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Local controllability of dynamic von Karman plates [1]

by

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1. Introduction and Statement of Main Result

Let Ω be a bounded region in $\Re^2$ with smooth boundary Γ. Let $X_0 = (x_0, y_0)$ be a fixed but otherwise arbitrary point of $\Re^2$, and set

$$\Gamma_+ = \{X \in \Gamma | (X - X_0) \cdot \nu > 0\}, \ \ \Gamma_- = \Gamma - \Gamma_+,$$

where ν denotes the unit normal to Γ pointing towards the exterior of Ω. Note that $\Gamma_\pm$ depend on the choice of X_0. We consider the following dynamic von Karman system consisting of

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$$\begin{cases} \ddot{w} - \gamma^2 \Delta \ddot{w} + \gamma^2 \Delta^2 w = [w, \chi] \text{ in } Q = \Omega \times (0,T), \\ w = \dfrac{\partial w}{\partial \nu} = 0 \text{ on } \Sigma_- = \Gamma_- \times (0,T), \\ \begin{cases} \gamma^2 [\Delta w + (1-\mu) P_1 w] = g_0, \\ \gamma^2 \left[\dfrac{\partial \Delta w}{\partial \nu} + (1-\mu) P_2 w - \dfrac{\partial \ddot{w}}{\partial \nu} \right] = -g_2 + \dfrac{\partial g_1}{\partial \tau} \\ \text{on } \Sigma_+ = \Gamma_+ \times (0,T), \end{cases} \\ w(\cdot, 0) = \dfrac{\partial w}{\partial t}(\cdot, 0) = 0 \text{ in } \Omega, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \Delta^2 \chi = -[w, w], \\ \chi = \dfrac{\partial \chi}{\partial \nu} = 0 \text{ on } \Sigma = \Gamma \times (0,T), \end{cases} \tag{1.2}$$

where

$$[\phi, \psi] = \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 \psi}{\partial x^2} - 2 \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial^2 \psi}{\partial x \partial y}.$$

In the above, $\dot{} = \partial / \partial t$, Δ is the ordinary Laplacian in $\Re^2$, γ^2 is a constant of order $O(h^2)$, h denoting the uniform thickness of the plate, and $\mu \in (0,1)$ is another constant (Poisson's ratio). ν is the unit normal to Γ pointing into the exterior of Ω, and τ is the positively oriented unit tangent vector to Γ_+. We specifically assume that $\Gamma_{\pm} \neq \emptyset$. P_1 and P_2 are boundary operators which satisfy the Green's formula

$$(\Delta^2 u, v)_{L^2(\Omega)} = a(u,v) + \int_\Gamma \left[v \left(\frac{\partial \Delta u}{\partial \nu} + (1-\mu) P_2 u \right) - (\Delta u + (1-\mu) P_1 u) \frac{\partial v}{\partial \nu} \right] d\Gamma \tag{1.3}$$

where

$$\begin{aligned} a(u,v) &= \int_\Omega \left[\left(\frac{\partial^2 v}{\partial x^2} \right)^2 + \left(\frac{\partial^2 v}{\partial y^2} \right)^2 + 2\mu \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 v}{\partial y^2} \right. \\ &+ \left. 2(1-\mu) \left(\frac{\partial^2 v}{\partial x \partial y} \right)^2 \right] dx dy. \end{aligned}$$

The specific forms of these operators may be found in [4] or in [9]. The above system defines the transverse deflection w and the so-called Airy stress function

χ of a thin, vibrating homogeneous, isotropic elastic plate in situations where w is not necessarily small relative to h. The quantities g_0, g_1 and g_2 are the *controls.* They correspond, respectively, to a bending moment about the tangent vector to Γ, a twisting moment about the normal to Γ and to an edge shear force acting perpendicularly to the faces of the plate. Our purpose here is to consider the *reachability problem* for (1.1), (1.2), which is to identify the *reachable set*

$$\mathcal{R}_T = \{(w(T), \dot{w}(T)) | (g_0, g_1, g_2) \in \mathcal{C}\},$$

where $\mathcal{C}$ is a given space of controls.

