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Some spectral solutions for a class of LQR problems

by

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A sufficient, and in some cases necessary, condition is given for finding the eigenvalues and eigenvectors of a class Infinite Horizon Linear Quadratic Regulator problems in a Hilbert space. The uncontrolled systems are assumed to be conservative with their controls introduced by linear operators which in the most interesting cases are unbounded. (MR and Zbl Classification 93D15, 93C20, 35L).

Introduction

In this paper we apply complex analysis techniques to obtain solutions of linear quadratic regular (LQR) problems in Hilbert space $\mathcal{H}$. We assume the uncontrolled systems generate uniformly bounded C_0–semigroups on $\mathcal{H}$, the controls are introduced through linear operators, which may be unbounded, and that the cost functionals are positive semidefinite in the space variables and positive definite in the control variables. A well–known class of systems which conforms to this description is linear elastic systems governed by hyperbolic partial differential equations with controls on their boundaries. (See [6] for a comprehensive survey of such systems).

It is well–known (see e.g., [6]) when an LQR problem is optimizable, its optimal controls are feedback controls which generate a linear system. In this paper we make no assumption on the optimizability of the system. Instead we seek solutions which are of the form $x_0 e^{\lambda_0 t}$, $\operatorname{Re}\lambda_0 < 0$, $x_0 \in \mathcal{H}$, which are optimal. We call these solutions spectral solutions. If x_0 is nontrivial, then the corresponding λ_0 is called an eigenvalue of the problem.

The main contribution of this paper is a sufficient condition for the existence of nontrivial spectral solutions which presents a constructive method for obtaining them. This condition is given in Theorem 2.10. Bearing in mind that such a condition may be vacuous, we give, in Theorem 2.12, criteria for Theorem 2.10 to also be necessary condition.

Section 3 presents a method for computing certain optimal solutions of LQR problems in terms of contour integration in the left half plane $\operatorname{Re}\lambda < 0$. This method is the analogue of spectral approximation by eigenvectors of the solutions to linear operator equations in a Hilbert space.

In Section 4 we restrict the problem to the special case of abstract linear elastic systems. That is systems representable in the form

$$\ddot{x}(t) + A^2(x(t)) = Bu(t), \tag{0.1}$$

where A represents a symmetric positive semidefinite unbounded operator on $\mathcal{H}$ and B is a possibly unbounded linear mapping from a Hilbert space H into $\mathcal{H}$. For such systems the sufficient condition of Theorem 2.10 reduces to an eigenvalue problem (Equations (4.13) and (4.14)).

Three special cases of the LQR problem are considered. One observation based on these cases is that the velocity term, $\dot{x}(t)$, appears to be the most important component of the cost functional so far as stabilization is concerned.

In Example 4.2 we specialize the system to one where the control is one–dimensional. For these systems Theorem 2.10 reduces to finding the roots of a symmetric meromorphic function which lie in $\operatorname{Re}\lambda < 0$. Theorem 4.4 presents a criterion for Theorem 2.10 to also be a necessary condition. This criterion consists of the ability to construct a certain entire function, q, of order one (see e.g. [1]) which cancels the poles of the operator $(\lambda^2 I + A^2)^{-1}$ and is such that $(\lambda^2 I + A^2)^{-1} q(\lambda)$ is a finite Laplace transform (F.L.T.) (see e.g., [3]).

Examples 4.5 and 4.6 are explicit examples of systems which satisfy Theorem 4.4 and for which Theorem 2.10 is necessary and sufficient. Example 4.5 is the one–dimensional wave equation with Neumann boundary conditions on one end. Example 4.6 is a two–dimensional beam which can have a finite number of pointwise controllers at interior points and is neither approximately controllable nor weakly stabilizable. Nevertheless the spectrum of an associated LQR problem is nonempty. There is one appendix.

1. Preliminaries

1. $\mathbb{R}$ will denote real numbers, $\mathbb{R}^+$ the positive real numbers and $\mathcal{C}$ the complex numbers.

2. $\mathcal{H}$ is a complex Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. H is a complex Hilbert space with the inner product $((\cdot,\cdot))$ and norm $\|\cdot\|_H$. V and V' (the dual of V) are Banach spaces such that $V \subset \mathcal{H} \subset V'$ and V and $\mathcal{H}$ are dense in their successors under continuous injection. The norm on V is denoted by $|\cdot|_V$ and that on V' by $|\cdot|$. If $v' \in V'$ we denote the functional on V generated by v' by the notation $\langle\cdot, v'\rangle$, and observe that if $v' \in \mathcal{H}$

$$\langle v, v'\rangle = (v, v') \tag{1.1}$$

for all $v \in V$.

3. Let X and Y be complex Banach spaces. We denote the Banach space of continuous linear mappings from X into Y by $[X, Y]$ and if $X = Y$ by $[X]$. The identity mapping on all Banach spaces will be denoted by I and the zero mapping by $\mathcal{O}$. The adjoint of any Banach space X, except V, will be denoted by X^*. If $Q \in [X, Y]$ the adjoint in $[Y^*, X^*]$ will be denoted by Q^*.

4. $L_2(\mathbb{R}^+, X)$ will denote the space of all Bochner square–integrable mappings from $\mathbb{R}^+$ into X.

5. $S(t)$, $t \in \mathbb{R}^+$, will denote a c_0–semigroup of operators on V' with infinitesimal generator α. It is assumed $S(t)$ is uniformly bounded on V', that $S(t) : \mathcal{H} \to \mathcal{H}$ for all $t \in \mathbb{R}^+$ and is also uniformly bounded there, that the domain of α, considered as the generator of $S(t)$ on $\mathcal{H}$, $\mathcal{D}(\alpha)$, is in V, and that α^{-1} is a bounded linear operator on $\mathcal{H}$ and on V'. The adjoint of α in $\mathcal{H}$ is denoted by α^*.

6. $\beta \in [H, V']$ is such that $\alpha^{-1}\beta \in [H, V]$.

