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Optimality criterion and adjoint problem interpretation for nonlinear elastic plates

by

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The paper deals with the maximum stiffness (minimum compliance) design problem for nonlinear solid elastic plates in bending, assuming the plate thickness distribution as design variable. The constitutive law, written in terms of both local variables and characteristics, is given in the most general explicit form, through an expansion in Taylor's series of the local elastic potential function.

The optimization problem is formulated both in general and polar coordinates, and the necessary conditions for stationarity, with their interpretation as governing equations of an adjoint structure, are given as well.

Some mechanical features of such an adjoint structure are outlined, and, in particular, the influence of approximations on the actual constitutive relationship is shown. The particular case of a moment-curvature constitutive law expanded in power series, but truncated up to the second order, has been studied, and the orthotropic nature of the adjoint material pointed out.

1. Introduction

Optimal design problems in linear elastic field have been widely studied in the past, and several approaches to solution procedures have been presented in literature. Classically, the problems are formulated in a variational form, which enables one to define an adjoint problem, particularly simple and useful in the linear case.

The introduction of nonlinearities in the constitutive law makes the definition of the adjoint material more complex, and the solution of the problem via optimality criterion method much more difficult. Even in the case of one–dimensional flexural systems (see [1], [2] and [3]) the adjoint stress-strain relationship shows a deeply different behavior with respect to the actual one.

The two–dimensional case has been studied by Mroz, Kamat and Plaut [2], where again the non–homogeneous and linear nature of adjoint material is shown.

In the present paper, and in the spirit of a former work (see [4]) dealing with the interpretation of adjoint constitutive law for nonlinear hyperelastic continua, some results on plates in bending will be presented.

Starting from a general expression of the stress–strain relationship for the material, that is, assuming the existence of a local elastic potential, in Section 2 some classical results of nonlinear elasticity (see Rainer [6]) are applied in order to express the constitutive law in a general and quasi–explicit form.

In Section 3, the minimum compliance problem for a plate showing nonlinear behavior is formulated in general coordinates and the equations governing the adjoint problem derived. It is also shown how the adjoint constitutive law can be completely defined only through the introduction of some approximations in the model, and how such a law is strictly related to the choice of the adopted approximation. In particular, it is shown that if a quadratic dependence of moments on curvatures is assumed, then the adjoint material is linear, non–homogeneous and orthotropic, with principal axes of orthotropy defined by the principal directions of strain in the actual problem.

Finally, in Section 4, the formulation of the problem, with the before mentioned approximation, is specified in the case of a polar reference system, and the adjoint state equations derived as well.

2. The constitutive law

Dealing with a general theory of nonlinear elastic structures, the choice of a suitable representation of stress–strain relationship (in terms of local variables or of their averages over the cross sections) usually represents the keystone of a significant formulation (see, as an example in the field of optimal design, the choices made in [3]). In fact, a constitutive law of this kind needs to be carefully considered, in order to fulfil both generality purposes and compactness of its analytical representation, with the goal, in other words, of being comprehensive of the particular cases, and keeping in mind the necessity of an easy manipulation in the computations to be performed.

The most natural choice consists in the assumption of existence of an elastic potential (se, e.g., Mroz, Kamat, Plaut [2]) and in expressing the stress–strain relationship through its derivatives with respect to strains.

On the other hand, in optimal structural design problems, for an explicit and correct interpretation of the mechanical behavior of the adjoint problem, a so general expression of the constitutive law enables one to highlight only some features of the problem, leaving in obscurity further difficulties which can arise as a consequence of the approximations obviously needed in order to solve practical problems.

To this purpose, in this Section, making use of classical results in nonlinear elasticity (see Rainer [6]), a completely general expression of the elastic potential will be given, which, although not really explicit, can be easily explicitated when approximations are needed.

Therefore, let the material be assumed hyperelastic, so that an elastic potential $\tilde{\mathcal{U}} = \tilde{\mathcal{U}}(\epsilon)$ exists, which is function of the strains only and such that, in matrix form

$$\sigma = \frac{\partial \tilde{\mathcal{U}}}{\partial \epsilon}. \tag{2.1}$$

Moreover the material is assumed to be nonlinear; in such a case, the most general expression of potential $\tilde{\mathcal{U}}$ can be given through the following expansion in power series

$$\tilde{\mathcal{U}}(\epsilon) = \tilde{\mathcal{A}}_0 + \tilde{\mathcal{B}}_0\epsilon + \tilde{\mathcal{C}}_0\epsilon^2 + \tilde{\mathcal{D}}_0\epsilon^3 + \cdots \tag{2.2}$$

where dots indicate an infinity of similar terms. In Rel. (2.2) the coefficients of strains depend on the material constants only, and $\tilde{\mathcal{A}}_0$ is the value of the scalar function $\tilde{\mathcal{U}}$ at the reference state.