In the case of *linear* elastic plate dynamics, there is an extensive literature devoted to the reachability problem (equivalent to the *exact controllability problem* in the linear case); see, e.g., [9], [10], [11], [12], [13], [15, Chapter IV]. A recent result for the linear analog of (1.1), i.e., when the right side of (1.1) is equal to zero, is

$$\begin{aligned} \{(w(T), \dot{w}(T)) | \, g_i \in L^2(\Sigma_+), \quad & i = 0,1,2\} \supset H^2_{\Gamma_-}(\Omega) \times H^1_{\Gamma_-}(\Omega), \\ & T > T_0, \end{aligned} \tag{1.4}$$

where $T_0 > 0$ depends on Ω and γ and where

$$H^1_{\Gamma_-}(\Omega) = \{\phi | \, \phi \in H^1(\Omega), \, \phi|_{\Gamma_-} = 0\},$$

$$H^2_{\Gamma_-}(\Omega) = \{\phi | \, \phi \in H^2(\Omega), \, \phi|_{\Gamma_-} = \partial\phi/\partial\nu|_{\Gamma_-} = 0\},$$

$H^k(\Omega)$ denoting the standard Sobolev space of order k based on $L^2(\Omega)$. A proof of (1.4) may be found in [6], and is based on a related result in [9, Chapter V] which states that

$$\begin{aligned} \{(w(T), \dot{w}(T)) | \, g_i \;&\in\; H^{-1}(0,T; L^2(\Gamma_+)), \, i = 0,1,2\} \supset \\ &\supset\; H^1_{\Gamma_-}(\Omega) \times (H^2_{\Gamma_-}(\Omega))', \, T > T_0, \end{aligned}$$

where $(H^2_{\Gamma_-}(\Omega))'$ is the dual space of $H^2_{\Gamma_-}(\Omega)$ with respect to $H^1_{\Gamma_-}(\Omega)$.

Global reachability results for certain semilinear plate problems, e.g.,

$$\begin{cases} \ddot{w} + \Delta^2 w = f(w) \ \text{ in } Q, \\ w = g_1, \ \ \Delta w = g_2 \ \text{ on } \Sigma, \end{cases} \tag{1.5}$$

have been given in [14]. If the function $f : \Re \mapsto \Re$ satisfies

$$||f||_{W^{1,\infty}(\Re)} \leq \text{Constant},$$

it is proved that

$$\begin{aligned}\{(w(T),\dot{w}(T)) \mid (g_1,g_2) &\in H^m(\Sigma)\times H^{1/4}(0,T;L^2(\Gamma))\} \supset \\ &\supset (H^2(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega)\end{aligned}$$

for every $T > 0$, where $m > 0$ is arbitrary. We are unaware of other exact controllability/reachability results, either local or global, for nonlinear plate problems besides those in [14].

In this paper, a *local* reachability result for the system (1.1), (1.2), analogous to (1.4), will be established, namely

Theorem 1.1 *There is is ball S_r of radius r and centered at* $(0,0)$ *in* $H^2_{\Gamma_-}(\Omega)\times H^1_{\Gamma_-}(\Omega)$ *such that*

$$\{(w(T),\dot{w}(T)) \mid g_i \in L^2(\Sigma_+),\ i=0,1,2\} \supset S_r,\ \ T > T_0, \tag{1.6}$$

where w, χ satisfy (1.1), (1.2) *and where T_0 is the same as in* (1.4).

We do not assert the stronger statement (1.4) for the solution of (1.1), (1.2) and, indeed, (1.4) is probably *false* for these dynamics. However, it is probably true, although not proved here, that (1.6) holds for *every* ball S_r in $H^2_{\Gamma_-}(\Omega)\times H^1_{\Gamma_-}(\Omega)$ for some T_0 *depending on* r. A result of this type has been proved in [8] for a dynamic nonlinear beam system that is the one-dimensional analog of the von Karman system considered here.

2. Outline of the Proof of Theorem 1.1

Denote by $\mathcal{L}(X,Y)$ the space of bounded operators from X to Y, both Banach spaces. Let G be the Green's operator for Δ^2 subject to Dirichlet boundary conditions, that is,

$$Gf=\phi \Longleftrightarrow \Delta^2\phi = f \ \text{ in } \Omega,\ \ \phi=\frac{\partial\phi}{\partial\nu}=0 \ \text{ on } \Gamma.$$

