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Shape optimal structural design using mixed elements and minimum compliance techniques

by

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In this paper mixed elements are applied to the optimal shape design of two dimensional elastic structures. The mixed finite element model is based on a eight node mixed isoparametric quadratic element, whose degrees of freedom are two displacements and three stresses, per node. The design objective is to minimize compliance of the structure subject to an area constraint. are calculated using formulae obtained by the variational method. The corresponding nonlinear programming problem is solved using the method of sequential convex programming and the modified method of feasible directions, available in the commercial program ADS (Automated Design Synthesis). The formulation developed is applied to the optimal shape design of two dimensional elasticity problems and the advantages and disadvantages of the mixed elements are discussed with reference to applications.

1. Introduction

Shape optimal design differs from structural optimization with fixed since the domain in which state equations are defined constitutes the unknown of the problem. Consequently, in shape optimal the representation of the boundary and the finite element model must be adequate as design proceeds. The use of Bezier curves and B-splines to define the design boundary in shape optimization is commonly used [1]-[2].

The regularity of the finite element mesh is important to obtain numerical solution of the problem with enough accuracy. The poor results in analysis implies bad results in sensitivity analysis, which is one of the essential ingredients to obtain a good optimization solution. Boundary element methods have been used to overcome this problem [3] since distortion in boundary is much smaller than the correspoding distortion of the domain.

In a finite element design model, the necessary quality of the mesh, must be guaranteed by the use of an automatic adaptive mesh generator [2],[4] as design proceeds.

It is well known that efficient shape optimization requires a good sensitivity analysis. Accurate sensitivity values provides a good relation between shape perturbation and corresponding variations of the objective function and constraints. Sensitivity analysis can be calculated from analytic or numerical differentiation of finite element equations. These procedures are generally designated as discret methods and have been considered by Zienkiewicz and Campbell [5], Francavilla et al. [6] and Braibant and Fleury [7], among others.

Alternatively the gradients can be obtained by analytical expressions for sensitivity of objective and constraint functions or from the explicit expressions of optimality conditions. Haug et al. [8], Dems and Mróz [9] and Banichuk [10] have been developing this technique designated as continuum or variational method.

In this paper, formulae obtained by the variational method are used to calculate sensitivities of objective and constraint functions of the optimiztion problem. The design objective is to minimize compliance of the structure subjected to an area constraint. In this formulation values obtained on the boundary, namely displacements and stresses are required. Consequently, the employment of displacement finite elements has been shown to be inefficient, since the results obtained on the boundary are not, generally, accurate enough.

Mixed finite elements offer advantages over displacement finite elements, since in general the corresponding stresses are more accurate. Thus it is expected that mixed finite elements are more suitable to optimal design than displacement finite elements.

Only recently the mixed finite element has been applied to the optimal design of structures. The sensitivity analysis of beams and plates with static, dynamic and stability constraints, based on mixed formulations had been developed by Leal and Mota Soares [11]-[12]. The theory has been applied to minimum weight design of plates, subject to constraints on displacements, stresses, natural frequences or buckling stresses [13]. Also, Rodrigues [9] developed a variational formulation for shape optimal design of a two-dimensional linear elastic structures, using four node, isoparametric mixed finite element based on the functional of Hu-Washizu to interpolate the stress, strain and displacement fields.

The mixed finite element used in this paper, is the isoparametric quadratic element based on de Hellinger-Reissner's functional, with 8 nodes and 2 displacements and three stresses as degrees of freedom per node. The perturbation field on the boundary is interpolated with linear design elements. The mesh regularity of the finite element discretization is guaranteed by a mesh regenerator.

2. Mixed elements in bi-dimensional elasticity

Mixed elements are based on the Hellinger-Reissner principle. For elasticity this functional is given by

$$\begin{aligned} V_R(u_i,\sigma_{ij}) &= \int_\Omega \left[\tfrac{1}{2}\sigma_{ij}(u_{i,j}+u_{j,i}) - U_0^*(\sigma_{ij}) - b_i u_i\right] d\Omega \\ &- \int_{\Gamma_t} u_i \bar{t}_i d\Gamma - \int_{\Gamma_u} (u_i - \bar{u}_i) t_i d\Gamma \qquad \text{i,j=1,2} \end{aligned} \tag{1}$$

where Ω is the domain, Γ_t is the with known forces, Γ_u is the boundary with displacements and the superscript bar indicates the forces and displacements in the boundary, x_i are the global coordinates, u_i are displacements, σ_{ij} are stresses, b_i are body forces and t_i are tractions on boundary. U_0^* is the complementary energy density. Throughout this paper index notation and summation convention is used; besides, comma denotes differentiation with respect to $\mathbf{x}$, i.e., $u_{j,i} = \partial u_j / \partial x_i$.

