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Eigenvalues optimization - new view about the old problem

by

T. LEKSZYCKI

Institute of Fundamental Technological Research
Polish Academy of Sciences
ul. Świętokrzyska 21
00-049 Warszawa, Poland

Optimal design problem of vibrating structures with respect to their eigenvalues is under consideration. In reality a dissipation of energy is always present in mechanical structures therefore some important properties of viscoelastic systems are presented and necessity of new formulation of the problem is pointed out. The modified formulations with taking into account damping phenomenon are proposed and simple illustrative examples are discussed.

Introduction

There are many papers devoted to the problem of optimal design of natural vibration frequencies of elastic structures, see for example review papers by Olhoff at al, [1, 2, 3] and Mróz [4]. But perfect elastic material in reality does not exist and only few structural materials as e.g. aluminium alloys are characterized by very low dissipation of energy during mechanical vibration. In contrast, for numerous popular materials, as for example different composites, e.g. polymers with glass or carbon fiber, concrete, wood, various synthetic materials e.g. rubber, grounds and many others the dissipation of energy is notable and often has considerable influence on the motion of the structure. The present paper is devoted to the problem of optimization of eigenfrequencies of vibrating structures with damping. The aim is to show that in some situations the new formulation in comparision to the "elastic case", of synthesis is necessary. The particular but important case is considered in order to discuss some interesting dynamic properties of viscoelastic systems. These properties are discussed for the class of structures with material damping proportional to elasticity. This proportionality is described in detail in the following section. With such assumptions done one can derive a direct relation between complex eigenvalues of viscoelastic structure with "proportional damping", and eigenfrequencies of analogous system, but with no damping at all. This relation is discussed in detail for selected, simple model of viscoelastic material, in order to present important features of structures with damping arguing for necessity of new formulation of optimization problem. In spite of such simplification considered here case may be of some practical importance. Some new formulations of synthesis problem are discussed in the last part of the paper and a simple illustrative example is presented.

1 Some basic relations

Let us consider the body described by the equation of motion written in the form,

$$L^a \underline{Q}(x,t) + M\underline{\ddot{u}}(x,t) = 0 \qquad x \in \Omega \tag{1}$$

where $\underline{Q}(x,t)$ represents the generalized stresses, $\underline{u}(x,t)$ - displacements, M is the mass operator and L^a is the adjoint operator to linear differential operator L used in formula describing the relation between $\underline{u}(x,t)$ and generalized strains $\underline{q}(x,t)$. Ω is the system domain and dot denotes differentiation due to time. The relation between generalized strains and displacements has the form,

$$\underline{q}(x,t) = L\underline{u}(x,t) \tag{2}$$

The constitutive equation for the linear viscoelastic material may be written in differential form as follows,

$$D_Q^{(N)}\underline{Q}(x,t) = D_q^{(M)}\underline{q}(x,t) \qquad , \tag{3}$$

where $D_Q^{(N)}$ and $D_q^{(M)}$ denote linear differential operators,

$$D_Q^{(N)} = \sum_{k=0}^{N} \alpha_k \frac{d^k}{dt^k} \tag{4}$$

$$D_q^{(M)} = \sum_{k=0}^{M} \beta_k \frac{d^k}{dt^k} \tag{5}$$

and coefficients α_k and β_k in general, may depend on space coordinates x. A more general constitutive relation in integral form may be assumed instead of (3),

$$\underline{Q}(x,t) = \int_0^\infty k(\tau)\frac{d\underline{q}(x,t)}{dt}d\tau \tag{6}$$

where $k(\tau)$ is the relaxation function. Let us quote some selected relations valid for an arbitrary system made of material described with use of one of constitutive relations (3) or (6) in order to point out the difference between the general case and particular one discussed in the following sections. Let us assume the solution of equation of motion in the form,

$$\underline{U}(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t} u_j(x) \tag{7}$$

where $u_j(x) = u_{Rj}(x) + iu_{Ij}(x)$ is the comlex eigenfunction associated with the complex eigenvalue $\lambda_j = \alpha_j + i\omega_j$ and $i = \sqrt{-1}$. The real part α_j of λ_j represents the damping of vibrations and the imaginary part ω_j is the circular velocity related to eigenfrequency f_j by the relation $\omega_j = 2\pi f_j$. Substitution of (7) into the relations written above leads to the following equations governing the eigenproblem of viscoelastic systems. The index "j" is dropped below for the simplicity.