Now, making use of the Cayley–Hamilton theorem of algebra of matrices, and following Rainer (see [5] and [6]), one has

$$\epsilon^3 = J_3 I - J_2 \epsilon + J_1 \epsilon^2 \tag{2.3}$$

and

$$\epsilon^4 = J_1 J_3 I + (J_3 - J_1 J_2)\, \epsilon + \left(J_1^2 - J_2\right) \epsilon^2 \tag{2.4}$$

with similar relations for higher order terms.

In Rel.s (2.3) and (2.4) J_1, J_2 and J_3 denote the three invariants of the strain tensor, while I is the identity matrix.

In such a way, the development (2.2) can be rewritten as

$$\tilde{\mathcal{U}}\,(\epsilon) = \tilde{\mathcal{A}}_\epsilon + \tilde{\mathcal{B}}_\epsilon \epsilon + \tilde{\mathcal{C}}_\epsilon \epsilon^2 \tag{2.5}$$

where $\tilde{\mathcal{A}}_\epsilon$, $\tilde{\mathcal{B}}_\epsilon$ and $\tilde{\mathcal{C}}_\epsilon$ are now functions both of the material constants and of the three invariants J_1, J_2, and J_3.

If the material is homogeneous, as it will be assumed in the following, then strain coefficients do not depend on the point coordinates.

In such a way, by virtue of (2.1) and (2.5), it is possible to write the stress–strain relationship for a homogeneous, hyperelastic and nonlinear material as

$$\sigma = \mathcal{B}_\epsilon + \mathcal{C}_\epsilon \epsilon + \mathcal{D}_\epsilon \epsilon^2 \tag{2.6}$$

where the coefficients

$$\mathcal{B}_\epsilon = \frac{\partial \tilde{\mathcal{A}}_\epsilon}{\partial \epsilon} + \tilde{\mathcal{B}}_\epsilon \tag{2.7}$$

$$\mathcal{C}_\epsilon = \frac{\partial \tilde{\mathcal{B}}_\epsilon}{\partial \epsilon} + 2\tilde{\mathcal{C}}_\epsilon \tag{2.8}$$

$$\mathcal{D}_\epsilon = \frac{\partial \tilde{\mathcal{C}}_\epsilon}{\partial \epsilon} \tag{2.9}$$

have been defined.

Note that, in general, the term $\mathcal{B}_\epsilon$ in (2.6) is different from zero, even if the strains vanish, that is, $\mathcal{B}_\epsilon$ contains a self–equilibrated state of stress.

Leaving now the continuum formulation, in a general reference frame (O, x_1, x_2, x_3) a solid plate is considered, the midplane of which lies in the

(O, x_1, x_2) plane; the material of the plate is assumed to obey to the above shown constitutive law, where the coefficients are conceived as "reduced coefficients", according to the assumption $\sigma_{33} = 0$. For such a structural member, the stress–strain relationship can be rewritten in terms of characteristics, i.e., in terms of bending moments $\mathcal{M}$ and midplane curvatures χ.

Let one define the bending moments

$$\mathcal{M} = \int_H \sigma x_3 \, dx_3, \tag{2.10}$$

the local–generalized strain relationship

$$\epsilon = x_3 \chi \tag{2.11}$$

and the coefficients

$$\mathcal{B}_\chi = \int_H \mathcal{B}_\epsilon x_3 \, dx_3 \tag{2.12}$$

$$\mathcal{C}_\chi = \int_H \mathcal{C}_\epsilon x_3^2 dx_3 \tag{2.13}$$

$$\mathcal{D}_\chi = \int_H \mathcal{D}_\epsilon x_3^3 dx_3. \tag{2.14}$$

Note that in (2.12) to (2.14) H represents the total plate thickness, while in (2.11) the usual Kirchhoff's assumption for thin plates has been made.

Finally, by virtue of Rel.s (2.6) and (2.12) to (2.14), the constitutive law in terms of generalized variables can be expressed as

$$\mathcal{M} = \mathcal{B}_\chi + \mathcal{C}_\chi \chi + \mathcal{D}_\chi \chi^2 \tag{2.15}$$

or, using indices instead of matrices

$$\mathcal{M}^{ij} = \mathcal{B}_\chi^{ij} + \mathcal{C}_\chi^{ijhk} \chi_{hk} + \mathcal{D}_\chi^{ijhkrs} \chi_{hk} \chi_{rs} \tag{2.16}$$

where i, j, h, k, r, s range from 1 to 2.

In such a way, Rel. (2.15) (or Rel. (2.16)) represents the general moment–curvature relationship for homogeneous and hyperelastic plates in bending (and no matter for possible anisotropies). As in the continuum case, the coefficients $\mathcal{B}_\chi$, $\mathcal{C}_\chi$ and $\mathcal{D}_\chi$ are functions of the three invariants $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ of the curvature tensor. The forms of such functions are obviously unknown, because of their dependence on the infinity of terms of order higher than two in development of the elastic potential.

By the way, the expressions can be immediately explicitated if a truncated development in power series is assumed, that is, if only the terms up to the n^{th} power are considered.

An application of such a procedure and its consequences on the constitutive law of the adjoint problem will be explained in the next Sections, with reference to a sample optimization problem.