Then, in particular, $G\in\mathcal{L}(H^{-r}(\Omega), H^{4-r}(\Omega)\cap H_0^2(\Omega))$ for every $r\le 2$. We write (1.2) as

$$\chi = -G[w,w] \tag{2.1}$$

and substitute (2.1) into (1.1) to obtain the system

$$
\begin{cases}
\ddot{w} - \gamma^2 \Delta \ddot{w} + \gamma^2 \Delta^2 w = F(w) \text{ in } Q, \\
w = \dfrac{\partial w}{\partial \nu} = 0 \text{ on } \Sigma_-, \\
\begin{cases}
\gamma^2 [\Delta w + (1-\mu) P_1 w] = g_0, \\
\gamma^2 \left[\dfrac{\partial \Delta w}{\partial \nu} + (1-\mu) P_2 w - \dfrac{\partial \ddot{w}}{\partial \nu} \right] = -g_2 + \dfrac{\partial g_1}{\partial \tau} \text{ on } \Sigma_+,
\end{cases} \\
w(\cdot, 0) = \dfrac{\partial w}{\partial t}(\cdot, 0) = 0 \text{ in } \Omega,
\end{cases}
\tag{2.2}
$$

where

$$
F(w) = -[w, G[w,w]]. \tag{2.3}
$$

Our problem is therefore to show that the reachable set of (2.2) contains S_r for r small enough.

Lemma 2.1 *The trilinear mapping*

$$
(u, v, w) \mapsto [u, G[v, w]]
$$

is continuous from $(H^2(\Omega))^3$ *into* $H^{-\varepsilon}(\Omega)$ *for every* $\varepsilon > 0$.

Proof. The mapping

$$
(v, w) \mapsto [v, w] : (H^2(\Omega))^2 \mapsto L^1(\Omega)
$$

is continuous. But in dimension two, $L^1(\Omega) \subset H^{-1-\varepsilon}(\Omega)$ continuously for any $\varepsilon > 0$ [17] so that, if ∂^2 stands for any second order partial derivative,

$$
(v, w) \mapsto \partial^2 G[v, w] : (H^2(\Omega))^2 \mapsto H^{1-\varepsilon}(\Omega), \quad 0 < \varepsilon \le 1,
$$

is continuous. Since in dimension two the multiplication operator $(\psi, \phi) \mapsto \psi\phi$ is continuous from $L^2(\Omega) \times H^{1-\varepsilon}(\Omega)$ into $H^{-\varepsilon}(\Omega)$ for every $\varepsilon > 0$ [5, Theorem 1.4.4.2], it follows that

$$
(u, v, w) \mapsto \partial^2 u \partial^2 G[v, w] : (H^3(\Omega))^3 \mapsto H^{-\varepsilon}(\Omega)
$$

is continuous for every $\varepsilon > 0$. □

As a consequence of Lemma 2.2 the mapping $w \mapsto F(w) : \; H^2(\Omega) \mapsto H^{-\varepsilon}(\Omega)$ is continuous. It is also Fréchet differentiable, with derivative $DF(w) \in \mathcal{L}(H^2(\Omega), H^{-\varepsilon}(\Omega))$ given by

$$
DF(w)(\phi) = [\phi, G[w, w]] + 2[w, G[\phi, w]]. \tag{2.4}
$$

In fact, one easily sees that F is infinitely Fréchet differentiable and that

$$D^2F(w)(\phi,\psi) = 2[\phi, G[\psi, w]] + 2[\psi, G[\phi, w]] + 2[w, G[\phi, \psi]],$$

$$D^3F(w)(\phi,\psi,\lambda) = 2[\phi, G[\psi, \lambda]] + 2[\psi, G[\phi, \lambda]] + 2[\lambda, G[\phi, \psi]],$$

and $D^kF(w) = 0$ for $k \geq 4$.

Let us now outline the remainder of the proof of Theorem 1.1. We first write the system (2.2) as an abstract first order control system. Introduce the Hilbert spaces

$$H = H^1_{\Gamma_-}(\Omega), \quad V = H^2_{\Gamma_-}(\Omega)$$

with respective norms

$$\|\phi\|_H = \left(\int_\Omega (\phi^2 + \gamma^2|\nabla\phi|^2)dX\right)^{1/2}, \quad \|\phi\|_V = \left(\gamma^2\int_\Omega (\Delta\phi)^2 dX\right)^{1/2},$$

where $dX = dxdy$. We have $V \subset H \subset V'$ as usual. Let $\phi \in V$ and form the $L^2(\Omega)$ scalar product of ϕ with (2.2a). One obtains after some integrations by parts and with the aid of Green's formula (1.3) the variational equation

$$(\ddot{w}, \phi)_H + (w, \phi)_V = \int_\Omega F(w)\phi\, dX + \int_{\Gamma_+}\left(g_0\phi + g_1\frac{\partial\phi}{\partial\tau} + g_2\frac{\partial\phi}{\partial\nu}\right)d\Gamma. \tag{2.5}$$