Assuming that the geometric boundary conditions are satisfied in Γ_u

$$u_i = \bar{u}_i \tag{2}$$

the Hellinger-Reissner functional (1) for isotropic and linear elastic materials can be written in matricial form:

$$V_R(\mathbf{u}, \sigma) = \int_\Omega \left(\sigma^T \Delta \mathbf{u} - \frac{1}{2} \sigma^T \mathbf{C} \sigma - \mathbf{u}^T \mathbf{b} \right) d\Omega - \int_\Gamma \mathbf{u}^T \mathbf{t} d\Gamma \tag{3}$$

where

$\mathbf{u} = [\ u_1 \quad u_2\ ]^T$ is th vector of displacements

$\mathbf{b} = [\ b_1 \quad b_2\ ]^T$ is the vector of body forces

$\mathbf{t} = [\ t_1 \quad t_2\ ]^T$ is the vector of tractions on boundary

$\sigma = [\ \sigma_{11} \quad \sigma_{22} \quad \sigma_{12}\ ]^T$ is the vector of stresses

$$\Delta = \begin{bmatrix} \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x_1 \end{bmatrix}$$ is a differential operator

$$\mathbf{C} = \tfrac{1}{E^*} \begin{bmatrix} 1 & -\mathcal{V}^* & 0 \\ -\mathcal{V}^* & 1 & 0 \\ 0 & 0 & 2(1+\mathcal{V}^*) \end{bmatrix}$$ is the matrix of elastic properties of material

with

$$\begin{aligned} E^* &= E \\ \mathcal{V}^* &= \mathcal{V} \end{aligned} \tag{4}$$

in plane stress and

$$\begin{aligned} E^* &= \tfrac{E}{1-\mathcal{V}^2} \\ \mathcal{V}^* &= \tfrac{\mathcal{V}}{1-\mathcal{V}} \end{aligned} \tag{5}$$

in plane strain; E is Young's modulus and $\mathcal{V}$ is Poisson's ratio of the material.

Representing displacement and stress fields by

$$\mathbf{u} = \mathbf{N}\ \mathbf{q}_e \tag{6}$$

$$\sigma = \mathbf{L}\ \mathbf{m}_e \tag{7}$$

where $\mathbf{q}_e$ and $\mathbf{m}_e$ are the displacement and stress degrees of freedom of the element and $\mathbf{N}$ and $\mathbf{L}$ are the shape function matrices, given by

$$\begin{aligned} \mathbf{N} &= [\ \mathbf{N}_1 \ \ \mathbf{N}_2 \ \ \cdots \ \ \mathbf{N}_8\ ] \\ \mathbf{L} &= [\ \mathbf{L}_1 \ \ \mathbf{L}_2 \ \ \cdots \ \ \mathbf{L}_8\ ] \end{aligned} \tag{8}$$

with

$$\mathbf{N}_i = \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix}$$
$$\mathbf{L}_i = \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix}$$

where N_i are the corresponding element interpolation functions.

Introducing (6) and (7) in the functional (3) gives

$$\begin{aligned} V_R = \textstyle\sum_e \ \ (& \mathbf{m}_e^T \int_{\Omega_e} \mathbf{L}^T \Delta \mathbf{N} d\Omega \mathbf{q}_e - \tfrac{1}{2}\mathbf{m}_e^T \int_{\Omega_e} \mathbf{L}^T \mathbf{C}\ \mathbf{L}\ d\Omega \mathbf{m}_e \\ & -\mathbf{q}_e^T \int_{\Omega_e} \mathbf{N}^T \mathbf{b}\ d\Omega \ - \ \mathbf{q}_e^T \int_{\Gamma_e} \mathbf{N}^T \mathbf{t}\ d\Gamma\ ) \end{aligned} \tag{9}$$

or

$$V_R = \sum_e \left(\mathbf{m}_e^T\ \mathbf{H}_e\ \mathbf{q}_e \ - \ \frac{1}{2}\mathbf{m}_e^T\ \mathbf{G}_e\ \mathbf{m}_e \ - \ \mathbf{q}_e^T\ \mathbf{p}_{ve} \ - \mathbf{q}_e^T\ \mathbf{p}_{se} \right) \tag{10}$$

where

$$\begin{aligned} &\mathbf{G}_e = \textstyle\int_{\Omega_e} \mathbf{L}^T\ \mathbf{C}\ \mathbf{L}\ d\Omega && \text{is the element flexibility matrix} \\ &\mathbf{H}_e = \textstyle\int_{\Omega_e} \mathbf{L}^T \ \Delta\ \mathbf{N}\ d\Omega && \text{is the element flexibility/stiffness matrix} \\ &\mathbf{p}_{ve} = \textstyle\int_{\Omega_e} \mathbf{N}^T\ \mathbf{b}\ d\Omega && \text{is the element vector of body forces} \\ &\mathbf{p}_{se} = \textstyle\int_{\Gamma_e} \mathbf{N}^T\ \mathbf{t}\ d\Gamma && \text{is the element vector of boundary forces} \end{aligned} \tag{11}$$