3. The minimum compliance problem for nonlinear plates

With reference to the nonlinear plate formerly defined, a classical optimization problem will be formulated in this Section, taking into account the moment–curvature relationship in its most general expression. The necessary conditions for the optimal solution of the problem will be derived, and their interpretation as governing equations of an adjoint problem shown as well.

As it is well known, (see for example Mroz, Kamat, Plaut [2]), the adjoint structure shows a constitutive law deeply different from the one of the actual problem, but it seems to be remarkable that the adjoint stiffness coefficients do not guarantee the isotropy of the material, because of their non–explicit representation. If these anisotropies do exist, then their nature can be recognized only through a suitable approximation of the actual constitutive law, without forgetting a possible dependence of anisotropic properties on the approximation itself.

For a preassigned load distribution $p(x_1, x_2)$ acting in x_3 direction and for a given amount of structural volume available, the optimal design problem is formulated as the search for the maximum of the global stiffness of the plate, i.e., for the minimum of the work performed by the external loads (compliance). The function $t(x_1, x_2)$ is assumed as design variable and represents the total plate thickness for solid plates and the thickness of the external reinforcing layers for sandwich plates.

If $w(x_1, x_2)$ is the displacement function of the plate midplane, in x_3 direction, such a problem can be stated as follows. Find

$$inf \int_{\Omega} p\, w\, d\Omega \tag{3.1}$$

where the *infimum* has to be computed with respect to the state variables $\mathcal{M}^{ij}$,

χ_{hk}, w and to the design variable t, and in complying with equilibrium and compatibility conditions and constitutive relationship.

The equilibrium equation and the compatibility conditions read respectively

$$\mathcal{M}^{ij}\big|_{ij} + p\,(x_1, x_2) = 0 \tag{3.2}$$

$$\chi_{hk} + w|_{hk} = 0 \tag{3.3}$$

while the constitutive law is given by Rel. (2.5). The technological requirements can be conceived in the form

$$0 \le \underline{t} \le t\,(x_1, x_2) \le \bar{t} \tag{3.4}$$

and the condition on the total volume is given as

$$\int_\Omega t\, d\Omega = \mathcal{V}. \tag{3.5}$$

In inequalities (3.4), $\underline{t}$ and $\bar{t}$ are prescribed lower and upper bounds, respectively, for the thickness distribution of the plate.

By means of Lagrangian multiplier method (see Save, Prager [7]), the constrained *inf* problem stated above can be rewritten as a unconstrained *inf sup* problem in the following way. Find

$$inf\ sup\ \mathcal{L} \tag{3.6}$$

where the augmented Lagrangian $\mathcal{L}$ reads

$$\begin{aligned}
\mathcal{L} \;=\; & \int_\Omega pw\, d\Omega + \int_\Omega w^* \left(\mathcal{M}^{ij}\big|_{ij} + p\right) d\Omega \\
+ \; & \int_\Omega \mathcal{M}^{*hk} \left(\chi_{hk} + w|_{hk}\right) d\Omega \\
+ \; & \int_\Omega \chi^*_{ij} \left(\mathcal{M}^{ij} - \mathcal{B}^{ij}_\chi - \mathcal{C}^{ijhk}_\chi \chi_{hk} - \mathcal{D}^{ijhkrs}_\chi \chi_{hk}\chi_{rs}\right) d\Omega \\
+ \; & \int_\Omega \mu_1 \left(t - \bar{t}\right) d\Omega + \int_\Omega \mu_2 \left(\underline{t} - t\right) d\Omega + \lambda \left(\int_\Omega t\, d\Omega - \mathcal{V}\right)
\end{aligned} \tag{3.7}$$

obtained appending to the objective function the constraints (3.2) to (3.5) and (2.15) through the Lagrangian multipliers w^*, $\mathcal{M}^{*hk}$, χ^{*hk}, μ_1, μ_2 and λ. Observe that the choice of the symbolic representation for multipliers has been made in view of their interpretation in the stationarity conditions. Moreover, in (3.6) the *infimum* must be computed again with respect to state functions and design variable, while the *supremum* has to be found with respect to the Lagrangian multipliers.

Finally, it must be noticed that while w^*, $\mathcal{M}^{*hk}$, χ^*_{ij} and λ, associated with equality constraints, are not sign–restricted, the multipliers μ_1 and μ_2, associated to inequalities, must be non–negative.

As usual, stationarity of (3.7) with respect to the Lagrangian multipliers return the constraints together with the so–called orthogonality (or switching) conditions

$$\mu_1\left(t-\bar{t}\right)=0 \tag{3.8}$$

$$\mu_2\left(\underline{t}-t\right)=0. \tag{3.9}$$

Rel.s (3.8) and (3.9) state that the multipliers μ_1 and μ_2, in each point of the plate, cannot be simultaneously greather than zero, and can be positive only if the corresponding bound on thickness function is attained.