Set $U = (L^2(\Gamma_+))^3$ and suppose that $\mathrm{g} = (g_0, g_1, g_2) \in U$. Since

$$\left|\int_{\Gamma_+}\left(g_0\phi + g_1\frac{\partial\phi}{\partial\tau} + g_2\frac{\partial\phi}{\partial\nu}\right)d\Gamma\right| \leq C\|g\|_U\|\phi\|_V,$$

an operator $B \in \mathcal{L}(U, V')$ is defined by

$$\langle B\mathrm{g}, \phi\rangle = \int_{\Gamma_+}\left(g_0\phi + g_1\frac{\partial\phi}{\partial\tau} + g_2\frac{\partial\phi}{\partial\nu}\right)d\Gamma, \quad \forall\phi \in \mathcal{V}, \tag{2.6}$$

where $\langle v', v\rangle$ denotes the duality pairing between elements $v' \in V'$ and $v \in V$. In addition, if $w \in V$ we have, by virtue of Lemma 2.1,

$$\begin{aligned}\left|\int_\Omega F(w)\phi\, dX\right| &\leq \|F(w)\|_{H^{-\epsilon}(\Omega)}\|\phi\|_{H_0^\epsilon(\Omega)}\\ &\leq C\|F(w)\|_{H^{-\epsilon}(\Omega)}\|\phi\|_H\end{aligned}$$

for $\varepsilon < 1/2$, since $H^\epsilon(\Omega) = H_0^\epsilon(\Omega)$ for such ε. Therefore, there is an $\widetilde{F}(w) \in H$ such that

$$\int_\Omega F(w)\phi\, dX = (\widetilde{F}(w), \phi)_H, \quad \forall w \in V, \forall \phi \in H. \tag{2.7}$$

(In fact, $\widetilde{F}(w)$ is defined by

$$\widetilde{F}(w) - \gamma^2 \Delta \widetilde{F}(w) = F(w) \text{ in } \Omega, \quad \widetilde{F}(w)|_{\Gamma_-} = 0, \quad \left.\frac{\partial \widetilde{F}(w)}{\partial \nu}\right|_{\Gamma_+} = 0.)$$

The mapping $w \mapsto \widetilde{F}(w) : V \mapsto H$ is infinitely Fréchet differentiable with, for example,

$$(D\widetilde{F}(w)\phi, \psi)_H = \textstyle\int_\Omega \{[\phi, G[w,w]] + 2[w, G[\phi, w]]\}\psi\, dX,$$
$$\forall w, \phi \in V, \ \forall \psi \in H.$$

It follows from (2.5)–(2.7) that the system (2.2) may be written

$$\begin{cases} \ddot{w} + Aw = \widetilde{F}(w) + B\mathrm{g} \text{ in } V', \\ w(0) = \dot{w}(0) = 0, \end{cases}$$

where A is the Riesz isomorphism of V onto V'. By setting $w_1 = w$, $w_2 = \dot{w}$,

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad \mathcal{F}(\mathbf{w}) = \begin{pmatrix} 0 \\ \widetilde{F}(w_1) \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} 0 & -A \\ I & 0 \end{pmatrix}, \quad \mathcal{B}\mathrm{g} = \begin{pmatrix} 0 \\ B\mathrm{g} \end{pmatrix},$$

the last system becomes

$$\dot{\mathbf{w}} = \mathcal{A}\mathbf{w} + \mathcal{F}(\mathbf{w}) + \mathcal{B}\mathrm{g} \text{ in } \mathcal{H}, \quad \mathbf{w}(0) = 0, \tag{2.8}$$

where

$$\mathcal{H} = H \times V', \quad \mathcal{V} = V \times H,$$

$$\mathcal{B} \in \mathcal{L}(U, \mathcal{H}),$$

$$\mathbf{w} \mapsto \mathcal{F}(\mathbf{w}) : \mathcal{V} \mapsto \mathcal{V} \text{ is infinitely Fréchet differentiable.}$$