Representing by $\mathbf{f}_e$ the element force vector, the equation (10) can be written as

$$V_R \ = \ \sum_e \left(\mathbf{m}_e^T\ \mathbf{H}_e\ \mathbf{q}_e \ - \ \frac{1}{2}\ \mathbf{m}_e^T\ \mathbf{G}_e\ \mathbf{m}_e \ - \ \mathbf{q}_e^T \mathbf{f}_e \right) \tag{12}$$

The stationary condition of the Reissner's functional (12) leads to an equation at element level:

$$\begin{bmatrix} -\mathbf{G}_e & \mathbf{H}_e \\ \mathbf{H}_e^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{m}_e \\ \mathbf{q}_e \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_e \end{bmatrix} \tag{13}$$

Assembling in the usual way equations (13) for all elements, we obtain the global equation for bi-dimensional elasticity

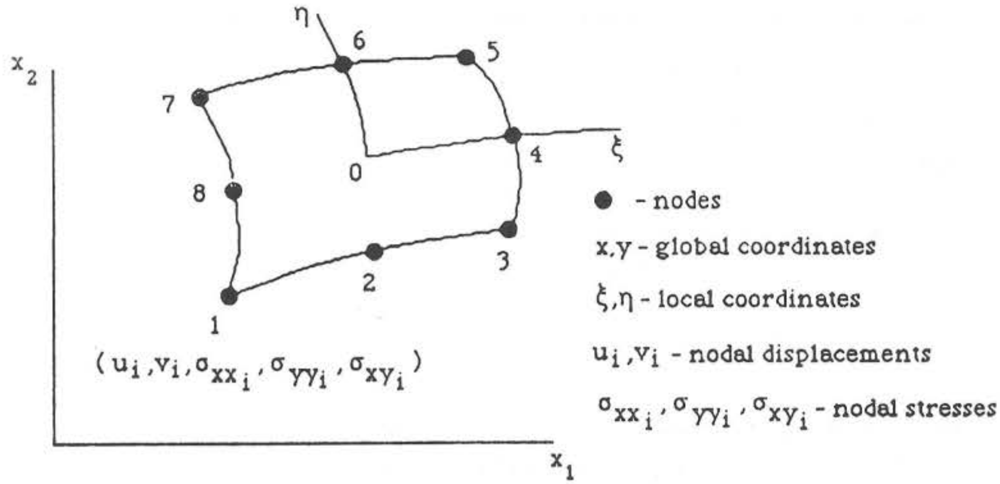

Figure 1. – Isoparametric quadratic mixed element

$$\begin{bmatrix} -\mathbf{G} & \mathbf{H} \\ \mathbf{H}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{m} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix} \tag{14}$$

where $\mathbf{m}$ is the global vector of stress degrees of freedom and $\mathbf{q}$ is the global vector of displacements degrees of freedom.

The mixed element used in this paper has 8 nodes and 2 displacements and 3 stresses degrees of freedom per node. This mixed isoparametric quadratic element is represented in Fig. 1.

The corresponding element interpolation functions are:

$$N_i = \frac{1}{4}(1+\xi_0)\,(1+\eta_0)\,(\xi_0+\eta_0-1) \tag{15}$$

for the nodes 1,3,5 and 7,

$$N_i = \frac{1}{2}(1-\xi^2)\,(1+\eta_0) \tag{16}$$

for the nodes 4 and 8,

$$N_i = \frac{1}{2}(1+\xi_0)\,(1-\eta^2) \tag{17}$$

for the nodes 2 and 6.

In these expressions we have

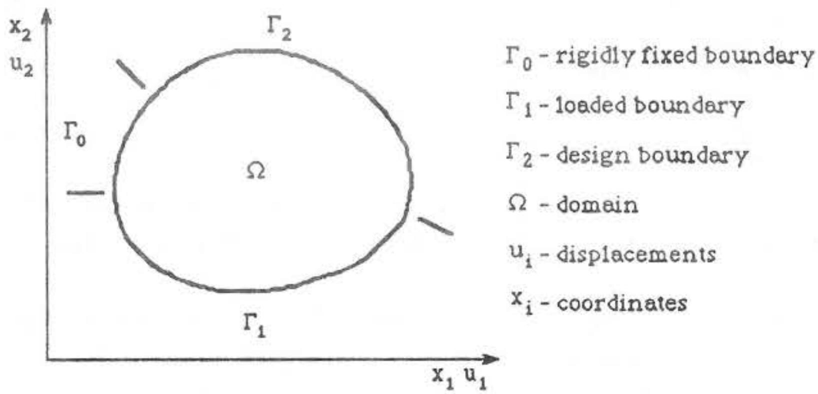

Figure 2. – **Bi-dimensional elastic structure**

$$\xi_0 = \xi_i\,\xi \qquad \eta_0 = \eta_i\,\eta \tag{18}$$

where ξ_i and η_i are the local coordinates or the node i.

The element matrices are calculated substituting equations (8 and 15-18) into equations(11). The matrices are integrated numerically using 3*3 and 3 Gaussian points.