The optimality condition, i.e., the stationarity of $\mathcal{L}$ with respect to the design variable t, is given by

$$-\chi^*_{ij}\left(\mathcal{B}^{ij}_{\chi,t}+\mathcal{C}^{ijhk}_{\chi,t}\chi_{hk}+\mathcal{D}^{ijhkrs}_{\chi,t}\chi_{hk}\chi_{rs}\right)+\mu_1-\mu_2+\lambda=0. \tag{3.10}$$

Now, the stationarity conditions obtained through a variation of displacement function w and bending moments $\mathcal{M}^{ij}$ read respectively

$$\left.\mathcal{M}^{*hk}\right|_{hk}+p=0 \tag{3.11}$$

$$\chi^*_{ij}+\left.w^*\right|_{ij}=0. \tag{3.12}$$

It clearly appears that if $\mathcal{M}^{*hk}$, χ^*_{ij} and w^* are interpreted as moment, curvature and displacement distribution respectively, then Rel.s (3.11) and (3.12) can be seen as equilibrium equation and compatibility condition of an adjoint thin plate which shows the same load condition as the actual one. It must be outlined that the adjoint load distribution is equal to the actual one only as a consequence of the particular optimization problem considered, but this is not the case in the most general situation.

From this standpoint, the adjoint constitutive law can be obtained as the stationarity condition of Lagrangian $\mathcal{L}$ with respect to actual curvatures χ_{hk}, as

$$\begin{aligned}\mathcal{M}^{*hk}=&\left(\mathcal{B}^{ij}_{\chi,\chi_{hk}}+\mathcal{C}^{ijmn}_{\chi,\chi_{hk}}\chi_{mn}+\mathcal{C}^{ijhk}_{\chi}+\right.\\&\left.+\mathcal{D}^{ijmnrs}_{\chi,\chi_{hk}}\chi_{mn}\chi_{rs}+2\mathcal{D}^{ijhkrs}_{\chi}\chi_{rs}\right)\chi^*_{ij}\end{aligned} \tag{3.13}$$

(where the derivatives of curvature coefficients with respect to curvatures are written in a symbolic way), or, in more compact form, as

$$\mathcal{M}^{*hk} = \mathcal{C}^{*hkij}\chi^*_{ij}. \tag{3.14}$$

Now, by virtue of symmetry of actual stress and strain tensors and because of its hyperelastic nature, the following symmetries for the fourth order tensor $\mathcal{C}_\chi^{ijhk}$ and for the sixth order tensor $\mathcal{D}_\chi^{ijhkrs}$ (see Rovati, Taliercio [4]) hold

$$\mathcal{C}_\chi^{ijhk} = \mathcal{C}_\chi^{jihk} = \mathcal{C}_\chi^{ijkh} = \mathcal{C}_\chi^{hkij} \tag{3.15}$$

$$\begin{aligned}\mathcal{D}_\chi^{ijhkrs} &= \mathcal{D}_\chi^{jihkrs} = \mathcal{D}_\chi^{ijkhrs} = \mathcal{D}_\chi^{ijhksr} = \mathcal{D}_\chi^{hkijrs} \\ &= \mathcal{D}_\chi^{hkrsij} = \mathcal{D}_\chi^{rsijhk} = \mathcal{D}_\chi^{rshkij} = \mathcal{D}_\chi^{ijrshk}.\end{aligned} \tag{3.16}$$

By comparing Rel.s (3.13) and (3.14), symmetries (3.15) and (3.16) enable one to obtain the following equality

$$\mathcal{C}^{*ijhk} = \mathcal{C}^{*hkij} \tag{3.17}$$

that represents the existence condition of an elastic potential from which the adjoint constitutive law derives. Thus the adjoint material is hyperelastic.

Moreover, since the elasticity tensor $\mathcal{C}^{*ijhk}$ (see. Rel. (3.13)) does not depend on the adjoint curvatures χ^*_{hk}, the adjoint material is linear.

On the other hand, $\mathcal{C}^{*ijhk}$ depends on the actual state of strain, i.e., on the curvatures χ_{hk}; in general, the curvature distribution is not uniform over the plate, so that the adjoint material is non–homogeneous.

Now, if Rel. (3.13) is left in its general form, no further properties of the adjoint constitutive law can be shown, and, in particular, it is impossible to understand whether the adjoint material is isotropic or, conversely, it shows general anisotropy or special elastic symmetries. Furthermore, the complex dependence of coefficients $\mathcal{B}_\chi$, $\mathcal{C}_\chi$ and $\mathcal{D}_\chi$ on actual curvatures makes a practical application of the optimality criterion method unacceptable.