It is standard theory that the operator $\mathcal{A}$ is skew-adjoint as an operator in $\mathcal{H}$ with domain $\mathcal{V}$ and also as an operator in $\mathcal{V}$ with domain $D_A \times V$, where

$$D_A = \{\phi |\, A\phi \in H\}.$$

Now we employ a strategy that originated with [16] in the context of finite dimensional control systems and which has frequently been used in infinite dimensional settings (c.f. [2]). The idea is to write

$$\mathbf{w} = \mathbf{u} + \mathbf{v}$$

where

$$\dot{\mathbf{u}} = \mathcal{A}\mathbf{u} + \mathcal{B}\mathbf{g}, \quad \mathbf{u}(0) = 0, \tag{2.9}$$

$$\dot{\mathbf{v}} = \mathcal{A}\mathbf{v} + \mathcal{F}(\mathbf{u} + \mathbf{v}), \quad \mathbf{v}(0) = 0. \tag{2.10}$$

If $\mathbf{g} \in L^2(0,T;U)$ then $\mathcal{B}\mathbf{g} \in L^2(0,T;\mathcal{H})$ so that (2.9) has a unique mild solution $\mathbf{u} \in C([0,T];\mathcal{H})$. Moreover, it is known ([6, Section 4], c.f. [9, Chapter V]) that there exists $T_0 > 0$ such that for all $T > T_0$

$$\{\mathbf{u}(T) | \mathbf{g} \in \mathcal{U}\} \supset \mathcal{V},$$

where we have set

$$\mathcal{U} = L^2(0,T;U).$$

Let $\mathbf{u}^0 \in \mathcal{V}$. If $T > T_0$, there is at least one control $\mathbf{g} \in \mathcal{U}$ such that the corresponding solution of (2.9) satisfies $\mathbf{u}(T) = \mathbf{u}^0$. We want to substitute this solution into (2.10). However, this may not be possible since the only regularity of $\mathbf{u}$ that is assured *a priori* is $\mathbf{u} \in C([0,T];\mathcal{H})$, but the function $\mathcal{F}$ in (2.10) is defined only on $\mathcal{V}$. In order to make things work, we need to choose a control that drives 0 to $\mathbf{u}^0$ in a special way.

Proposition 2.2 *There exists $T_0 > 0$ such that for every $T > T_0$ and $\mathbf{u}^0 \in \mathcal{V}$ we may choose a control $\mathbf{g} \in \mathcal{U}$ such that the corresponding solution of* (2.9) *satisfies*

(i) $\mathbf{u} \in C(0,T];\mathcal{V}), \quad \mathbf{u}(T) = \mathbf{u}^0$;

(ii) *the mapping $\mathbf{u}^0 \mapsto \mathbf{u} : \mathcal{V} \mapsto C([0,T];\mathcal{V})$ is linear and continuous.*

Let $T > T_0$, $\mathbf{u}^0$, $\mathbf{u}$ and $\mathbf{g}$ be the quantities in Proposition 2.2, and substitute $\mathbf{u}$ into (2.10). The corresponding solution satisfies $\mathbf{v} \in C([0,T];\mathcal{V})$, and we can prove

Proposition 2.3 *There is a number $R_0 > 0$ such that if $0 < R < R_0$ and $\|\mathbf{u}^0\|_{\mathcal{V}} < R$, the solution of* (2.10) *satisfies*

$$\|\mathbf{v}\|_{L^\infty(0,T;\mathcal{V})} < R.$$

By virtue of Propositions 2.2 and 2.3 we may consider for every positive $R < R_0$ the mapping $K : \mathbf{u}^0 \mapsto -\mathbf{v}(T) : S_R \mapsto S_R$, where S_R is the open ball in $\mathcal{V}$ of radius R, centered at the origin. The reachable set of (2.8) therefore contains

$$\{\mathbf{u}^0 - K\mathbf{u}^0 : \mathbf{u}^0 \in S_R\}, \ \ \forall R < R_0.$$

Proposition 2.4 *One has $(I - K)S_R \supset S_r$ for some $R > 0$ and $r > 0$.*

Theorem 1.1 follows immediately from Proposition 2.4.