3. Shape optimal design based on minimum compliance

In the optimization of elastic structures, the compliance, which is the work done by external force, has been used to measure the global stiffness of the structures.

For the two dimensional linear elastic structure described in Fig. 2, the objective is to determine the domain Ω such that the compliance is minimized. Γ_0 is the boundary with known displacements, Γ_1 is the boundary with known tractions and Γ_2 is the design boundary. We assume that we have neither body forces nor boundary tractions on Γ_2 and that Γ_0 is held rigidly fixed ($u_i = 0$).

The problem is defined as the minimization of the compliance

$$\Psi_0 = \frac{1}{2}\int_{\Gamma} t_i u_i \, d\Gamma \tag{19}$$

subject to the area constraint

$$\Psi_1 = \int_{\Omega} d\Omega - A \leq 0 \tag{20}$$

where A is the pre-defined area. The minimization problem is also subject to the condition for equilibrium which is defined by the stationarity condition of Reissner's functional (1.

The solution for this nonlinear programming problem (19-20) needs the first variation of objective and constraint functionals. Using the general formulation of Haug *et al.* [8], based on representing the modifications in design by a velocity field and applying the material derivative concept, the first variation of compliance (19), for unloaded boundaries and without body forces, is

$$\delta\Psi_0 = -\int_{\Gamma_2} U V_n \, d\Gamma \tag{21}$$

where U is the strain energy density and V_n is the normal perturbation field of the domain, defined on Γ_2. The first variation of the area constraint (20) is

$$\delta\Psi_1 = \int_{\Gamma_2} V_n \, d\Gamma \tag{22}$$

The first variation of the compliance (21) can be efficiently obtained with mixed finite elements. Since mixed elements can provide better boundary results, we may expect more accurate sensitivity results than obtained by displacement finite elements.

The specific strain energy at one boundary point of an unloaded boundary is given by

$$U = \frac{1}{2}\sigma_{ss}\epsilon_{ss} = \frac{1}{2E^*}\sigma_{ss}^2 \tag{23}$$

where σ_{ss} is the tangential stress, which is the first invariant of the stress tensor

$$\sigma_{ss} = \sigma_{11} + \sigma_{22}. \tag{24}$$

The mixed element used is an isoparametric eight node mixed element with two displacement and three stress degrees of freedom per node. Thus, the variation of tangential stress on the element side is quadratic. The boundary geometry is described by linear design elements. The nodes of each design element are coincident with the extreme nodes of one side of the mixed element, as shown

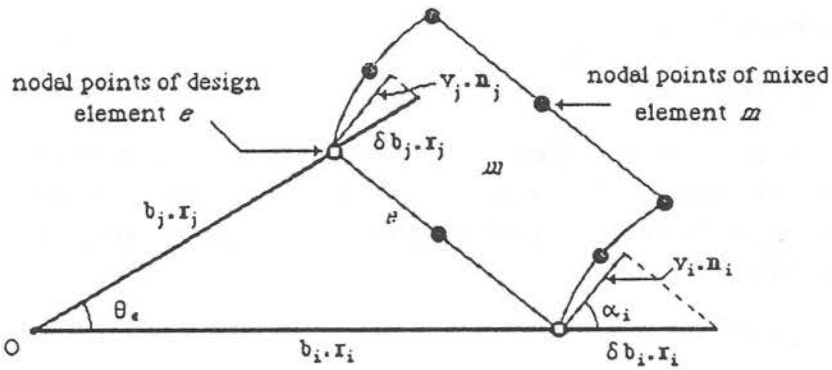

Figure 3. – Design variables and its correlation with mixed finite element mesh

in Fig. 3. The design variables are defined as the norm of the position vector of the interpolation nodes with respect to a pre-defined origin 0.

The tangential stress σ_{ss} on the side of mixed element m, can be described by the quadratic shape functions L_i as

$$\sigma_{ss} = L_i \sigma_{ss_i} \tag{25}$$

where σ_{ss_i} is the tangential stress of each nodal point and the shape functions L_i are:

$$\begin{aligned} L_1(\xi) &= \tfrac{1}{2}(\xi^2 - \xi) \\ L_2(\xi) &= (1 - \xi^2) \\ L_3(\xi) &= \tfrac{1}{2}(\xi^2 + \xi) \end{aligned} \tag{26}$$

where ξ is the local (tangential) coordinate of the side of the element.

The normal boundary perturbation V_n on linear geometric design element e, can be described by the linear shape functions N_i^L as

$$V_n = N_i^L v_i \tag{27}$$

where v_i is the normal perturbation of each nodal point and the shape functions N_i^L are:

$$\begin{aligned} N_1^L(\xi) &= \tfrac{1}{2}(1 - \xi) \\ N_2^L(\xi) &= \tfrac{1}{2}(1 + \xi) \end{aligned} \tag{28}$$

where ξ is the local (tangential) coordinate of the element.