For these reasons, a step forward can be made only by introducing some approximations on the actual moment–curvature relationship. A natural and simple approximation can be found by assuming coefficients $\mathcal{B}_\chi$, $\mathcal{C}_\chi$ and $\mathcal{D}_\chi$ in Rel. (2.15) to be independent of curvature χ, that means, in other words, to write bending moments $\mathcal{M}$ as an expansion in power series of curvatures, up to the second order, as

$$\mathcal{M} = \mathcal{B} + \mathcal{C}\chi + \mathcal{D}\chi^2 \tag{3.18}$$

or, in index notation

$$\mathcal{M}^{ij} = \mathcal{B}^{ij} + \mathcal{C}^{ijhk}\chi_{hk} + \mathcal{D}^{ijhkrs}\chi_{hk}\chi_{rs}. \tag{3.19}$$

The constitutive relationship written as in (3.18) or (3.19) represents, of course, a first approximation, but, on the other hand, it preserves the nonlinear nature of the material, and, moreover, can be handled, as it will be seen later, in a much easier way.

Under such an assumption, the adjoint moment–curvature relationship can be rewritten in the simpler form

$$\mathcal{M}^{*hk} = \left(\mathcal{C}^{ijhk} + 2\mathcal{D}^{ijhkrs}\chi_{rs}\right)\chi^*_{ij} \tag{3.20}$$

and therefore the adjoint tensor of stiffness coefficients reads

$$\mathcal{C}^{*ijhk} = \mathcal{C}^{ijhk} + 2\mathcal{D}^{ijhkrs}\chi_{rs}. \tag{3.21}$$

Now, by virtue of the approximation we have made, and if the actual material is isotropic, it can be shown that tensor $\mathcal{C}^{*ijhkrs}$, at the optimum, is orthotropic (see Rovati, Taliercio [4] for the general three–dimensional case), with principal directions of orthotropy directed as the principal directions of curvature in the actual plate.

In order to proof such a property (neglecting in our considerations the isotropic contribution of $\mathcal{C}^{ijhk}$ in (3.21)), consider the following expression for actual curvatures, written in terms of their principal values $\chi_{(R)}$, $(R = 1, 2)$, at the optimum

$$\chi_{RS} = \chi_{(R)}\delta_{RS} \tag{3.22}$$

(no summation over R) where δ_{RS} is the Kronecker delta.

Since it is possible to show (see again Ref. [4]) that the sixth order tensor $\mathcal{D}^{ijhkrs}$ depends, in the isotropic case, on three material constants only (indicated here as α, β and γ), the second term of the right hand side of Rel. (3.21) can be rewriten as

$$\mathcal{D}^{IJHKRS}\chi_{RS} = \sum_{R=1}^{2}\chi_R\mathcal{D}^{IJHKRR} \tag{3.23}$$

or explicitly in the form

$$\begin{aligned}\mathcal{D}^{IJHKRS}\chi_{RS} &= \left[\alpha g^{IJ}g^{HK} + \beta\left(g^{IH}g^{JK} + g^{IK}g^{JH}\right)\right]\mathcal{J}_1\\ &\quad +2\sum_{R=1}^{2}\chi_R\left\{\beta\left(g^{IJ}g^{HR}g^{KR} + g^{HK}g^{IR}g^{JR}\right)\right.\\ &\left.+\gamma\left[g^{JR}\left(g^{IH}g^{KR} + g^{IK}g^{HR}\right) + g^{IR}\left(g^{JH}g^{KR} + g^{JK}g^{HR}\right)\right]\right\}\end{aligned} \tag{3.24}$$

where g^{IJ} are the components of the metric tensor. It can be recognized that, in (3.24), $\mathcal{J}_1$, which is the linear invariant of the actual curvature tensor, multiply an isotropic term, so that such a term does not enter in this discussion. Conversely, it can be easily verified that the second term in (3.23) vanishes if any index is repeated an odd number of times, that is, the term $\mathcal{D}^{IJHKRS}\chi_{RS}$, and then $\mathcal{C}^{*ijhk}$, is orthotropic at the optimum, with planes of elastic symmetry defined by the principal directions of curvature in the actual structural problem.

In such a way, it has been shown how much the adjoint constitutive law differs from the actual one, and how deep is the influence of the approximations that can be done on the most general expression of stress-strain (moment–curvature) relationship. The implications that such differences may have on the numerical solutions of optimal structural design problems via optimality criteria methods are not the subject of the present work, but the great computational effort needed, in comparison with linear elastic problems, is well clear, mainly because of a not simple updating of the constitutive nature of the adjoint structure, step by step, in dependence of the actual strain (curvature) field.

4. Governing equations and stationarity conditions in polar coordinates

In the former Section, the general and approximated formulations of the minimum compliance problem for elastic nonlinear plates in bending has been formulated, in general coordinates. In the present Section, the same problem will be specified assuming a polar system ρ, θ as reference frame, and the necessary conditions for the optimal solution, i.e., the equations governing the behavior of an adjoint structural system, will be derived as well.