3. Proofs of Propositions 2.2–2.4

Proof of Proposition 2.2. The control $\mathbf{g}$ in question is constructed using *stabilization operators* and a technique due to D. L. Russell [18]. More specifically, we first solve the backwards problem (which is possible since $\mathcal{A}$ is skew-adjoint)

$$\dot{\widetilde{\mathbf{u}}} = \mathcal{A}\widetilde{\mathbf{u}} + \mathcal{B}\widetilde{\mathbf{g}}, \ \ 0 \le t < T, \ \ \widetilde{\mathbf{u}}(T) = \widetilde{\mathbf{u}}^0 \in \mathcal{V},$$

using *feedback controls*

$$\widetilde{g}_0 = (m \cdot \nu)\frac{\partial \widetilde{u}_2}{\partial \nu}, \ \ \widetilde{g}_1 = (m \cdot \nu)\frac{\partial \widetilde{u}_2}{\partial \tau}, \ \ \widetilde{g}_2 = (m \cdot \nu)\widetilde{u}_2,$$

where $m = X - X_0$, X_0 being the point used in the definition of $\Gamma_\pm$.
(Recall that $\widetilde{\mathbf{u}} = (\widetilde{u}_1, \widetilde{u}_2)$, $\widetilde{u}_2 = \dot{\widetilde{u}}_1$.) It is proved in [7] that $\widetilde{\mathbf{g}} \in \mathcal{U}$, that $\widetilde{\mathbf{u}} \in C([0,T];\mathcal{V})$ and, most importantly,

$$\|\widetilde{\mathbf{u}}(t)\|_{\mathcal{V}} \le Ce^{-\omega(T-t)}\|\widetilde{\mathbf{u}}^0\|_{\mathcal{V}}, \ \ 0 \le t \le T, \tag{3.1}$$

where $C \ge 1$ and $\omega > 0$ are constants.

One next solves the forward problem

$$\dot{\widehat{\mathbf{u}}} = \mathcal{A}\widehat{\mathbf{u}} + \mathcal{B}\widehat{\mathbf{g}}, \ \ \widehat{\mathbf{u}}(0) = \widetilde{\mathbf{u}}(0) \in \mathcal{V},$$

using feedback controls

$$\widehat{g}_0 = -(m \cdot \nu)\frac{\partial \widehat{u}_2}{\partial \nu}, \ \ \widehat{g}_1 = -(m \cdot \nu)\frac{\partial \widehat{u}_2}{\partial \tau}, \ \ \widehat{g}_2 = -(m \cdot \nu)\widehat{u}_2.$$

One has

$$\|\widehat{\mathbf{u}}(t)\|_{\mathcal{V}} \le Ce^{-\omega t}\|\widetilde{\mathbf{u}}(0)\|_{\mathcal{V}}, \ \ t \ge 0,$$

so that from (3.1)

$$\|\widehat{\mathbf{u}}(T)\|_{\mathcal{V}} \le C^2 e^{-2\omega T} \|\widetilde{\mathbf{u}}^0\|_{\mathcal{V}}.$$

Therefore, if $C^2 e^{-2\omega T} < 1$, i.e., if $T > T_0 := (\log C)/\omega$, the map $L_T : \widetilde{\mathbf{u}}^0 \mapsto \widehat{\mathbf{u}}(T)$ is a contraction on $\mathcal{V}$. Set

$$\mathbf{u} = \widetilde{\mathbf{u}} - \widehat{\mathbf{u}}, \quad \mathbf{g} = \widetilde{\mathbf{g}} - \widehat{\mathbf{g}}.$$

Then $\mathbf{u}$ satisfies

$$\dot{\mathbf{u}} = \mathcal{A}\mathbf{u} + \mathcal{B}\mathbf{g}, \quad \mathbf{u}(T) = (I - L_T)\widetilde{\mathbf{u}}^0.$$

Because L_T is a contraction on $\mathcal{V}$ for $T > T_0$, given $\mathbf{u}^0 \in \mathcal{V}$ we may choose $\widetilde{\mathbf{u}}^0$ such that $(I - L_T)\widetilde{\mathbf{u}}^0 = \mathbf{u}^0$. Therefore $\mathbf{g} \in \mathcal{U}$ is a control that steers 0 to $\mathbf{u}^0$ such that the corresponding solution $\mathbf{u} \in C([0,T];\mathcal{V})$. The map $\mathbf{u}^0 \mapsto \mathbf{u}$ is obviously linear and

$$\begin{aligned} \|\mathbf{u}\|_{L^\infty(0,T;\mathcal{V})} &\le \|\widetilde{\mathbf{u}}\|_{L^\infty(0,T;\mathcal{V})} + \|\widehat{\mathbf{u}}\|_{L^\infty(0,T;\mathcal{V})} \\ &\le C(1 + e^{-\omega T})\|(I - L_T)^{-1}\|\|\mathbf{u}^0\|_{\mathcal{V}}. \quad \square \end{aligned}$$

In what follows the specific form of $\mathcal{F}(w)$ will not be required. All that is needed is that $\mathcal{F}$ be Fréchet differentiable with $D\mathcal{F}$ locally Lipschitz continuous, and that $\mathcal{F}(0) = 0$ and $D\mathcal{F}(0) = 0$.

