omega_\lambda = \{y = \frac{1}{\lambda}(x - x_0) + x_0 : x \in \Omega\}.$$

Since Ω is convex, we have that $\overline{\Omega} \subset \Omega_\lambda$. Given $u \in U_{ad}$ we take $u_\lambda : \Omega_\lambda \longrightarrow R$ defined by $u_\lambda(x) = u(x_0 + \lambda(x - x_0))$. Then it is obvious that $u_\lambda|_{\overline{\Omega}} \in U_{ad}$ and $u_\lambda \to u$ uniformly in $\overline{\Omega}$ when $\lambda \to 1$. Now if we extend u_λ by zero to R^2 and we make the convolution of u_λ with a regularizing sequence $\{\phi_j\}$, we get a sequence $\{u_{\lambda j}\} \subset C^\infty(\overline{\Omega})$ that converges to u_λ uniformly in $\overline{\Omega}$. Furthermore if we take $j_\lambda \in N$ such that $\phi_j(x-y) = 0 \; \forall (x,y) \in \overline{\Omega} \times \Omega_\lambda^c$ for every $j \ge j_\lambda$, then we deduce for all $x \in \overline{\Omega}$

$$a \leq \int_{\Omega_\lambda} \phi_j(x-y)a dy \leq \int_{\Omega_\lambda} \phi_j(x-y)u_\lambda(y)dy = u_{\lambda j}(x) \leq \int_{\Omega_\lambda} \phi_j(x-y)b dy = b$$

and

$$\|\nabla u_{\lambda j}(x)\| = \left\| \int_{\Omega_\lambda} \phi_j(x-y)\nabla u_\lambda(y)dy \right\| \leq \int_{\Omega_\lambda} \phi_j(x-y)\|\nabla u_\lambda(y)\|dy \leq \lambda c < c.$$

Then $u_{\lambda j}|_{\overline{\Omega}} \in U_{ad}$ and we can take a subsequence converging to u. Thus we have proved that $C^\infty(\overline{\Omega}) \cap U_{ad}$ is dense in U_{ad} for the uniform convergence topology. To conclude the proof let us take a function $u \in C^\infty(\overline{\Omega}) \cap U_{ad}$ and let us see that it can be approximated by a sequence of $C^\infty(\overline{\Omega}) \cap U_0$. For each $t \in (0,1)$ let $u_t : \overline{\Omega} \longrightarrow R$ be the function

$$u_t(x) = \frac{ta + (1-t)b}{t^2 a + (1-t^2)b}[tu(x) + (1-t)b].$$

Then $u_t \in C^\infty(\overline{\Omega}) \cap U_0$ and $u_t \to u$ uniformly in $\overline{\Omega}$ when $t \to 1$. □

Now we can prove the following result

LEMMA 3 *1. For every $u \in U_{ad}$ there exists a sequence $\{u_h\}$, with $u_h \in U_{adh}$, such that*

$$\lim_{h \to 0} \|u - u_h\|_{C(\overline{\Omega})} = 0.$$

2. If $\{u_h\}$ is a sequence that converges uniformly towards u and if $u_h \in U_{adh}$ for every $h > 0$, then $u \in U_{ad}$.

Proof. To prove the first part, thanks to lemma 2 it follows that it is enough to consider an element $u \in C^\infty(\overline{\Omega}) \cap U_0$ and show that it can be approximated uniformly by elements of U_{adh}. So given u let us take

$$u_h = \sum_{j=0}^{n(h)} u(x_j)e_j.$$

Because of the regularity of u we get from the interpolation theory (Ciarlet [6])

$$\|u - u_h\|_{W^{1,\infty}(\Omega)} \leq Ch\|u\|_{W^{2,\infty}(\Omega)}. \tag{4.1}$$