If each point of the plate midplane is identified through a radius ρ and an angle θ, and if $\mathcal{M}_\rho$ and $\mathcal{M}_\theta$ indicate the moments acting in radial and its orthogonal directions respectively, being $\mathcal{M}_{\rho\theta}$ the torsional moment, then the equilibrium equation reads

$$\frac{1}{\rho}\left(\mathcal{M}_\rho \rho\right)_{,\rho\rho} - \frac{1}{\rho}\mathcal{M}_{\theta,\rho} + \frac{1}{\rho^2}\mathcal{M}_{\theta,\theta\theta} + \frac{2}{\rho^2}\left(\mathcal{M}_{\rho\theta}\rho\right)_{,\rho\theta} + p = 0 \tag{4.1}$$

Analogously, if $\chi_{\rho\rho}$ and $\chi_{\theta\theta}$ are the midplane curvatures in ρ and θ directions, $\chi_{\rho\theta}$ is the torsional midplane curvature and $w(\rho,\theta)$ is the normal displacement, the compatibility conditions must be written in the form

$$\chi_{\rho\rho} + w_{,\rho\rho} = 0 \tag{4.2}$$

$$\chi_{\theta\theta} + \frac{1}{\rho} w_{,\rho} + \frac{1}{\rho^2} w_{,\theta\theta} = 0 \tag{4.3}$$

$$\chi_{\rho\theta} + 2\left(\frac{1}{\rho} w\right)_{,\rho\theta} = 0 \tag{4.4}$$

Then, it is again assumed that the approximate moment–curvature relationship (3.19) holds

$$\mathcal{M}^{ij} = \mathcal{B}^{ij} + \mathcal{C}^{ijhk}\chi_{hk} + \mathcal{D}^{ijhkrs}\chi_{hk}\chi_{rs} \tag{4.5}$$

where indices i, j, k, r, s range now between ρ and θ.

If the minimum compliance problem for the plate is again dealt with, then objective function and technological constraints are given by Rel.s (3.2) and (3.3), while state equations by Rel.s (4.1) to (4.5). In such a way, the Lagrangian functional takes the form

$$\begin{aligned} \mathcal{L} \;=\; & \int_\Omega p w \,\rho \,d\rho\, d\theta + \int_\Omega w^* \left[\frac{1}{\rho}\left(\mathcal{M}_\rho \rho\right)_{,\rho\rho} - \frac{1}{\rho}\mathcal{M}_{\theta,\rho} \right. \\ + \; & \left. \frac{1}{\rho^2}\mathcal{M}_{\theta,\theta\theta} + \frac{2}{\rho^2}\left(\mathcal{M}_{\rho\theta}\rho\right)_{,\rho\theta} + p\right] \rho \,d\rho\, d\theta \\ + \; & \int_\Omega \left\{\mathcal{M}^*_{\rho\rho}\left(\chi_{\rho\rho} + w_{,\rho\rho}\right) + \mathcal{M}^*_{\theta\theta}\left(\chi_{\theta\theta} + \frac{1}{\rho}w_{,\rho} + \frac{1}{\rho^2}w_{,\theta\theta}\right)\right. \\ + \; & \left. \mathcal{M}^*_{\rho\theta}\left[\chi_{\rho\theta} + 2\left(\frac{1}{\rho}w\right)_{,\rho\theta}\right]\right\} \rho \,d\rho\, d\theta \\ + \; & \int_\Omega \chi^*_{ij}\left(\mathcal{M}^{ij} - \mathcal{B}^{ij} - \mathcal{C}^{ijhk}\chi_{hk} - \mathcal{D}^{ijhkrs}\chi_{hk}\chi_{rs}\right) \rho \,d\rho\, d\theta \\ + \; & \int_\Omega \left[\mu_1\left(t - \bar{t}\right) + \mu_2\left(\underline{t} - t\right)\right] \rho \,d\rho\, d\theta + \lambda\left\{\int_\Omega t\rho \,d\rho\, d\theta - \mathcal{V}\right\}. \end{aligned} \tag{4.6}$$

Also in the present case the symbols for the Lagrangian multipliers have been chosen in a suitable way, that is, to make clearer the interpretation of adjoint variables.

As usual, stationarity of (4.6) with respect to multipliers returns the constraints and the switching conditions, while the stationarity with respect to $\mathcal{M}_{ij}$ $(i, j = \rho, \theta)$ furnishes the adjoint compatibility conditions

$$w^*_{,\rho\rho} + \chi^*_{\rho\rho} = 0 \tag{4.7}$$

$$\frac{1}{\rho} w^*_{,\rho} + \frac{1}{\rho^2} w^*_{,\theta\theta} + \chi^*_{\theta\theta} = 0 \tag{4.8}$$

$$2\left(\frac{1}{\rho} w^*\right)_{,\rho\theta} + \chi^*_{\rho\theta} = 0. \tag{4.9}$$

The adjoint constitutive law is given by a variation of (4.6) with respect to actual curvatures, that reads

$$\mathcal{M}^{*hk} - \mathcal{C}^{ijhk}\chi^*_{ij} - 2\mathcal{D}^{ijhkrs}\chi^*_{ij}\chi_{rs} = 0 \tag{4.10}$$

which shows the same structure and the same mechanical features as (3.20).