As shown in Fig. 2

$$v_i = \delta b_i(\mathbf{r}_i \cdot \mathbf{n}_i) \tag{29}$$

where, $\mathbf{r}_i$ and b_i are, respectively, the unit vector and the norm of the position vector of the respective interpolation node i, δb_i is the variation of design variable b_i and $\mathbf{n}_i$ is the unit normal vector to the boundary in node i. So, the normal perturbation field is

$$V_n = N_i^L \, \delta b_i \, cos\, \alpha_i \tag{30}$$

Introducing (23) and (30) in expression (21) we obtain, for geometric element e, in matricial form

$$\delta\Psi_0 = -\frac{l_e}{4E^*}\,(\mathbf{s}^T\,\mathbf{M}\,\mathbf{v}) \tag{31}$$

where l_e is the element length and

$$\begin{aligned} \mathbf{s} &= [\,\sigma_{ss_1}^2\ \sigma_{ss_2}^2\ \sigma_{ss_3}^2\ 2\sigma_{ss_1}\sigma_{ss_2}\ 2\sigma_{ss_1}\sigma_{ss_2}\ 2\sigma_{ss_2}\sigma_{ss_3}\,]^T \\ \mathbf{M} &= \int_{-1}^{+1} \mathbf{L}^*\,\mathbf{N}^{L^T}\, d\xi \end{aligned} \tag{32}$$

with

$$\begin{aligned} \mathbf{L}^* &= [\ L_1^2 \quad L_2^2 \quad L_3^2 \quad L_1L_2 \quad L_1L_3 \quad L_2L_3\ ]^T \\ \mathbf{N}^L &= [\ N_1^L \quad N_2^L\ ]^T \end{aligned}$$

and, finally,

$$\mathbf{v} = [\,\delta b_i \, cos\, \alpha_i \quad \delta b_j\, cos\, \alpha_j\,]^T$$

where i and j represent nodal points of element e. Equation (32) is numerically integrated by 3 Gauss points.

The first variation of the area constraint (22) can be obtained for element e, as shown in Fig. 4, by the expression:

$$\delta\Psi_{l_e} = \int_{\Gamma_e} V_n \, d\Gamma \, = \, \frac{1}{2}\, sen\, \theta_e \, \left(b_i\,\delta b_j \, + \, b_j\,\delta b_i\right) \tag{33}$$

The nonlinear programming problem 19-20) is solved using the method of sequential convex programming [15] and the modified method of feasible directions, available in the commercially available programme ADS (Automated Design Synthesis) [16].

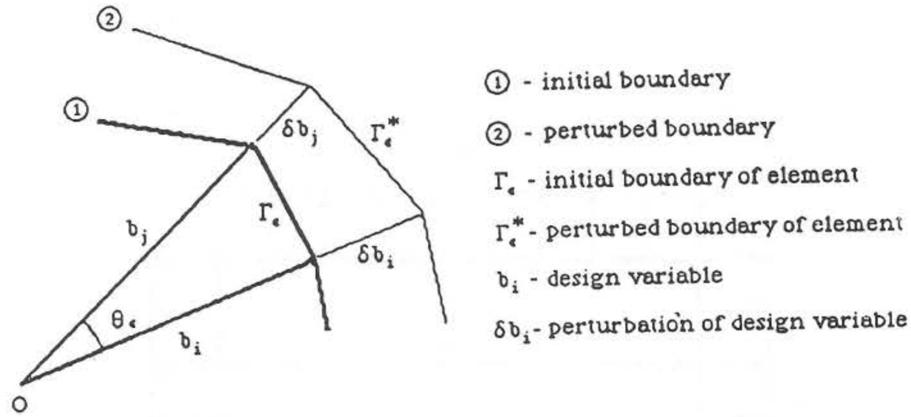

Figure 4. – Influence of perturbation of design variables in the variation of the area

4. Applications

Consider the problem represented in Fig. 5

For the infinite plate the analytical solution [10] is a circular hole, when the load $T_1 = T_2$ and an elliptical hole with a semi-axis ratio equal to the ratio of the external applied forces when $T_1 \neq T_2$. Since it is known the analytical solution of the problem, the example treated is a good test to check the numerical procedure developed.

The problem data are the following:

- plane stress

- material elastic properties:

$E = 200GPa$

$\nu = 0.3$

- applied forces:

first load case: $T_1 = T_2 = 100\ MPa$

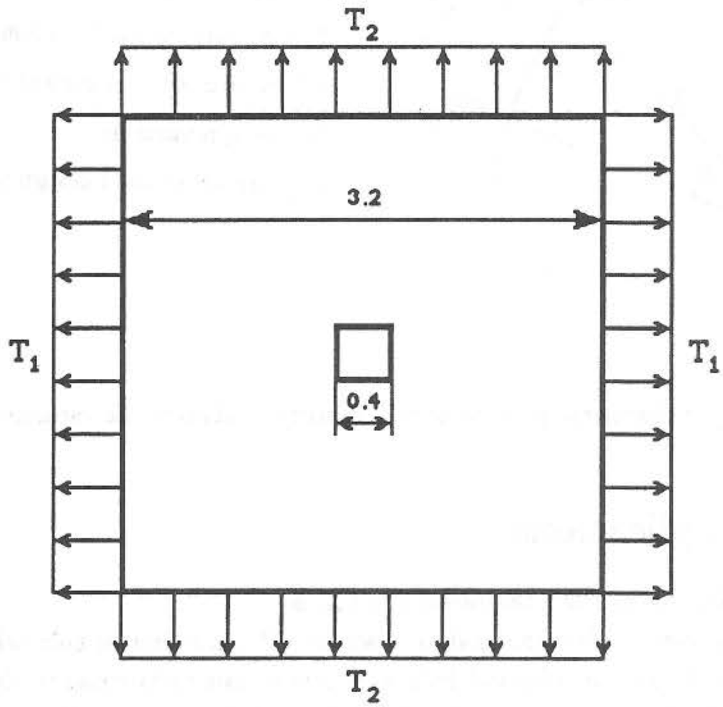

Figure 5. – Square plate with hole

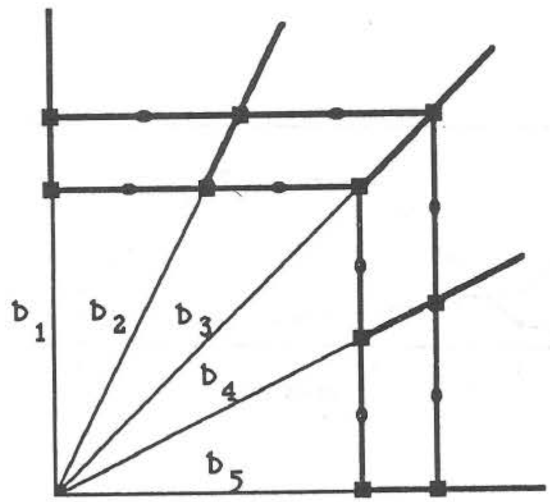

Figure 6. - Design variables

second load case $T_1 = 75\ MPa$ $T_2 = 100\ MPa$

- objective function is the compliance

- maximum admissible area is equal to the initial area of the hole.

The problem is solved modelling 1/4 of the plate with 16 mixed quadratic elements. It is used 5 design variables as represented in Fig.6.

For the first load case, $T_1 = T_2$, results are presented in Fig. 7 to 9.

In Fig. 7, are shown the evolution of the compliance and area constraint. The constraint value presented is obtained by the expression:

$$\Psi_{1rep} = \frac{\Psi_1}{\int_\Omega d\Omega} * 10^6 \tag{34}$$

In Fig. 8, are shown initial and final meshes. The mesh regenerator updates only 8 elements of the sub-domain near the design boundary.

The evolution of hole design is presented in Fig. 9.

For the second load case, $T_1 = 0.75T_2$, results are presented in Fig. 10 to 12. In Fig.fig10, are shown the evolution of the compliance and area constraint. The constraint value presented is obtained by the expression (34).

In Fig. 11, are shown initial and final meshes. The mesh regenerator updates only 8 elements of the sub-domain near the design boundary.

The evolution of hole design is presented in Fig. 12.

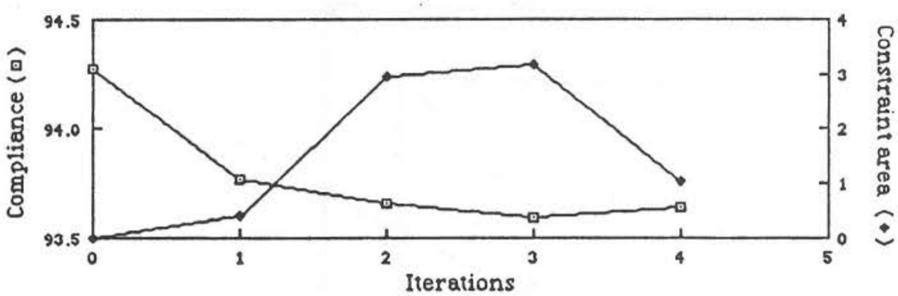

Figure 7. – Evolution of objective and constraint functions. First load case.