Let $\epsilon = \epsilon(u)$ as in the definition of U_0, then from the previous inequality we deduce

$$\exists h_0 > 0 \text{ such that } \|u - u_h\|_{W^{1,\infty}(\Omega)} \leq \epsilon/2 \quad \forall h < h_0$$

hence $u_h \in U_{adh}$ $\forall h < h_0$. Moreover the equation 4.1 implies $\|u - u_h\|_{C(\overline{\Omega})} \to 0$ and so the proof of the first part is complete. The second part of lemma is immediate. □

4.3. Convergence analysis

In this Section we are going to consider two different situations. The first one consists in assuming the Slater hypothesis 3.7, in this case we will prove that (P_h) has at least one solution $\overline{u}_h$ for h small enough and that $\overline{u}_h$ converges to a solution of the continuous problem. The other situation occurs when we do not assume 3.7. In this case we will see that any limit of a sequence of points satisfying the optimality system for the discrete problem verifies the optimality conditions 3.1–3.4 and 3.5 or 3.6. Before stating these results we need to stablish the next lemma:

LEMMA 4 *Given $u_h, u \in A(\Omega)$ such that $\|u - u_h\|_{C(\overline{\Omega})} \to 0$ when $h \to 0$, then the following equalities are satisfied:*

1. $\lim_{h\to 0} J_h(u_h) = J(u)$.
2. $\lim_{h\to 0} \|y_h(u_h) - y_u\|_{H_0^2(\Omega)} = 0$.

Proof. The first equality is obtained in the following way:

$$|J(u) - J_h(u_h)| = \left| \int_{\Omega_h} [u(x) - u_h(x)]dx + \int_{\Omega\setminus\Omega_h} u(x)dx \right| \leq$$

$$\int_{\Omega_h} |u(x) - u_h(x)|dx + \int_{\Omega\setminus\Omega_h} |u(x)|dx \leq$$

$$\text{meas}(\Omega_h)\|u - u_h\|_{C(\overline{\Omega})} + \text{meas}(\Omega\setminus\Omega_h)\|u\|_{C(\overline{\Omega})} \to 0 \text{ if } h \to 0,$$

where $\Omega_h = \Omega = (0, L)$ in the one dimensional case.

For the second equality we must remember that $y_h(u_h) \in V_{0h}$ is extended by zero to Ω and then $y_h(u_h) \in H_0^2(\Omega)$. In order to simplify the notation let us denote $y_h = y_h(u_h)$. Now from the uniform convergence of u_h to u we get the existence of a constant $c_1 > 0$ such that $u_h(x) \geq c_1\ \forall x \in \overline{\Omega}$. Then the bilinear forms a_{u_h} are uniformly coercive and consequently from the definition of y_h we deduce that $\{y_h\}_{h>0}$ is a bounded sequence in $H_0^2(\Omega)$. Thus we can get a subsequence $\{y_{h_k}\}$ converging weakly to an element $y \in H_0^2(\Omega)$. Let us prove that $y = y_u$ or what is the same $A_u y = f$. Let be $\phi \in D(\Omega)$ and let ϕ_h be the element of V_{0h} which interpolates ϕ, so we have $\phi_h \to \phi$ in $H_0^2(\Omega)$. Then, using the uniform convergence of u_h to u, the weak convergence of y_{h_k} to y and the strong convergence of ϕ_h to ϕ, we get

$$< A_u y, \phi >= \lim_{k\to\infty} < A_{u_{h_k}} y_{h_k}, \phi_{h_k} >= \lim_{k\to\infty} a_{u_{h_k}}(y_{h_k}, \phi_{h_k}) =$$

$$\lim_{k\to\infty} < f, \phi_{h_k} >=< f, \phi > .$$

Since $\phi \in D(\Omega)$ is arbitrary we deduce that $A_u y = f$ and so $y = y_u$. Hence y_h converges to y weakly in $H_0^2(\Omega)$ when $h \to 0$. It remains to prove the strong convergence. For it we begin noting that the uniform convergence of $\{u_h\}$ implies

$$| < (A_u - A_{u_h})y_h, y_h > | \le c_2 \|u - u_h\|_{C(\overline{\Omega})} \|y_h\|^2_{H_0^2(\Omega)} \le$$
$$c_3 \|u - u_h\|_{C(\overline{\Omega})} \to 0.$$