7. $W \in [\mathcal{H}]$ is symmetric positive semidefinite and $W : V \to V$.

8. If $q \in L_2(\mathbb{R}^+, X)$, we denote its Laplace transform by $\hat{q}$, i.e.

$$\hat{q}(s) = \int_0^\infty e^{-st} q(t)\,dt, \tag{1.2}$$

where s is such that (1.2) converges absolutely in the norm of X.

2. Statement and discussion of the problem

Let $u \in L_2(\mathbb{R}^+, H)$ and $x_0 \in \mathcal{H}$. We consider mild solutions of the differential equation

$$\dot{x}(t, x_0, u) = \alpha x(t, x_0, u) + \beta u(t), \quad t \geq 0, \tag{2.1}$$

in $\mathcal{H}$.

DEFINITION 2.1 If $u \in L_2(\mathbb{R}^+, H)$, $x_0 \in \mathcal{H}$ and the formal Laplace transform of (2.1),

$$\hat{x}(s) = (sI - \alpha)^{-1}[x_0 + \beta \hat{u}(s)], \tag{2.2}$$

is the Laplace transform of a mapping $x : \mathbb{R}^+ \to \mathcal{H}$, then $x(t)$ is called a mild solution of (2.1). An equivalent statement is that

$$x(t) = S(t)x_0 + \int_0^t S(t - \sigma)\beta u(\sigma) d\sigma \tag{2.3}$$

is a continuous mapping from $\mathbb{R}^+$ into $\mathcal{H}$ which has a Laplace transform.

REMARK 2.2 Since $\beta \in [H, V']$ the term under the integral sign in (2.3) is for each pair t, σ, $0 \leq \sigma \leq t$, in V', but not necessarily in $\mathcal{H}$. Hence (2.1) is not quaranteed á priori to possess a solution for all $u \in L_2(\mathbb{R}^+, H)$. However, for $\operatorname{Re} s > 0$, $(sI - \alpha)^{-1}\beta \in [H, V]$, and thus $\int_0^t S(t - \sigma)\beta u(\sigma) d\sigma \in \mathcal{H}$ for all $t \geq 0$ since $u \in L_2(\mathbb{R}^+, H)$.

We now consider the optimal control problem. Given $x_0 \in \mathcal{H}$, minimize the functional

$$C(u, x_0) = \int_0^\infty [(Wx(t, x_0, u), x(t, x_0, u)) + ((u(t), u(t)))] dt \tag{2.4}$$

over $L_2(\mathbb{R}^+, H)$ subject to the constraint that $x(t, x_0, u)$ be a mild solution of (2.1). If for some $u \in L_2(\mathbb{R}^+, H)$ $\quad C(u, x_0) < \infty$, we define

$$m(x_0) = \inf_{m \in L_2(\mathbb{R}^+, H)} C(u, x_0). \tag{2.5}$$

DEFINITION 2.3 We define the subset $\mathcal{U} \subset L_2(\mathbb{R}^+, H)$ as the set of $h \in L_2(\mathbb{R}^+, H)$ such that mappings

$$y(t, h) = \int_0^t S(t - \sigma)\beta h(\sigma) d\sigma \tag{2.6}$$

satisfy the condition

$$\int_0^\infty (Wy(t,h), y(t,h))dt < \infty. \tag{2.7}$$

PROPERTY 2.4 *$\mathcal{U}$ is a subspace of $L_2(\mathbb{R}^+, H)$.*

PROOF. The proof is a consequence of fact that the positive square root of (2.7) defines a seminorm on $\mathcal{U}$ and that $y(t, \cdot)$ is linear in h.

PROPERTY 2.5 *Let $C(u, x_0) < \infty$. Then a sufficient condition that there exists $u_0 \in L_2(\mathbb{R}^+, H)$ such that*

$$m(x_0) = C(u_0, x_0) \tag{2.8}$$

is that

$$C(u_0 + h, x_0) - C(u_0, x_0) \geq 0 \tag{2.9}$$

for all $h \in \mathcal{U}$

PROOF Assume (2.9) is satisfied. Let

$$C(u, x_0) < \infty \tag{2.10}$$

and let

$$h = u - u_0 \tag{2.11}$$

Since

$$x(t, x_0, u) = x(t, x_0, u_0) + y(t, h) \tag{2.12}$$

and (2.10) holds, it follows from the quadratic nature of (2.4) that $y(t, h)$ satisfies (2.7) and hence $h \in \mathcal{U}$. But this implies that

$$C(u, x_0) - C(u_0, x_0) \geq 0 \tag{2.13}$$

Since u is arbitrary this proves that $C(u_0, x_0) = m(x_0)$.

PROPERTY 2.6 *If $C(u_0, x_0) = m(x_0)$, then u_0 is unique.*

PROOF. The proof is a consequence of the convexity of $(Wx(t, x_0, u), x(t, x_0, u))$ as a function of u and the strict convexity of the norm squared in H.

THEOREM 2.7 *A sufficient condition that $C(u_0, x_0) = m(x_0)$, is that*

$$\int_0^\infty [(Wx(t, x_0, u_0), y(t, h) + ((u(t), h(t)))]dt = 0 \tag{2.14}$$

for all $h \in \mathcal{U}$.

PROOF. By direct computation

$$\begin{aligned} C(u_0 + h, x_0) - C(u_0, x_0) = \\ = \int_0^\infty [(Wy(t, h), y(t, h)) + ((h(t), h(t)))]dt \\ + 2\operatorname{Re}\int_0^\infty [(Wx(t, x_0, u_0), y(t, h)) + ((u_0(t), h(t)))]dt. \end{aligned} \tag{2.15}$$

Hence if (2.14) is satisfied

$$C(u_0 + h, x_0) - C(u_0, x_0) \geq 0$$

for all $h \in \mathcal{U}$ and by Property 2.5 this implies $C(u_0, x_0) = m(x_0)$.