Proof of Proposition 2.3. Since $D\mathcal{F}$ is continuous with $D\mathcal{F}(0) = 0$, there is a real, continuous function $R \mapsto \omega_R$ such that

$$\|\mathcal{F}(\mathbf{z}) - \mathcal{F}(\mathbf{v})\|_{\mathcal{V}} \le \omega_R \|\mathbf{z} - \mathbf{v}\|_{\mathcal{V}}, \quad \forall \mathbf{z}, \mathbf{v} \in S_R,$$

and $\omega_R \to 0$ as $R \to 0$. (In fact, $\omega_R = O(R^2)$ for the specific function $\mathcal{F}$ arising from (2.3).) Define

$$\overline{\mathcal{F}}(t, \mathbf{v}) = \mathcal{F}(\mathbf{v} + \mathbf{u}(t)), \quad 0 \le t \le T, \quad \mathbf{v} \in \mathcal{V}.$$

Then $t \mapsto \overline{\mathcal{F}}(t, \mathbf{v})$ is continuous from $[0,T]$ into $\mathcal{V}$, and $\mathbf{v} \mapsto \overline{\mathcal{F}}(t, \mathbf{v}) : \mathcal{V} \mapsto \mathcal{V}$ is locally Lipschitz continuous, uniformly in $t \in [0,T]$, with Lipschitz constant $\overline{\omega}_R = \omega_{(1+M)R}$. Here M is a constant such that

$$\|\mathbf{u}\|_{L^\infty(0,T;\mathcal{V})} \le M \|\mathbf{u}^0\|_{\mathcal{V}}.$$

Let $\overline{\mathcal{F}}_R(t, \mathbf{v})$ be an $\mathcal{V}$-valued function defined on $[0,T] \times \mathcal{V}$ such that (i) $t \to \overline{\mathcal{F}}_R(t, \mathbf{v})$ is continuous for every $\mathbf{v} \in \mathcal{V}$, (ii) $\mathbf{v} \to \overline{\mathcal{F}}_R(t, \mathbf{v})$ is globally Lipschitz continuous uniformly in t, and (iii) $\overline{\mathcal{F}}_R(t, \mathbf{v}) = \overline{\mathcal{F}}(t, \mathbf{v})$ on $[0,T] \times S_R$. Consider the problem

$$\dot{\mathbf{v}}_R = \mathcal{A}\mathbf{v}_R + \overline{\mathcal{F}}_R(t, \mathbf{v}_R), \quad \mathbf{v}_R(0) = 0. \tag{3.2}$$

From standard theory it follows that this problem has a unique mild solution $\mathbf{v}_R \in C([0,T]; \mathcal{V})$, (In fact, $\mathbf{v}_R$ is Lipschitz continuous on $[0,T]$ and is a strong solution in the sense of [1, Chapter III]. It is strongly differentiable and satisfies (3.2) almost everywhere on $[0,T]$.) We wish to show that

$$\|\mathbf{v}_R\|_{L^\infty(0,T;\mathcal{V})} < R$$

if R is sufficiently small, since then $\overline{\mathcal{F}}_R(t, \mathbf{v}_R(t)) = \overline{\mathcal{F}}(t, \mathbf{v}_R(t))$ for $0 \le t \le T$, and the proposition will be proved.