Moreover, stationarity with respect to displacement function w gives the adjoint equilibrium equation

$$p\rho + \left(\rho\mathcal{M}^*_\rho\right)_{,\rho\rho} - \mathcal{M}^*_{\theta,\rho} + \frac{1}{\rho}\mathcal{M}^*_{\theta,\theta\theta} + \frac{2}{\rho}\left(\mathcal{M}^*_{\rho\theta}\rho\right)_{,\rho\theta} = 0. \tag{4.11}$$

Finally, the optimality condition, i.e., stationarity of $\mathcal{L}$ with respect to the design variable t is expressed as

$$-\chi^*_{ij}\left(\mathcal{B}^{ij}_{,t} + \mathcal{C}^{ijhk}_{,t}\chi_{hk} + \mathcal{D}^{ijhkrs}_{,t}\chi_{hk}\chi_{rs}\right) + \mu_1 - \mu_2 + \lambda = 0 \tag{4.12}$$

that is in the usual form of mutual energy derived with respect to the design variable. Also in this case it must be observed that, if the bounds on maximum and minimum thickness are not attained, then the optimal solution is characterized by a constant value of such a mutual energy. On the other hand, it must be noticed, once again, that the solution of the optimality condition through the simultaneous solutions of both actual and adjoint structures is in this case much more complicated than in the case of linear elastic structures, because of the particular mechanical behavior of the adjoint problem.

As a further illustrative example, the above formulated problem can be rewritten in a much simpler form, if the analysis is restricted to particular geometries. If material and geometric conditions of axisymmetry are considered, then every derivative operator with respect to the variable θ vanishes as well as torsional moment $M_{\rho\theta}$ and curvature $\chi_{\rho\theta}$. Then, equilibrium and compatibility conditions read respectively

$$\frac{1}{\rho}\left(M_\rho\rho\right)'' - \frac{1}{\rho}M'_\theta + p = 0 \tag{4.13}$$

$$\chi_{\rho\rho} = -w'' \tag{4.14}$$

$$\chi_{\theta\theta} = -\frac{1}{\rho}\,w' \tag{4.15}$$

where a prime denotes derivative with respect to the independent variable ρ.

Then, from Rel.s (4.7), (4.8) and (4.11) the compatibility and equilibrium equations of the adjoint problem read

$$w^{*\prime\prime} + \chi^{*}_{\rho\rho} = 0 \tag{4.16}$$

$$\frac{1}{\rho}\, w^{*\prime} + \chi^{*}_{\theta\theta} = 0 \tag{4.17}$$

$$p\,\rho + \left(\rho M^{*}_{\rho}\right)'' - M^{*\prime}_{\theta} = 0. \tag{4.18}$$

It must be remembered that, in the axisymmetric case, the principal directions of generalized strain and stress (curvature and moment) in each point of the plate are collinear with the radial and circumferential directions. The adjoint material is again orthotropic, with constitutive law given by Rel. (4.10) and principal directions of orthotropy directed along the principal directions of strain.

In this case, from Rel.s (3.21) and (3.24) an explicit form can be provided for the elastic coefficients of the adjoint problem

$$\begin{aligned} C^{*\rho\rho\rho\rho} &= C^{\rho\rho\rho\rho} + 2\left[D^{\rho\rho\rho\rho\rho\rho}\chi_{\rho\rho} + D^{\rho\rho\rho\rho\theta\theta}\chi_{\theta\theta}\right] = \\ &= C^{\rho\rho\rho\rho} + 2\left[(\alpha+2\beta)\left(\chi_{\rho\rho}+\chi_{\theta\theta}\right) + 4\left(\beta+2\gamma\right)\chi_{\rho\rho}\right] \end{aligned} \tag{4.19}$$

$$\begin{aligned} C^{*\theta\theta\theta\theta} &= C^{\theta\theta\theta\theta} + 2\left[D^{\theta\theta\theta\theta\rho\rho}\chi_{\rho\rho} + D^{\theta\theta\theta\theta\theta\theta}\chi_{\theta\theta}\right] = \\ &= C^{\theta\theta\theta\theta} + 2\left[(\alpha+2\beta)\left(\chi_{\rho\rho}+\chi_{\theta\theta}\right) + 4\left(\beta+2\gamma\right)\chi_{\theta\theta}\right] \end{aligned} \tag{4.20}$$

$$\begin{aligned} C^{*\rho\rho\theta\theta} &= C^{\rho\rho\theta\theta} + 2\left[D^{\rho\rho\theta\theta\rho\rho}\chi_{\rho\rho} + D^{\rho\rho\theta\theta\theta\theta}\chi_{\theta\theta}\right] = \\ &= C^{\rho\rho\theta\theta} + 2\left(\alpha+2\beta\right)\left(\chi_{\rho\rho}+\chi_{\theta\theta}\right) = C^{*\theta\theta\rho\rho} \end{aligned} \tag{4.21}$$

Consider now the problem of a circular plate with piecewise constant thickness, i.e., a plate constituted by rings of constant thickness. If M is the total number of rings and R_m $(m = 1, \ldots, M)$ denotes the radius of the boundary between the m–th and the $(m+1)$–th element, the condition on the total volume of Rel. (4.5) has to be modified in the form

$$\pi \sum_{m=1}^{M} \left(R_m^2 - R_{m-1}^2\right) t_m = \mathcal{V} \tag{4.22}$$

The state functions can be suitably referred ring by ring and the continuity conditions at the internal boundaries $\rho = R_m$ are to be considered for displacement and slope functions w and w', for bending moment M_ρ and for shear forces $(\rho M_\rho)' - M_\theta$.