DEFINITION 2.8 A point $x_0 \in \mathcal{H}$ for which there exists $u_0 \in L_2(\mathbb{R}^+, H)$ and $\lambda_0 \in \mathcal{C}$ such that

$$0 < C(u_0, x_0) = m(x_0) < \infty \tag{2.16}$$

and

$$x(t, x_0, u_0) = x_0 e^{\lambda_0 t} \tag{2.17}$$

is called a *spectral point* of the optimization problem (2.4). The point λ_0 is called an *eigenvalue* of the problem (2.4).

PROPERTY 2.9 *If $x_0 \neq \mathcal{O}$ is a spectral point of the problem (2.4), $(\lambda_0 I - \alpha)^{-1}$ exists, $\|WS(\cdot)x_0\| \notin L_2(\mathbb{R}^+, \mathcal{H})$ and $(Wx_0, x_0) > 0$, then the eigenvalue λ_0 satisfies $\operatorname{Re}\lambda_0 < 0$, and there exists $q_0 \in H$ such that*

$$x_0 = (\lambda_0 I - \alpha)^{-1}\beta q_0 \tag{2.18}$$

and

$$u_0(t) = e^{\lambda_0 t} q_0. \tag{2.19}$$

PROOF. The Laplace transform of (2.17) must satisfy (2.2). That is

$$\frac{x_0}{\lambda - \lambda_0} = (\lambda I - \alpha)^{-1}[x_0 + \beta \hat{u}_0(\lambda)]. \tag{2.20}$$

Thus

$$\frac{x_0}{\lambda-\lambda_0}-(\lambda I-\alpha)^{-1}x_0 = \frac{(\lambda_0 I-\alpha)}{\lambda-\lambda_0}(\lambda I-\alpha)^{-1}x_0=(\lambda I-\alpha)^{-1}\beta\hat{u}_0(\lambda), \tag{2.21}$$

and hence

$$\frac{x_0}{\lambda-\lambda_0}=(\lambda_0 I-\alpha)^{-1}\beta\hat{u}_0(\lambda). \tag{2.22}$$

Equation (2.22) implies that

$$\beta\hat{u}_0(\lambda)=\frac{(\lambda_0 I-\alpha)}{\lambda-\lambda_0}x_0. \tag{2.23}$$

Consequently

$$\beta u_0(t)=(\lambda_0 I-\alpha)x_0e^{\lambda_0 t} \tag{2.24}$$

and $u_0(t)$ must be of the form

$$u_0(t)=q_0e^{\lambda_0 t}+h_0(t);\quad x_0=(\lambda_0 I-\alpha)^{-1}\beta q_0, \tag{2.25}$$

where $h_0\in\mathcal{U}$ is such that

$$y(t,h_0)\equiv\mathcal{O} \tag{2.26}$$

because $\beta h_0(t)=\mathcal{O}$ a.e. on $\mathbb{R}^+$. Moreover, since $(Wx_0,x_0)>0$, it follows that

$$0<\int_0^\infty(Wx_0e^{\lambda_0 t},x_0e^{\lambda_0 t})dt=\frac{-1}{2\operatorname{Re}\lambda_0}(Wx_0,x_0).$$

Thus $\operatorname{Re}\lambda_0<0$ and

$$C(q_0e^{\lambda_0 t},x_0)<\infty \tag{2.27}$$

But then, if $h_0(t)\not\equiv 0$ a.e. on $\mathbb{R}^+$,

$$\mu_0(t)-h_0(t)=e^{\lambda_0 t}q_0\in\mathcal{U},$$

since $h_0\in\mathcal{U}$.

However this is impossible because

$$\int_0^t S(t-\sigma)\beta q_0e^{\lambda_0\sigma}d\sigma=x_0e^{\lambda_0 t}-S(t)x_0 \tag{2.28}$$

and $\|S(\cdot)x_0\|\notin L_2(\mathbb{R}^+,\mathcal{H})$. Thus $h_0(t)=0$ a.e. on $\mathbb{R}^+$,

$$x_0(t) = (\lambda_0 I - \alpha)^{-1}\beta q_0 e^{\lambda_0 t} \tag{2.29}$$

and

$$u_0(t) = q_0 e^{\lambda_0 t}. \tag{2.30}$$

THEOREM 2.10 *If* $x_0 \neq \mathcal{O}$ *in* $\mathcal{H}$ *satisfies*

$$x_0 = (\lambda I - \alpha)^{-1}\beta\beta^*(\lambda I + \alpha^*)^{-1}W x_0 \tag{2.31}$$

for some $\lambda \in \mathcal{C}$ *with* $\operatorname{Re}\lambda < 0$, *then the pair*

$$u_0(t) = e^{\lambda t}\beta^*(\lambda I + \alpha^*)^{-1}W x_0 \tag{2.32}$$

and

$$x(t, t_0, u_0) = x_0 e^{\lambda t} \tag{2.33}$$

are optimal for the problem (2.4).

PROOF. (i) Observe that if x_0 satisfies (2.31) and u_0 and $x(t, t_0, u_0)$ satisfy (2.32) and (2.33), then $(\lambda I - \alpha)x_0 \in V'$,

$$\hat{x}(s) = \frac{x_0}{s - \lambda} \tag{2.34}$$

and

$$\begin{aligned} (sI - \alpha)^{-1}[x_0 + \beta\hat{u}_0(s)] &= (sI - \alpha)^{-1}\left[I + \frac{\beta\beta^*(\lambda I + \alpha^*)^{-1}W}{s - \lambda}\right]x_0 \\ &= (sI - \alpha)^{-1}\left[x_0 + \frac{(\lambda I - \alpha)}{s - \lambda}x_0\right] \\ &= \frac{x_0}{s - \lambda} \end{aligned} \tag{2.35}$$

Thus $x_0 e^{\lambda t}$ and u_0 form a mild solution of (2.1) and clearly $C(u_0, x_0) < \infty$.