From this relation and the coercivity of a_u we obtain

$$c_4 \|y - y_h\|^2_{H_0^2(\Omega)} \le < A_u(y - y_h), y - y_h >=$$

$$< f, y - y_h > - < f, y_h > + < A_u y_h, y_h >=$$

$$< f, y - y_h > - < f, y_h > + < A_{u_h} y_h, y_h > + < (A_u - A_{u_h})y_h, y_h >=$$

$$< f, y - y_h > + < (A_u - A_{u_h})y_h, y_h > \to 0.$$

Finally we are ready to study the convergence of the discrete optimal design problem.

THEOREM 5 *Suppose that the Slater condition 3.7 is satisfied, then there exists $h_0 > 0$ such that (P_h) has at least one solution $\overline{u}_h$ for each $h \le h_0$. Moreover there exist subsequences $\{\overline{u}_{h_k}\}_{k\in N}$ of $\{\overline{u}_h\}_{h\le h_0}$ and elements $\tilde{u} \in U_{ad}$ such that*

$$\lim_{k\to\infty} \|\overline{u}_{h_k} - \tilde{u}\|_{C(\overline{\Omega})} = 0.$$

Each one of these limit points is a solution of problem (P_δ). Finally we have

$$\lim_{h\to 0} J_h(\overline{u}_h) = \min(P_\delta).$$

Proof. We will prove that the set of elements $u_h \in U_{adh}$ such that $|y_h(u_h)(x_j)| \le \delta$ for all $j \in I_h$ is nonempty. Then we can argue as in the proof of theorem 2 and to obtain the existence of a solution.

Let $\overline{u}$ be the solution of (P_δ) and $(u_0, z_0) \in U_{ad} \times H_0^2(\Omega)$ the pair considered in the Slater condition 3.7. For every $\lambda \in (0, 1)$ we take $u_\lambda = \overline{u} + \lambda(u_0 - \overline{u}) \in U_{ad}$ and $y_\lambda = y_{u_\lambda}$. Then we have that $\dfrac{y_\lambda - \overline{y}}{\lambda} \to z_0$ when $\lambda \to 0$. Therefore we deduce from 3.7 that there exists $\lambda_0 > 0$ such that

$$\overline{y} + \frac{y_\lambda - \overline{y}}{\lambda} \in B_\delta \quad \forall \lambda < \lambda_0$$

hence we get

$$y_\lambda = \lambda \left[\overline{y} + \frac{y_\lambda - \overline{y}}{\lambda} \right] + (1-\lambda)\overline{y} \in B_\delta \quad \forall \lambda < \lambda_0. \tag{4.2}$$

From lemmas 1 and 3 we deduce the existence of a sequence $\{u_{\lambda h}\}$, with $u_{\lambda h} \in U_{adh}$, such that $\|u_{\lambda h} - u_\lambda\|_{C(\overline{\Omega})} \to 0$ when $h \to 0$. Now from lemma 4 we obtain the convergence of $y_{\lambda h} = y_h(u_{\lambda h})$ towards y_λ in $H_0^2(\Omega)$, which implies that $y_{\lambda h} \to y_\lambda$ in $C(\overline{\Omega})$. So if $\lambda < \lambda_0$, from 4.2 we deduce the existence of $h_0 > 0$ such that $y_{\lambda h} \in B_\delta \; \forall h \leq h_0$ and therefore the set of admissible points is nonempty for every $h \leq h_0$, thus we conclude the proof of existence of solution.

To prove the second part of the theorem, first we note that $\{\overline{u}_h\}$ is a bounded sequence in $C^{0,1}(\overline{\Omega})$ and then from Ascoli's theorem we deduce the existence of uniformly convergent subsequences to elements $\tilde{u}$.