The complex state equation,

$$L^a Q(x) + \lambda^2 M u(x) = 0 \qquad x \in \Omega \tag{8}$$

The strain - displacement relation,

$$q(x) = Lu(x) \tag{9}$$

The complex constitutive equation,

$$Q(x) = Kq(x) \tag{10}$$

where $K = K_R + iK_I$ is the complex stiffness operator, described for the material model (3) in the form,

$$k = \frac{\sum_{k=0}^{M} \beta_k \lambda^k}{\sum_{k=0}^{N} \alpha_k \lambda^k} \tag{11}$$

For the model (6) it has the following form,

$$k = \int_0^{\infty} k(\tau) e^{\lambda \tau} d\tau \tag{12}$$

To complete the set of equations governing the eigenproblem let us introduce the static (13) and kinematic (14) boundary conditions on the parts Ω_1 and Ω_2 of external surface Ω_0 of the structure domain Ω,

$$B^a Q(x) = 0 \qquad x \in \Omega_1 \quad , \tag{13}$$

$$Bu(x) = 0 \qquad x \in \Omega_2 \quad , \tag{14}$$

where $\Omega_1 + \Omega_2 = \Omega_0$, and Ω_1 is a part of Ω_0 over which boundary conditions are described in terms of external forces but Ω_2 over which boundary conditions are described in terms of displacements. One can also consider the boundary conditions including nonstructural masses or nonrigid supports, then a simple modification of relations discussed in the following is necessary. The operators B and B^a defined on surface Ω_0 follow from the definition of adjoint operator L^a,

$$< Lu, KLu >_{\Omega} = < u, L^a KLu >_{\Omega} + < Bu, B^a KLu >_{\Omega_0} \tag{15}$$

where $< u,v >$ denotes scalar product of two complex vector functions $u(x)$, $v(x)$

$$< u(x), v(x) >_{\Omega} = \int_{\Omega} u(x) \bar{v}(x) dx \quad , \tag{16}$$

and $\overline{v}(x)$ is conjugate function of $v(x)$.
Let us define the complex potential energy $\Pi(u)$, as

$$\Pi(u) = U + iD + \lambda^2 T \tag{17}$$

where the complex elastic, dissipated and kinetic energies are introduced as follows,

$$U = \frac{1}{2} < K_R Lu(x) \ , \ L\bar{u}(x) >_{\Omega} \tag{18}$$

$$D = \frac{1}{2} < K_I L u(x) \, , \, L\bar{u}(x) >_\Omega \tag{19}$$

$$T = \frac{1}{2} < M u(x) \, , \, \bar{u}(x) >_\Omega \tag{20}$$

It is possible to show, see [5, 6, 7], that among all kinematical admissible displacement fields, those satisfying static conditions render the complex potential energy stationary. On the other hand, it is very easy to verify, that from the stationary condition of Π,

$$\delta\Pi(u) = 0 \tag{21}$$

result static relations i.e. state equation (8) and boundary condition (13), (stationary of complex functional has the meaning here that the stationary condition of real and imaginary parts of considered functional are satisfied simultaneously). One can also prove the stationary property of some functional describing complex eigenvalue. Let us assume, there exists some functional $\lambda(u)$ provided by the relation,

$$< KLu(x), L\overline{u}(x) >_\Omega + < \lambda^2 M u(x), \overline{u}(x) >_\Omega = 0 \tag{22}$$

where K is of course a function of λ. It has been shown in [5, 6, 7] that in the class of kinematical admissible complex functions $u(x)$, functions that satisfy state equation (8) and static boundary condition (13), i.e. eigenfunction related to eigenvalue λ, make functional $\lambda(u)$ stationary and this stationary value is equal to complex eigenvalue λ. The stationary property of the functional $\lambda(u)$ is in force except the case of critical damping in the system. Similarly as for complex potential energy, one can also prove that the complex equilibrium equation (8) with the static boundary condition (13) follow from stationary condition of $\lambda(u)$. This theorem may be useful in formulating the problem of optimization of eigenvalues of elastic systems with damping. Let us restrict our considerations and concentrate on the systems made of Voigt viscoelastic material. The constitutive equation is a particular case of general relation (3), and has the form,

$$\underline{Q}(x,t) = K_e \underline{q}(x,t) + K_v \dot{\underline{q}}(x,t) \tag{23}$$

The complex stiffness operator follows from the eq.(23),

$$K = K_R + iK_I = K_e + \lambda K_v \tag{24}$$

and the complex potential energy can be introduced now in the slightly different, more convenient, form

$$\Pi(u) = B_1 + \lambda B_2 + \lambda^2 T \tag{25}$$

where

$$B_1 = \frac{1}{2} < K_e Lu(x), L\overline{u}(x) >_\Omega, \quad B_2 = \frac{1}{2} < K_v Lu(x), L\overline{u}(x) >_\Omega \tag{26}$$