(ii) Let $h \in \mathcal{U}$ then

$$\begin{aligned} &\int_0^\infty [(Wx(t, x_0, u_0), y(t, h)) + ((u_0(t), h(t)))]dt = \\ &= \int_0^\infty (Wx_0, e^{\bar{\lambda}t}\int_0^t S(t - \sigma)\beta h(\sigma)d\sigma)dt \\ &\quad + \int_0^\infty ((\beta^*(\lambda I + \alpha^\alpha)^{-1}Wx_0, e^{\bar{\lambda}t}h(t)))dt \\ &= -(Wx_0, (\bar{\lambda}I + \alpha)^{-1}\beta\hat{h}(-\lambda)) + (\beta^*(\lambda I + \alpha^*)^{-1}Wx_0, \hat{h}(-\bar{\lambda})) \\ &= 0 \end{aligned} \tag{2.36}$$

Thus by Theorem 2.7 the pair (2.32) and (2.33) are optimal for the problem (2.4).

REMARK 2.11 If x_0 satisfies (2.31) then it is a spectral point. The question naturally arises. Do all nontrivial spectral points satisfy an equation of the form (2.31)? In Section 4 we shall give an example of a class of systems all of whose spectral points are determined by (2.31). In the general case it is conceivable that there are only trivial solutions of (2.31). However, there is a condition which states that (2.31) is necessary and sufficient for x_0 to be a spectral point of the problem (2.4). This is given by the following theorem.

THEOREM 2.12 *If for all* $\lambda \in C$ *with* $\mathrm{Re}\,\lambda > 0$ *and any* $q_0 \in H$ *there exists* $h \in \mathcal{U}$ *such that*

$$h(\lambda) = q_0, \tag{2.37}$$

then Equation (2.31) determines the spectral points of the problem (2.4) for those points which satisfy the conditions of Property 2.9.

PROOF. Let the hypotheses of Property 2.9 be satisfied in addition to condition (2.37). Let x_0 be a spectral point for the Problem 2.4 with eigenvalue λ_0. The corresponding optimal control is then $u_0(t) = e^{\lambda_0 t} q_0$, where $q_0 \in H$ satisfies Equation (2.18). We choose $h_0 \in \mathcal{U}$ such that

$$h_0(-\bar{\lambda}_0) = -\beta^*(\lambda_0 I + \alpha^*)^{-1} W x_0 + q_0. \tag{2.38}$$

Then since $C(u_0 + h) - C(u_0) \geq 0$ for all $h \in \mathcal{U}$ it follows that

$$\mathrm{Re}\left[\int_0^\infty [(We^{\lambda_0 t} x_0, y(t, h_0)) + ((e^{\lambda_0 t} q_0, h_0(t)))]dt\right] = 0 \tag{2.39}$$

Using (2.38) we can express (2.39) in the form

$$\| - \beta^*(\lambda_0 I + \alpha^*)^{-1} W x_0 + q_0\|_H^2 = 0. \tag{2.40}$$

Hence

$$q_0 = \beta^*(\lambda_0 I + \alpha^*)^{-1} W x_0. \tag{2.41}$$

Substituting (2.41) into (2.18) shows that x_0 satisfies (2.31).

3. A class of optimal solutions

In this section we construct a class of optimal solutions for the functional (2.4) using contour integration techniques.

Definition 3.1 A mild solution of (2.1), $x(t, x_0, u_0(t))$, and its controller, $u_0(t)$, are called an *optimal pair* if

$$C(u_0, x_0) = m(x_0) < \infty \tag{3.1}$$

The mapping u_0 is termed *the optimal control.*

Let

$$S_1(\lambda) = [(\lambda I - \alpha) - \beta\beta^*(\lambda I + \alpha^*)^{-1}W]^{-1} \tag{3.2}$$

wherever the inverse exists. Since

$$S_1(\lambda) = (\lambda I - \alpha)^{-1}[I - \beta\beta^*(\lambda I + \alpha^*)^{-1}W(\lambda I - \alpha)^{-1}]^{-1}, \tag{3.3}$$

it is clear that $S_1(\lambda) \in [\mathcal{H}]$.

Property 3.2 *Let $x_0 \in \mathcal{H}$ and Γ be a Jordan curve in $\mathrm{Re}\,\lambda < 0$ such that $S_1(\lambda)$ exists on Γ. Define*

$$x_\Gamma = \frac{1}{2\pi i}\int_\Gamma S_1(\lambda)x_0 d\lambda. \tag{3.4}$$

Then the pair

$$u_\Gamma(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\beta^*(\lambda I + \alpha^*)^{-1}W S_1(\lambda)x_0 d\lambda \tag{3.5}$$

and

$$x_\Gamma(t, x_\Gamma, u_\Gamma) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}S_1(\lambda)x_0 d\lambda. \tag{3.6}$$

are an optimal pair for the functional (2.4)

Proof. (i) We first show that for $\mathrm{Re}\,s > 0$ the Laplace transform of (3.5) and (3.6) satisfy (2.2).

Observe that for $\mathrm{Re}\,s > 0$

$$\begin{aligned}(sI - \alpha)^{-1}[x_\Gamma + \beta\hat{u}_\Gamma(s)] &= \\ \frac{1}{2\pi i}\int_\Gamma (sI - \alpha)^{-1}\left[I + \frac{\beta\beta^*(\lambda I + \alpha^*)^{-1}}{s - \lambda}\right] S_1(\lambda)x_0 d\lambda\end{aligned} \tag{3.7}$$

Using (3.3)

$$\begin{aligned}&(sI - \alpha)^{-1}\left[I - \frac{(\lambda I - \alpha)}{s - \lambda} + \frac{(\lambda I - \alpha) + \beta\beta^*(\lambda I + \alpha^\lambda)^{-1}}{s - \lambda}\right] S_1(\lambda)x_0 = \\ &= (sI - \alpha)^{-1}\left[I - \frac{(\lambda I - \alpha)}{s - \lambda}S_1(\lambda) - \frac{I}{s - \lambda}\right] x_0 \\ &= \left[\frac{S_1(\lambda)}{s - \lambda} - \frac{(sI - \alpha)^{-1}}{s - \lambda}\right] x_0.\end{aligned} \tag{3.8}$$

Since $\frac{(sI-\alpha)^{-1}}{s-\lambda}$ is analytic on Γ and its interior, (3.7), with its integrand represented by (3.8), satisfies

$$(sI-\alpha)^{-1}[x_\Gamma+\beta\hat{u}_\Gamma(s)] = \frac{1}{2\pi i}\int_\Gamma \frac{S_1(\lambda)}{s-\lambda}x_0 d\lambda = \hat{x}_\Gamma(s, x_\Gamma, u_\Gamma), \tag{3.9}$$

which proves that (3.5) and (3.6) satisfy (2.2).