Now let us take $\lambda < \lambda_0$. Since $\overline{u}_{h_k}$ is a solution of (P_{h_k}) and $u_{\lambda h_k}$ is an admissible point for $h < h_0$ we get with the aid of lemma 4 that

$$J_{h_k}(\overline{u}_{h_k}) \leq J_{h_k}(u_{\lambda h_k}) \Longrightarrow J(\tilde{u}) = \lim_{k\to\infty} J_{h_k}(\overline{u}_{h_k}) \leq \lim_{k\to\infty} J_{h_k}(u_{\lambda h_k}) = J(u_\lambda)$$

hence

$$J(\tilde{u}) \leq \lim_{\lambda \to 0} J(u_\lambda) = J(\overline{u}) = \min\,(P_\delta).$$

It remains to prove that $\tilde{u}$ is an admissible point to deduce that it is a solution of (P_δ). From lemmas 1 and 3 we obtain that $\tilde{u} \in U_{ad}$ and from lemma 4 we get that $\overline{y}_{h_k} = y_{h_k}(\overline{u}_{h_k})$ converges uniformly to $y_{\tilde{u}}$ in $\overline{\Omega}$, then $|\overline{y}_{h_k}(x_j)| \leq \delta \; \forall j \in I_{h_k}$ implies that $y_{\tilde{u}} \in \overline{B}_\delta$, so $\tilde{u}$ is an admissible point. □

We have seen that the Slater condition is not only useful to derive the optimality system but it is a stability condition that guaranties the discrete set of admissible points is nonempty for h small enough and at the same time it allows us to prove the convergence of the discretizations. When the Slater condition is not satisfied the usual process to prove the previous results consists of changing δ by δ_h in such a way that $\delta_h \downarrow \delta$ when $h \to 0$. However this process can diminish the convergence order, which is obviously not desirable.

In practice we use algorithms to solve the discretized problem (P_h) that supply points satisfying the optimality necessary conditions. The following theorem states the optimality conditions for (P_h) and assures that points satisfying these conditions converge to points that satisfy the optimality conditions of (P_δ) given in the theorem 4.

THEOREM 6 *Let $\overline{u}_h$ be a solution of problem (P_h), then there exist a real number $\overline{\lambda}_h \geq 0$ and elements $\overline{\mu}_h = (\overline{\mu}_j)_{j \in I_h} \in R^{|I_h|}$ and $\overline{y}_h, \overline{p}_h \in V_{0h}$ satisfying*

$$\overline{\lambda}_h + \sum_{j \in I_h} |\overline{\mu}_j| = 1 \tag{4.3}$$

$$a_{\overline{u}_h}(\overline{y}_h, y_h) = < f, y_h > \quad \forall y_h \in V_{0h} \tag{4.4}$$

$$a_{\overline{u}_h}(\overline{p}_h, y_h) = \sum_{j \in I_h} \overline{\mu}_j y_h(x_j) \quad \forall y_h \in V_{0h} \tag{4.5}$$

$$\overline{\mu}_j \overline{y}_h(x_j) \geq 0 \quad \text{and} \quad \overline{\mu}_j(\overline{y}_h(x_j) - \delta) = 0 \quad \forall j \in I_h \tag{4.6}$$

and one of the following inequalities
BEAM:

$$\int_0^L (\overline{\lambda}_h - 2E\sigma\overline{u}_h\overline{y}_h''\overline{p}_h'')(u_h - \overline{u}_h)dx \geq 0 \quad \forall u_h \in U_{adh} \tag{4.7}$$