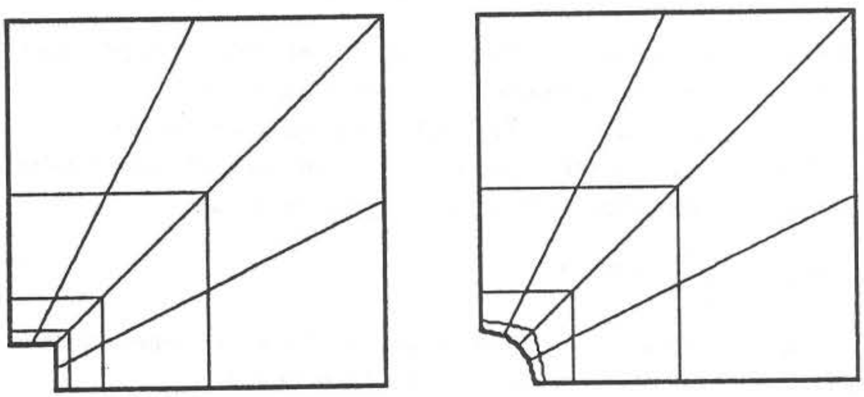

Figure 8. – Initial and final mesh. First load case

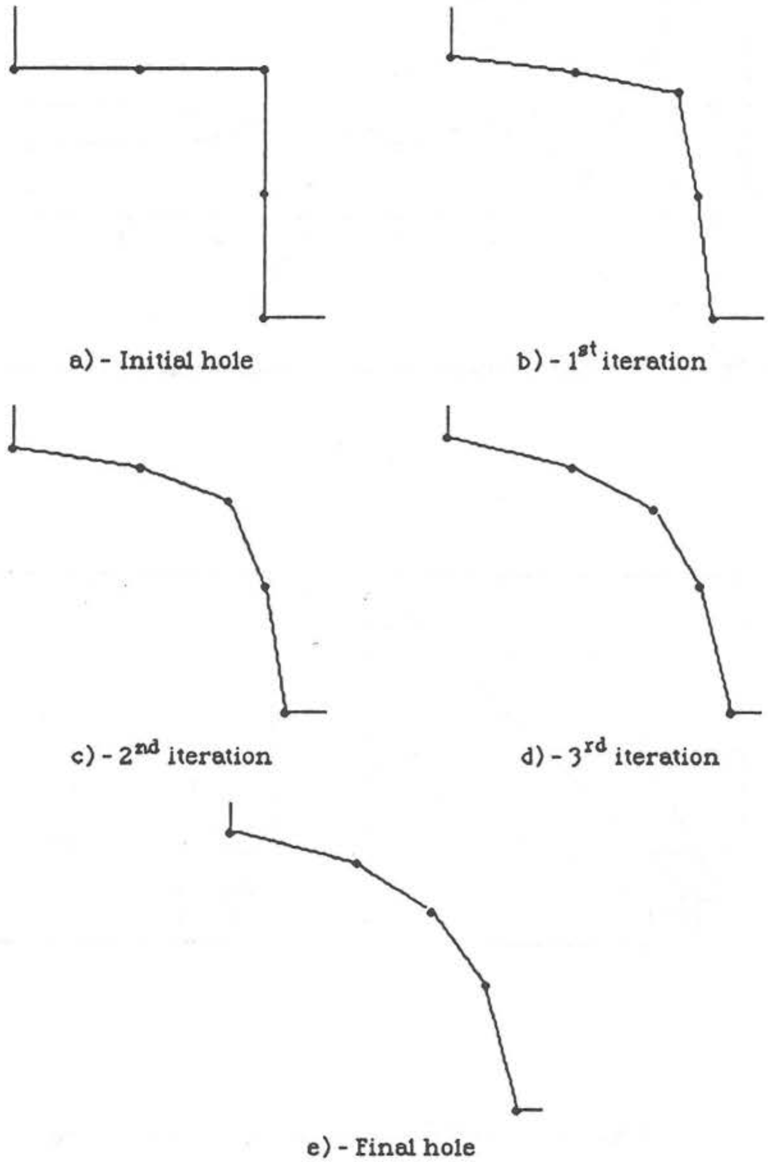

Figure 9. – Evolution of hole design. First load case

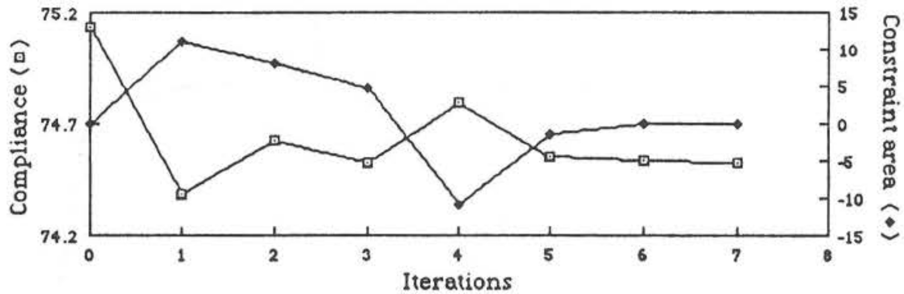

Figure 10. – Evolution of objective and constraint functions. Second load case.