(ii) To show that (3.5) and (3.6) are an optimal pair we observe that for any $h \in \mathcal{U}$ we can use Tonelli's Theorem (see e.g., [4]) to write

$$\begin{aligned}
&\int_0^\infty (W(xt, x_\Gamma, u_\Gamma), y(t,h))dt = \\
&= \frac{1}{2\pi i}\int_\Gamma\int_0^\infty (WS_1(\lambda)x_0, \int_0^t e^{\bar{\lambda}t}S(t-\sigma)\beta h(\sigma)d\sigma)dt d\lambda \\
&= -\frac{1}{2\pi i}\int_\Gamma (WS_1(\lambda)x_0, (\bar{\lambda}I+\alpha)^{-1}\beta\hat{h}(-\bar{\lambda}))d\lambda.
\end{aligned} \tag{3.10}$$

On the other hand, again using Tonelli's Theorem, we can write

$$\begin{aligned}
&\int_0^\infty ((u_\Gamma(t), h(t)))dt = \\
&= \frac{1}{2\pi i}\int_\Gamma\int_0^\infty ((\beta^*(\lambda I+\alpha^*)^{-1}WS_1(\lambda)x_0, e^{\bar{\lambda}t}h(t)))dt d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma ((\beta^*(\lambda I+\alpha^*)^{-1}WS_1(\lambda)x_0, \hat{h}(-\bar{\lambda})))d\lambda
\end{aligned} \tag{3.11}$$

Thus, since (3.10) is the negative of (3.11), Equation (2.14) in Theorem 2.7 is satisfied by the pair (3.5) and (3.6) and hence they are an optimal pair. The following theorem is now an obvious consequence of Property 3.2.

THEOREM 3.3 *If Γ_j, $1 \le j \le n$ are a finite sequence of Jordan contours in* $\operatorname{Re}\lambda < 0$ *such that $S_1(\lambda)$ exists on each Γ_j, then the functions*

$$x(t) = \sum_{j=1}^{n} \frac{1}{2\pi i}\int_{\Gamma_j} e^{\lambda t}S_1(\lambda)x_j d\lambda \tag{3.12}$$

and

$$u(t) = \sum_{j=1}^{n} \frac{1}{2\pi i}\int_{\Gamma_j} e^{\lambda t}\beta^*(\lambda I+\alpha^*)^{-1}WS_1(\lambda)x_j d\lambda \tag{3.13}$$

form an optimal pair for the functional (2.4)

4. Conservative elastic systems

Let A be a strictly positive self–adjoint operator with a dense domain, $\mathcal{D}(A)$, and a compact resolvent, $(\lambda I - A)^{-1}$, on a Hilbert space $\mathcal{H}_0$ whose inner product is denoted by $(\cdot,\cdot)$. We consider the following Banach spaces $\mathcal{H}_1 = \mathcal{D}(A)$ with the inner product

$$(x,y)_{\mathcal{H}_1} = (Ax, Ay) \quad \text{and norm } \|x\|_{\mathcal{H}_1} \tag{4.1}$$

$$V_0 \subset \mathcal{H}_1 \subset V_0', \tag{4.2}$$

where V_0 and $\mathcal{H}_0$ are dense in V_0' under continuous injection.

We define the Cartesian products

$$\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_0 \tag{4.3}$$

$$V = \mathcal{H}_1 \times V_0 \tag{4.4}$$

$$V' = \mathcal{H}_1 \times V_0' \tag{4.5}$$

The norms on $\mathcal{H}$, V and V' are the usual norms induced by the product topology.

We consider a Hilbert space H with the inner product $((\cdot,\cdot))$ and a mapping $B \in [H, V_0']$ such that

$$A^{-2}B \in [H, \mathcal{H}_1] \tag{4.6}$$

On $\mathcal{H}$ we consider the differential equation

$$\dot{x} = y, \quad x(0) = x_0 \in \mathcal{H}_1 \tag{4.7a}$$

$$\dot{y} = -A^2 x + Bu, \quad y(0) = y_0 \in \mathcal{H}_0 \tag{4.7b}$$

and the functional

$$C(u, x_0, y_0) = \int_0^\infty [\nu_1 \|x(t)\|^2_{\mathcal{H}_1} + \nu_2 \|y(t)\|^2 + ((u(t), u(t)))]dt \tag{4.8}$$

where γ_j, $j = 1,2$, are nonnegative constants.