Then the Lagrangian functional takes the form

$$\mathcal{L} \;=\; 2\pi \sum_{m=1}^{M} \left\{ \int_{R_{m-1}}^{R_m} p_{(m)} w_{(m)} \rho\, d\rho \right.$$

$$
\begin{aligned}
+ & \int_{R_{m-1}}^{R_m} w^*_{(m)} \left[\frac{1}{\rho}\left(M_\rho^{(m)}\rho\right)'' - \frac{1}{\rho}\left(M_\theta^{(m)}\right)' + p_m\right]\rho d\rho \\
+ & \int_{R_{m-1}}^{R_m} \left[M_{\rho\rho}^{(m)*}\left(\chi_{\rho\rho}^{(m)} - w''_{(m)}\right) + M_{\theta\theta}^{(m)*}\left(\chi_{\theta\theta}^{(m)} + \frac{1}{\rho}w'_{(m)}\right)\right]\rho d\rho \\
+ & \int_{R_{m-1}}^{R_m} \chi_{ij}^{(m)*}\left(M_{(m)}^{ij} - B_{(m)}^{ij} - C_{(m)}^{ijhk}\chi_{hk}^{(m)}\right. \\
- & \left. D_{(m)}^{ijhkrs}\chi_{hk}^{(m)}\chi_{rs}^{(m)}\right)\rho d\rho\Big\} + \sum_{m=1}^{M}\left[\mu_{1m}\left(t_m - \bar{t}\right)\right. \\
+ & \left.\mu_{2m}\left(\underline{t} - t_m\right)\right] + \lambda\left[\pi\sum_{m=1}^{M}\left(R_m^2 - R_{m-1}^2\right)t_m - V\right] \qquad (4.23)
\end{aligned}
$$

The stationarity conditions of $\mathcal{L}$ provide the state equations for the adjoint problem, for which the relevant continuity conditions at the internal boundaries R_m are found as natural boundary conditions.

Finally, the optimality criterion reads

$$
\begin{aligned}
-2\pi\int_{R_{m-1}}^{R_m} \chi_{ij}^{(m)*}\left(B_{,t_m}^{ij} + C_{,t_m}^{ijhk}\chi_{hk}^{(m)} + D_{,t_m}^{ijhkrs}\chi_{hk}^{(m)}\chi_{rs}^{(m)}\right)\rho d\rho \\
+\mu_{1m} - \mu_{2m} + \lambda\pi\left(R_m^2 - R_{m-1}^2\right) = 0 \qquad (4.24)
\end{aligned}
$$

where the mutual energy derived with respect to the design variable is integrated over every ring of constant thickness.

5. Conclusions

As in linear elastic case, the variational formulation of an optimal design problem in nonlinear field leads to the definition of an adjoint problem. However, the constitutive law of this adjoint problem has been shown to be linear, even if the actual is nonlinear, and non–homogeneous. In particular, a strict dependence of the adjoint stiffness coefficients on the actual strain (curvature) field has been pointed out. All these features, which are well clear in the general formulation of stress-strain relationship here developed, can be enriched of more informations when the generality is left.

An example of the influence of approximations on the adjoint constitutive law has been carried out in this paper, where a quadratic dependence of stresses (bending moments) on strains (curvatures) has been assumed, leading to an adjoint constitutive law which shows linearity, non–homogeneity and, as a consequence of the approximation, also orthotropy.

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References

[1] CINQUINI C. Optimality Criteria for Materials with Nonlinear Behaviour: Application to Beams in Bending, *Eng. Struct.*, **6**, (1984) 61–64.

[2] MRÓZ Z., KAMAT M. P., PLAUT R. H. Sensitivity Analysis and Optimal Design of Nonlinear Beams and Plates, *J. Struct. Mech.*, **13**, (1985) 3 & 4, 245–266.

[3] ROVATI M., CINQUINI C. Closed–Form Solutions in Optimal Design of Structures with Nonlinear Behavior, *Mech. Struct. & Mach.*, **16**, (1988–89), 4, 407–422.

[4] ROVATI M., TALIERCIO A. On the Adjoint Constitutive Law in Nonlinear Elastic Optimal Design, *Meccanica*, **22**, (1987), 3, 150–156.

[5] RAINER M. A Mathematical Theory of Dilatancy, *Am. J. Math.*, **67**, (1945), 350–362.

[6] RAINER M. Elasticity Beyond the Elastic Limit, *Am. J. Math.*, **70**, (1948), 433–446.

[7] SAVE M., PRAGER W. Structural Optimization, **1**: Optimality Criteria, Plenum Press, New York, 1985.