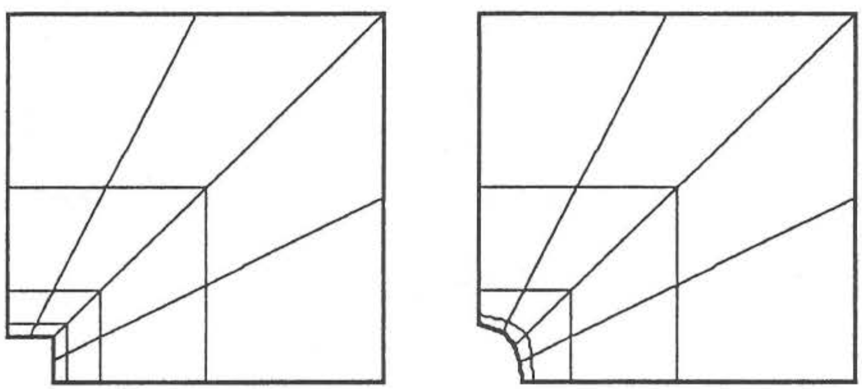

Figure 11. – Initial and final mesh. Second load case

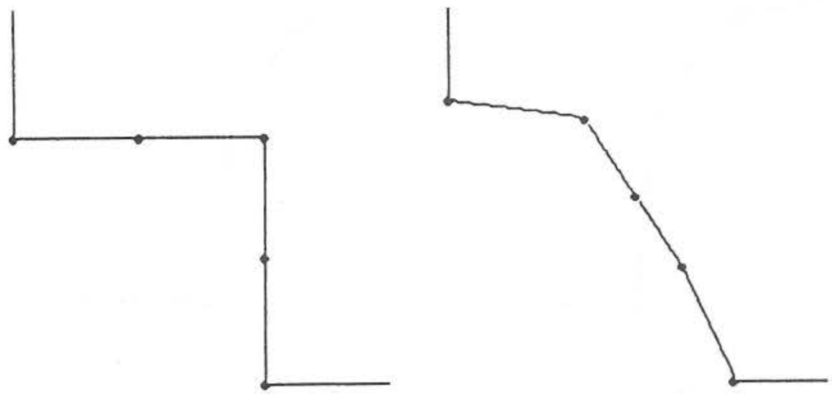

Figure 12. a) – initial hole b) – 1^{st} iterations

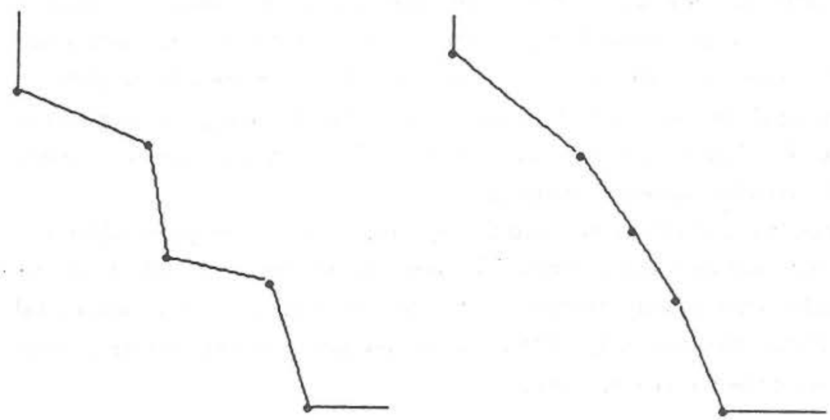

Figure 12. c) – 2^{nd} iteration d) – 3^{rd} iteration

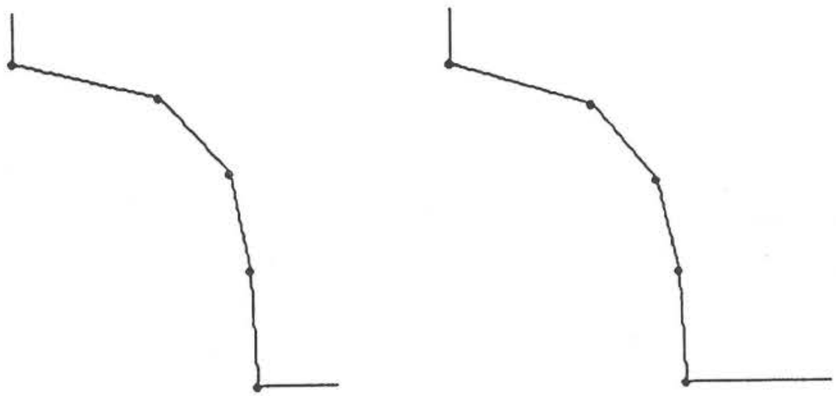

Figure 12. e) – 4^{th} iteration f) – $5^{th}, 6^{th}$ iterations and final hole

Figure 12. – Evolution of hole design. Second load case.

5. Concluding remarks

In the applications, the final design is almost equal to the analytical solution for infinite plates[10]. For bi-axial equal loads, final design is an excellent approximation to the circle, presenting in design variables a maximum variation of 0.16% . The mean stress factor on the boundary of the hole is 2.025 which has an 1% error comparing with the analytical solution of the infinite plate.

For bi-axial unequal loads, final design is a polygon having a strong resemblance to the ellipse. The semi-axis ratio is 0.73 which compares favourably with 0.75 given by analytical solution.

Considering that the model used is very simple, only 5 design variables and 4 linear elements describing design boundary, the results are excellent. It can be concluded that mixed elements can be efficient in shape optimal structural design. Mixed elements may offer some advantages over displacements finite element since the stresses are more accurate.

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