Let $\mu_R \ge \overline{\omega}_R$ denote the Lipschitz constant for $\overline{\mathcal{F}}_R$. Since A is skew-adjoint we have

$$\begin{aligned}
(\dot{\mathbf{v}}_R(t), \mathbf{v}_R(t))_{\mathcal{V}} &= (\overline{\mathcal{F}}_R(t, \mathbf{v}_R(t)), \mathbf{v}_R(t))_{\mathcal{V}} \\
&\le \mu_R \|\mathbf{v}_R(t)\|_{\mathcal{V}}^2 + (\overline{\mathcal{F}}_R(t,0), \mathbf{v}_R(t))_{\mathcal{V}} \\
&= \mu_R \|\mathbf{v}_R(t)\|_{\mathcal{V}}^2 + (\mathcal{F}(\mathbf{u}(t)), \mathbf{v}_R(t))_{\mathcal{V}},
\end{aligned}$$

hence

$$\begin{aligned}
\|\mathbf{v}_R\|_{L^\infty(0,T;\mathcal{V})} &\le \frac{e^{2\mu_R T} - 1}{\mu_R} \|\mathcal{F}(\mathbf{u})\|_{L^\infty(0,T;\mathcal{V})} \\
&\le \frac{e^{2\mu_R T} - 1}{\mu_R} M\omega_{MR} \|\mathbf{u}^0\|_{\mathcal{V}}.
\end{aligned}$$

Since $\omega_{MR} \to 0$ as $R \to 0$, it follows that there is an $R_0 > 0$ such that

$$\frac{e^{2\mu_R T} - 1}{\mu_R} M\omega_{MR} \|\mathbf{u}^0\|_{\mathcal{V}} < R$$

if $0 < R < R_0$ and $\|\mathbf{u}^0\|_{\mathcal{V}} < R$. □

Proof of Proposition 2.4. The mapping $\mathbf{u}^0 \to \mathbf{u}^0 - K(\mathbf{u}^0) \equiv G(\mathbf{u}^0)$ may be written

$$G(\mathbf{u}^0) = \mathbf{u}^0 + \int_0^T S(T-s)\mathcal{F}(\mathbf{v}(s) + \mathbf{u}(s))ds,$$

where $S(t)$ is the unitary group on $\mathcal{V}$ generated by the skew- adjoint operator $\mathcal{A}$. Therefore the Fréchet derivative of G is

$$DG(\mathbf{u}^0) = I + \int_0^T S(T-s)D\mathcal{F}(\mathbf{v}(s) + \mathbf{u}(s)) \circ [D\mathbf{v}(s) + D\mathbf{u}(s)]ds,$$

provided $D\mathbf{u}$ and $D\mathbf{v}$ exist, where these represent the Fréchet derivatives of the indicated variables with respect to $\mathbf{u}^0$. (We have supressed the writing of the argument $\mathbf{u}^0$ in $D\mathbf{v}(s)$ and in $D\mathbf{u}(s)$; we should, more properly, have written $D\mathbf{v}(\mathbf{u}^0)(s)$ and $D\mathbf{u}(\mathbf{u}^0)(s)$, respectively.)

According to Proposition 2.2(ii), the mapping $\mathbf{u}^0 \to \mathbf{u} : \mathcal{V} \to C([0,T];\mathcal{V})$ is linear and continuous and is, therefore, differentiable with constant derivative. As for $D\mathbf{v}$, we note that $\mathbf{v}$ itself satisfies

$$\mathbf{v}(t) = \int_0^t S(t-s)\mathcal{F}(\mathbf{v}(s) + \mathbf{u}(s))\, ds,$$

so that $D\mathbf{v}$, if it exists, must satisfy

$$D\mathbf{v}(t) = \int_0^t S(t-s)D\mathcal{F}(\mathbf{v}(s) + \mathbf{u}(s)) \circ [D\mathbf{v}(s) + D\mathbf{u}(s)]\, ds.$$

However, since $D\mathcal{F}(\mathbf{z})$ is Lipschitz continuous in $\mathbf{z}$ near $\mathbf{z} = 0$ and both $\mathbf{v}$ and $\mathbf{u}$ converge to zero in $L^\infty(0,T;\mathcal{V})$ as $\mathbf{u}^0 \to 0$ in $\mathcal{V}$, for small $\mathbf{u}^0$ the integral equation for $D\mathbf{v}$ has a unique solution, obtainable by successive approximations, which is continuous near $\mathbf{u}^0 = 0$. This shows that $DG(\mathbf{u}^0)$ exists and is continuous in a neighborhood of $\mathbf{u}^0 = 0$. Since $G(0) = 0$ and $DG(0) = I$, the Implicit Function Theorem [3, Chapter X] then guarantees that G is a homeomorphism of some neighborhood of 0 onto another neighborhood of 0 in $\mathcal{V}$. □

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