PLATE:

$$\int_{\Omega_h} \Big(\overline{\lambda}_h - D(\overline{u}_h, 1)[\nu\Delta\overline{y}_h\Delta\overline{p}_h + (1-\nu)(\partial^2_{x_1}\overline{y}_h\partial^2_{x_1}\overline{p}_h + \partial^2_{x_2}\overline{y}_h\partial^2_{x_2}\overline{p}_h +$$

$$2\partial^2_{x_1x_2}\overline{y}_h\partial^2_{x_1x_2}\overline{p}_h)]\Big)(u_h - \overline{u}_h)dx \geq 0 \; \forall u_h \in U_{adh} \tag{4.8}$$

where $|I_h|$ = Cardinal of I_h. Moreover for all sequence $\{(\overline{u}_h, \overline{\lambda}_h, \overline{\mu}_h, \overline{y}_h, \overline{p}_h)\}_h$ satisfying the optimality system 4.3-4.8, and assuming that $\overline{u}_h \in U_{adh}$, there exist subsequences $\{(\overline{u}_{h_k}, \overline{\lambda}_{h_k}, \overline{\mu}_{h_k}, \overline{y}_{h_k}, \overline{p}_{h_k})\}_k$ and elements $(\overline{u}, \overline{\lambda}, \overline{\mu}, \overline{y}, \overline{p})$ such that

$$\lim_{k\to\infty} \|\overline{u} - \overline{u}_{h_k}\|_{C(\overline{\Omega})} = \lim_{k\to\infty} \|\overline{y} - \overline{y}_{h_k}\|_{H^2_0(\Omega)} = \lim_{k\to\infty} |\overline{\lambda} - \overline{\lambda}_{h_k}| = 0 \tag{4.9}$$

$$\lim_{k\to\infty} \overline{p}_{h_k} = \overline{p} \text{ weakly in } H^2_0(\Omega) \tag{4.10}$$

$$\lim_{k\to\infty} \overline{\mu}_{h_k} = \lim_{k\to\infty} \sum_{j \in I_{h_k}} \overline{\mu}_j \delta_{[x_j]} = \overline{\mu} \quad \text{weakly}^\star \text{ in } M(\Omega) \tag{4.11}$$

Each one of these limit points satisfies the optimality system 3.1-3.4 and 3.5 or 3.6.

Proof. The optimality system 4.3–4.8 is obtained in a similar way as in the proof of theorem 4 by using the abstract result of theorem 3, taking in this case $U = U_h$, $Z = R^{|I_h|}$, $K = U_{adh}$, $C_\delta = \{\xi \in R^{|I_h|} : |\xi_i| \leq \delta, i \in I_h\}$, $J = J_h$ and

$G(u_h) = (y_h(u_h)(x_j))_{j\in I_h}$. The equality 4.3 is obtained dividing $\overline{\mu}_h$, $\overline{\lambda}_h$ and $\overline{p}_h$ by a suitable constant.

Since $\overline{u}_h \in U_{adh}$ $\forall h$ we have that $\{\overline{u}_h\}$ is a bounded sequence in $C^{0,1}(\overline{\Omega})$, then using again the Ascoli's theorem we deduce the existence of a subsequence $\{u_{h_k}\}$ converging uniformly to an element $\overline{u} \in U_{ad}$. Then $\{y_{h_k}\}$ converges to $\overline{y} = y_{\overline{u}}$ in $H_0^2(\Omega)$ (Lemma 4).

On the other hand the equation 4.3 implies that $\{\overline{\lambda}_{h_k}\}$ and $\{\overline{\mu}_{h_k}\}$ are bounded sequences in R and $M(\Omega)$ respectively, where we are identifying $\overline{\mu}_h$ with the element of $M(\Omega)$ defined by

$$\overline{\mu}_h = \sum_{j\in I_h} \overline{\mu}_j \delta_{[x_j]}.$$

So we can extract two subsequences, denoted still in the same way, converging to $\overline{\lambda} \geq 0$ and $\overline{\mu} \in M(\Omega)$ respectively. Now using the equation 4.5 and arguing as in the proof of lemma 3 and remembering that the inclusion $M(\Omega) \subset H_0^2(\Omega)$ is continuous, we deduce that $\overline{p}_{h_k} \to \overline{p}$ weakly in $H_0^2(\Omega)$, where $\overline{p}$ is the state associated to $\overline{\mu}$. Now it is easy to pass from 4.3–4.8 to the system 3.1–3.6, which concludes the proof. □

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