The system (4.7) and (4.8) conforms to the model (2.1), (2.4) given in Section 2. In this case α, α^{-1} and β are the following operators defined on the Cartesian product space

$$\alpha = \begin{pmatrix} \mathcal{O} & I \\ -A^2 & \mathcal{O} \end{pmatrix} \tag{4.9}$$

$$\alpha^{-1} = \begin{pmatrix} \mathcal{O} & -A^{-2} \\ I & \mathcal{O} \end{pmatrix} \tag{4.10}$$

$$\beta = \begin{pmatrix} \mathcal{O} \\ B \end{pmatrix}. \tag{4.11}$$

Notice that

$$\alpha^{-1}\beta = \begin{pmatrix} -A^{-2}B \\ \mathcal{O} \end{pmatrix} \quad \text{in } [H, V].$$

In this case

$$W = \begin{pmatrix} \nu_1 I & \mathcal{O} \\ \mathcal{O} & \nu_2 I \end{pmatrix}. \tag{4.12}$$

For the above system Equation (2.30) reduces to the equation

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -\tau(\lambda) & \lambda\tau(\lambda) \\ -\lambda\tau(\lambda) & \lambda^2\tau(\lambda) \end{pmatrix} \begin{pmatrix} \nu_1 I & \mathcal{O} \\ \mathcal{O} & \nu_2 I \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{4.13}$$

where

$$\tau(\lambda) = (\lambda^2 I + A^2)^{-1} BB^* (\lambda^2 I + A^2)^{-1}. \tag{4.14}$$

Notice that from (4.14) we can obtain the equalities

$$\tau(-\lambda) = \tau(\lambda) \text{ and} \tag{4.15}$$

$$\tau^*(\lambda) = \tau(\bar{\lambda}). \tag{4.16}$$

Some special cases

I: $\nu_1 = 1$, $\nu_2 = 0$. The Equation (2.30) becomes

$$x_0 = -\tau(\lambda)x_0, \quad y_0 = -\lambda\tau(\lambda)x_0. \tag{4.17}$$

Thus

$$x_0 = -(\lambda^2 I + A^2)^{-1} BB^* (\lambda^2 I + A^2)^{-1} x_0$$

II: $\nu_1 = 0$, $\nu_2 = 1$. The Equation (2.30) satisfies

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \mathcal{O} & \lambda\tau(\lambda) \\ \mathcal{O} & \lambda^2\tau(\lambda) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

or

$$x_0 = \lambda\tau(\lambda)y_0, \quad y_0 = \lambda^2\tau(\lambda)y_0.$$

Since $x(t) = x_0e^{\lambda t}$ and $\dot{x}(t) = \lambda e^{\lambda t}x_0 = y(t)$,

$$x_0 = \lambda^2\tau(\lambda)y_0. \tag{4.18}$$

III: $\nu_1 = \nu_2 = 1$. Then (2.30) reduces to

$$x_0 = (\lambda^2 - 1)\tau(\lambda)x_0. \tag{4.19}$$

Cases II and III seem to indicate that the dominant term in the cost functional (4.8) is $\|y(t)\|$ and not $\|x(t)\|_{\mathcal{H}_1}$. We shall give below an example where this is clearly the case, but we do not know if this is true in the general case. However, the following observation seems to indicate that $\|y(t)\|$ is the dominant term in (4.8).

OBSERVATION 4.1 Assume $\nu_1 = 1$ and $\nu_2 = 0$ in (4.12). Let

$$\mathcal{H} = \mathcal{H}_0 \times \mathcal{H}_0 \tag{4.20}$$

and

$$A^{-1}B \in [H, \mathcal{H}_0]. \tag{4.21}$$

Then for $x_0 \in \mathcal{H}_0$ and $y_0 \in \mathcal{H}_0$ we can express the control problem (4.7), (4.8) in the form

$$\dot{x} = Ay, \quad \dot{y} = -Ax + A^{-1}Bu, \tag{4.22}$$

with $C(u, x_0, y_0)$ given by (4.8), where $\nu_1 = 1$, $\nu_2 = 0$.

The system (4.22) has now a bounded control input. Since the homogeneous system (i.e. $u(t) = 0$) is conservative and the cost functional contains only bounded operators it is not plausible for the system to be made uniformly stable via a feedback based on optimization of the infinite horizon problem.

EXAMPLE 4.2 Let $H = \mathbb{R}$ and $B = b \in V'$. (This is a rank–one control problem). We assume $\nu_1 = \nu_2 = 1$ and $A^{-2}b \in V$. Then (2.30) reduces to

$$x_0 = (\lambda^2 - 1)\langle(\lambda^2 I + A^2)^{-1}x_0, b\rangle(\lambda^2 I + A^2)^{-1}b, \quad \operatorname{Re}\lambda < 0. \tag{4.23}$$

The corresponding eigenvalues are easily computed to satisfy the equation

$$1=(\lambda^2-1)\langle((\lambda^2 I+A^2)^{-2}b,b\rangle,\quad \operatorname{Re}\lambda<0. \tag{4.24}$$

If $\nu_1 = 0$ and $\nu_2 = 1$ the equivalent of (4.24) reduces to

$$1=\lambda^2\langle((\lambda^2 I+A^2)^{-2}b,b\rangle,\quad \operatorname{Re}\lambda<0. \tag{4.25}$$

If $\nu_1 = 1$ and $\nu_2 = 0$ the equivalent of (4.24) is

$$-1=\langle((\lambda^2 I+A^2)^{-2}b,b\rangle. \tag{4.26}$$

A condition for Theorem 2.12 to be satisfied.

Systems of the form (4.7) which arise in LQR problems involving linear hyperbolic partial differential equations with controls on their boundary often satisfy conditions of the following kind

$$(\lambda I-\alpha)^{-1}\beta=\sum_{j=1}^{\infty}\frac{1}{\lambda^2+\lambda_j^2}[\lambda A_j+B_j], \tag{4.27}$$

where $\{A_j\}$ and $\{B_j\}$ are in $[H,\mathcal{H}]$, $\{\lambda_j\}$ are real and

$$\sum_{j=1}^{\infty}\frac{1}{\lambda_j^2}<\infty,\quad \sup_j[|A_j|+|B_j|]<\infty. \tag{4.28}$$

(See e.g., [6], for a general survey of such systems, and [2] for properties related to the poles of the associated operators $(\lambda I-\alpha)^{-1}$.)

DEFINITION 4.3 A Bochner inegrable mapping f from a finite interval, $[0,T]$, into a Banach space X has a finite Laplace transform (F.L.T.) given by

$$\hat{f}(\lambda)=\int_0^T e^{\lambda t}f(t)dt. \tag{4.29}$$

The inverse transform of (4.29) has support on $[0,T]$, and $\hat{f}(\lambda)$ is an entire analytic function of exponential type which is L_2–integrable over $(-i\infty,i\infty)$ (see e.g., [1], p. 103, or [3]).