The generalized Rayleigh quotient can be defined, see [5, 6, 7], as

$$\lambda(u) = \frac{-B_2 + \sqrt{B_2^2 - 4TB_1}}{2T} \tag{27}$$

Let as restrict considerations to the very special case and assume that in the considered structure the "viscosity" is proportional to the "elasticity" with the proportionality factor c. That means, the following relation between coefficients K_e and K_e exists,

$$K_v = cK_e \tag{28}$$

The complex stiffness now has the form,

$$K = K_e(1 + c\lambda) \tag{29}$$

and the functional $\lambda(u)$ can be rewritten,

$$\lambda = -\frac{cB_1}{2K} \pm i\sqrt{\frac{B_1}{K} - [\frac{cB_1}{2K}]^2} \quad . \tag{30}$$

On the other hand, for this particular case, the complex eigenfunction $u(x)$ may be introduced in the form,

$$u(x) = zu_e(x) \tag{31}$$

where z represents some complex constant, and $u_e(x)$ is the eigenfunction of similar but elastic structure. Finally, the following relation between complex eigenvalue λ of viscoelastic system and eigenvalue ω_e of analogous system with no damping results from eqs. (30) and (31),

$$\lambda = -\frac{c\omega_e^2}{2} \pm i\frac{1}{2}\sqrt{(4 - c^2\omega_e^2)\omega_e^2} \quad . \tag{32}$$

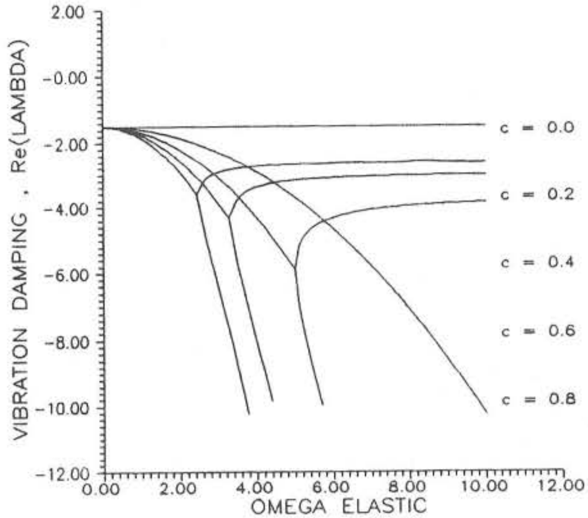

Figure 1: Real part of eigenvalue of viscoelastic system as a function of eigenvalue of related elastic system.

The dependence between $Re(\lambda)$ and ω_e is presented in Fig. 1 for different values of damping parameter c. Fig. 2 presents $Im(\lambda)$ as a function of ω_e. It follows from eq.(32) and Fig. 2 that for the Voigt material there exists some critical value ω_{cr}^0 of ω_e such, that all of the eigenvalues of viscoelastic system related to ω_e greather then ω_{cr}^0 are real.

The other important conclusion is that considered here structures, in contrast to the pure elastic systems, are characterized only by ***finite number of resonances***. This conclusion is of some practical importance, in particular in numerical analysis of vibrating systems as well as in formulating synthesis problems. One can also derive analogous relations for the other, more complex, material models. Then, the situation may be even more interesting and complicated for the reason that several domains depending on material constants are possible, where the eigenfrequencies of elastic system are associated with overdamped property of viscoelastic one. On the other hand for some other models of viscoelastic materials the "overdaming property" may not exist at all.

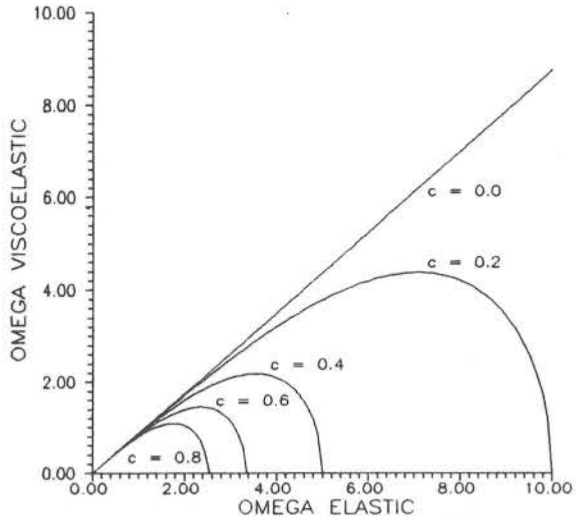

Figure 2: Imaginary part of eigenvalue of viscoelastic system as a function of eigenvalue of related elastic system.