THEOREM 4.4 *Let $q: C\to C$ be an entire function such that $q(\lambda)\not\equiv 0$ and such that the function*

$$q(\lambda)(\lambda I-\alpha)^{-1}\beta, \tag{4.30}$$

where $(\lambda I-\alpha)^{-1}\beta$ is given by (4.27), is a F.L.T. from C into $[H,\mathcal{H}]$, and such that, for any $\lambda_0\in \operatorname{Re}\lambda<0$, $\frac{q(\lambda)}{\lambda-\lambda_0}$ is in the Hardy–Lebesque space $H_2(0)$ (see e.g. [8]). Then Equation (2.30) determines the spectral points of the optimization problem for those points which satisfy Property 2.9.

PROOF. Let $\lambda_0 \in \operatorname{Re}\lambda < 0$. Define

$$q_1(\lambda) = q(\lambda) \tag{4.31}$$

if $q(-\lambda_0) \neq 0$, and the entire function

$$q_1(\lambda) = \frac{q(\lambda)}{(\lambda + \lambda_0)^m}, \quad q_1(\lambda_0) \neq 0, \tag{4.32}$$

if q has a zero of order m at $-\lambda_0$. Let $h_0 \in H$ are arbitrary. Then the function

$$\hat{h}(\lambda) = \frac{q_1(\lambda)}{\lambda - \lambda_0} h_0 \tag{4.33}$$

is the Laplace transform of some $h \in L_2(\mathbb{R}^+, H)$ as is the function

$$h_1(t) = h_0 e^{\lambda_0 t} \tag{4.34}$$

(see e.g., [8], p. 162–163).

Let $S_0(t)\beta$ denote the inverse Laplace transform of

$$q_1(\lambda)(\lambda I - \alpha)^{-1}\beta. \tag{4.35}$$

It is obvious that if (4.30) is a F.L.T. then (4.35) is also since it satisfies the F.L.T. version of the Paley–Wiener Theorem (see e.g., [1], p. 103). Thus, for some $T > 0$, $S_0(t)\beta \equiv 0$ if $t > T$, and for $t > T$

$$\begin{aligned} y(t,h) &= \int_0^t S(t-\sigma)\beta h(\sigma)d\sigma = \int_0^t S(\sigma)\beta h(t-\sigma)d\sigma \\ &= \int_0^t S_0(\sigma)\beta h_0 e^{\lambda_0(t-\sigma)}d\sigma = \int_0^T S_0(\sigma)\beta h_0 e^{\lambda_0(t-\sigma)}d\sigma. \end{aligned} \tag{4.36}$$

Hence $y(\cdot, h)$ defined by (4.36) is in $L_2(\mathbb{R}, H)$, $h \in \mathcal{U}$ and

$$\hat{h}(\lambda_0) = q_1(-\lambda_0)h_0. \tag{4.37}$$

Since h_0 is arbitrary Equation (2.31) yields the spectral points of (2.4) for those points satisfying Property 2.9.

EXAMPLE 4.5 Consider the problem

$$v_{tt}(x,t) = v_{xx}(x,t), \quad 0 < x < 1, \quad t > 0, \tag{4.38}$$

$$v_x(1,t) = \sqrt{2}u(t), \quad v(0,t) = 0, \quad u \in L_2(\mathbb{R}^+, \mathbb{R}), \tag{4.39}$$

plus appropriate initial conditions. Using a standard eigenvalue expansion (see e.g. [5], Chapter 4) the problem (4.38), (4.39) can be reduced to the following version of (2.1) defined on the sequence space $l^2 \times l^2$.

$$\ddot{x}_n + (n\pi + \frac{\pi}{2})^2 x_n = (-1)^n(2)u, \quad n = 0, 1, 2, \cdots. \tag{4.40}$$

For the system (4.39) α is the infinite matrix operator on $l^2 \times l^2$ defined by

$$\alpha = \left\{ \begin{pmatrix} \mathcal{O} & 1 \\ -(n\pi + \frac{\pi}{2})^2 & \mathcal{O} \end{pmatrix} \right\}_n, \quad n = 0, 1, 2, \cdots, \tag{4.41}$$

β is the vector in l^∞ given by

$$\beta = \{(-1)^n(2)\}, \quad n = 0, 1, 2, \cdots, \quad \text{and} \tag{4.42}$$

$$\{(\lambda I - \alpha)^{-1}\beta\}_n = \left\{ \frac{(-1)^n(2)}{\lambda^2 + (n\pi + \frac{\pi}{2})^2} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \right\}_n \tag{4.43}$$

$n = 0, 1, 2, \cdots$, is a bounded vector on $l^2 \times l^2$. Notice that (4.43) has the structure of (4.27), where $\lambda_j^2 = (j\pi + \frac{\pi}{2})^2$. Let

$$q(\lambda) = \frac{1}{\lambda}(1 - e^{-4\lambda}), \tag{4.44}$$

then it is trivial to show that all conditions of Theorem 4.4 are satisfied and that Equation (2.31) is a necessary and sufficient condition for a point $x_0 \neq \mathcal{O}$ to be a spectral point of the system (4.38), (4.39) when the cost functional is given by

$$C(u, x_0) = \int_0^\infty \left[\sum_{n=0}^{\infty} [x_n^2(t) + (\dot{x}_n(t))^2] + (u(t))^2 \right] dt, \tag{4.45}$$

For the system (4.38), (4.39) and (4.45) the equation for the eigenvalues corresponding to Equation (4.24) reduces to

$$\begin{aligned} 1 &= 4(\lambda^2 - 1) \sum_{n=0}^{\infty} \frac{1}{(\lambda^2 + (n\pi + \frac{\pi}{2})^2)^2} \\ &= -(\lambda^2 - 1) \frac{(\lambda \operatorname{sech}^2 \lambda - \tanh \lambda)}{\lambda^3}. \end{aligned} \tag{4.46}$$

(See Appendix A for the derivation of (4.46).) It is also not difficult to show that the solutions of (4.46) in $\operatorname{Re}\lambda < 0$ are asymptotic to the solutions of the equation

$$1 = -\operatorname{sech}^2 \lambda, \tag{4.47}$$

which satisfy $\operatorname{Re}\lambda = \frac{1}{2}\log(3-\sqrt{8}) < 0$.

EXAMPLE 4.6 In [9] Y. You considered, among other things, the controllability and stabilizability of the two–dimensional vibrating plate over the region,

$$\Gamma = \{(x,y):\ 0 < x < 1,\ 0 < y < 1\}, \tag{4.48}$$

given by the equation

$$w_{tt} + w_{xxxx} + 2w_{xx}w_{yy} + w_{yyyy} = \sum_{k=1}^{m} \delta(x-p_k, y-q_k) f_k(t), \tag{4.49}$$

$t > 0, \quad (x,y) \in \Gamma$ and

$$w = 0, \quad \frac{\partial^2 w}{\partial n^2} = 0 \tag{4.50}$$

on the boundary, $B(\Gamma)$, of Γ. In (4.49) $\delta(x-p_k, y-q_k)$ is the Dirac delta function concentrated at the point (p_k, q_k) in Γ and f_k is a control.

It was shown in [9], that no matter where the points $\{(p_k, q_k)\}$ are placed there is no system of finite controls, $\{f_k\}$, $1 \le k \le m$, such that (4.49)–(4.50) is either approximately controllable or weakly stabilizable. However, despite the negative result, Theorem 4.4 is still applicable in obtaining the spectral points of a quadratic cost problem of the form (4.7)–(4.8).

To see this observe that (4.49)–(4.50) can be translated into the form (4.7) (see e.g. [9], in particular the derivation of Equation (2.25)), where $H = \mathbb{R}^m$. For purposes of this example let us assume $H = \mathbb{R}$. In this case the resulting system is described by the infinite–dimensional system

$$\ddot{x}_{mn} + (m^2+n^2)^2\pi^4 x_{mn} = b_{mn} f(t) \tag{4.51}$$

$n, m = 1, 2, \cdots$, where for each m, n

$$b_{mn} = 2(\sin mp\pi)(\sin nq\pi),$$

$0 < p < 1$, $0 < q < 1$, fixed. Clearly $b_{mn} \neq 0$ for all (m,n).

We can rewrite (4.51) in the first order form as the infinite set of equations

$$\begin{aligned} \dot{x}_{mn} &= -y_{mn} \\ \dot{y}_{mn} &= -(m^2+n^2)^2\pi^4 x_{mn} + b_{mn} f(t) \end{aligned} \tag{4.52}$$

$m, n \geq 1$. The Laplace transform, $(\lambda I - \alpha)^{-1}\beta$, for this system is the bounded operator on $l^2 \times l^2$ whose components are

$$\{(\lambda I - \alpha)^{-1}\beta\}_{mn} = \left\{ \frac{b_{mn}}{\lambda^2 + (m^2+n^2)^2\pi^4} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \right\}_{mn}, \tag{4.53}$$

$m, n \geq 1$.

An entire function q which satisfies Theorem 4.4 for the system (4.53) is easily seen to be given by

$$q(\lambda) = \frac{1 - e^{\frac{-2\lambda}{\pi}}}{\lambda}. \tag{4.54}$$

In fact the inverse Laplace transform of $(\lambda I - \alpha)^{-1}\beta q(\lambda)$ for this example is the vector on $l^2 \times l^2$ whose components satisfy for $m, n \geq 1$

$$\begin{pmatrix} x_{mn}(t) \\ y_{mn}(t) \end{pmatrix} = \begin{pmatrix} \frac{1-\cos(m^2+n^2)^2\pi^2 t}{(m^2+n^2)^2\pi^4} \\ \frac{\sin(m^2+n^2)^2\pi^2 t}{(m^2+n^2)^2\pi^4} \end{pmatrix} b_{mn} \quad \text{if} \quad 0 \leq t \leq \frac{2}{\pi} \tag{4.55}$$

and

$$\begin{pmatrix} x_{mn}(t) \\ y_{mn}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{if} \quad t \geq \frac{2}{\pi}. \tag{4.56}$$

If we choose the quadratic cost function

$$C(u) = \sum_{m,n=1} \int_0^\infty |\dot{x}_{mn}(t)|^2 dt + \int_0^\infty |f(t)|^2 dt, \tag{4.57}$$

then, on the basis of Example 4.2, the spectrum of the resulting LQR problem is given by the solution of

$$1 = \lambda^2 \sum_{m,n=1} \frac{b_{mn}^2}{[\lambda^2 + (m^2+n^2)^2\pi^4]^2}, \quad \operatorname{Re}\lambda < 0. \tag{4.58}$$

Clearly since $b_{mn} \neq 0$ for some m and n it follows that (4.58) can have at least one solution with $\operatorname{Re}\lambda < 0$. Hence even though a system is neither approximately controllable nor weakly stabilizable the spectrum of a corresponding LQR problem need not be empty. For example if the right side of (4.58) has a value $\tau > 1$ when $\lambda = -1$, then clearly (4.58) has at least one solution with $\operatorname{Re}\lambda < 0$.

Appendix A.

Using the methods in [7], p. 64, it is easy to show that

$$\frac{\tanh \lambda}{\lambda} = 2 \sum_{n=0}^{\infty} \frac{1}{\lambda^2 + (n\pi + \frac{\pi}{2})^2}. \tag{A.1}$$

Thus

$$\left(\frac{\tanh \lambda}{\lambda}\right)' = \frac{\lambda \operatorname{sech}^2 \lambda - \tanh \lambda}{\lambda^2} = -4\lambda \sum_{n=0}^{\infty} \frac{1}{[\lambda^2 + (n\pi + \frac{\pi}{2})^2]^2}, \tag{A.2}$$

From which (4.43) follows.

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