2 Optimization of complex eigenvalues of vibrating elastic systems with damping.

These observations argue that modified or new formulation of optimization problem with respect to natural frequencies of structures with damping is necessary, since the direct utilization of the results obtained for pure elastic systems is not possible. Moreover, one can expect that the solutions of synthesis problem should differ one from the other depending on the level of material damping. Big part of the literature devoted to optimization of vibrating elastic systems is related to the problem of maximization of the lowest natural vibration frequency of the structure, in order to put the resonances out of the working region, see e.g. [1, 2, 3]. This can be done for example with use of shape optimization. For the elastic systems the increase of stiffness during maximization of j-th eigenfrequency related with j-th mode of vibration may weaken the structure with regard to the other eigenmode. This sometimes leads to the necessity of including multimodal solutions in formulation of synthesis problem. When the

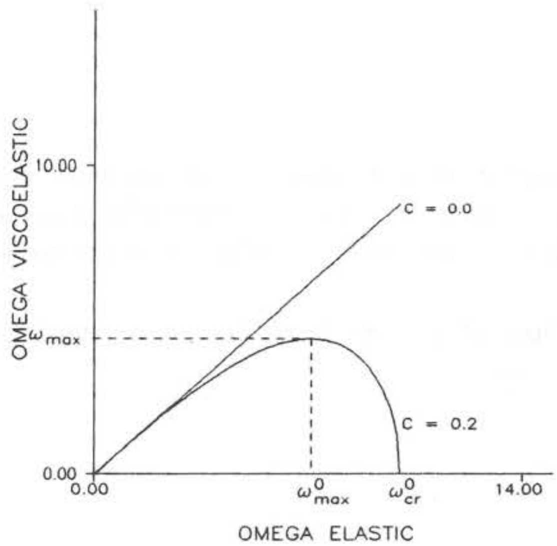

Figure 3: Domais for optimization of frequency of damped free vibration

viscoelastic structures are considered the effects of changes in design parameters or functions are more complicated. They depend on the region including eigenvalue of elastic system related with considered complex eigenvalue, see Fig. 3. For the domain $(0, \omega^0_{max})$ the increase of "elastic eigenvalue" cause associated with these changes increase of eigenfrequency of viscoelastic system. The effect is contrary in the next domain, i.e. for $\omega_e \in (\omega^0_{max}, \omega^0_{cr})$. Finally the system with damping associated with elastic system represented by ω_e greater then the critical value ω^0_{cr} has the "overdamped" property. It means that "viscoelastic" eigenvalues are real. Such variety of situations creates a possibility for different formulations of synthesis with taking into account the effect of energy dissipation. Three example are quoted below. The notation used in the following considerations reefers to the Fig. 3.

Let us consider the cost function or functional C. It may represent volume of material, cost of design etc. Let d represents the design variable.

2.1 Optimization of overdamped systems.

Minimize the cost C,

$$\min_d C = C_{min} \tag{33}$$

under the constraints

$$\left(\omega_j^0\right)^2 \geq \omega_{cr}^2 \tag{34}$$

where ω_j^0 represents eigenvalues of the elastic system associated with considered one. Some modification of this formulation related with imposing another constraint for eigenvalues may be sometimes usefull in numerical computations.

2.2 Optimization of the system with maximal possible eigenfrequency.

Minimize the cost C,

$$\min_d C = C_{min} \tag{35}$$

under the constraints

$$\min_j \left(\omega_j^0\right)^2 = \left(\omega_k^0\right)^2 = \omega_0^2 \tag{36}$$

$$\left(\omega_j^0\right)^2 \geq \omega_{cr}^2 \quad j = 1, \ldots, k-1, \ldots, \infty \tag{37}$$

2.3 Maximization of the lowest eigenfrequency.

$$\max_d \min_j \left(Im\lambda_j\right)^2 \quad for \ \{j: 0 < \omega_j^0 < \omega_{cr}\} \tag{38}$$

with the constraint imposed on the cost C,

$$C = C_0 \tag{39}$$

where λ_j is a complex eigenvalue of viscoelastic structure.

3 Simple example of beam optimization - optimization of cross-section of viscoelastic vibrating beam.

Let us consider simply supported viscoelastic beam of constant cross-section along the beam shown in Fig. 4. The length of the beam is L, the material

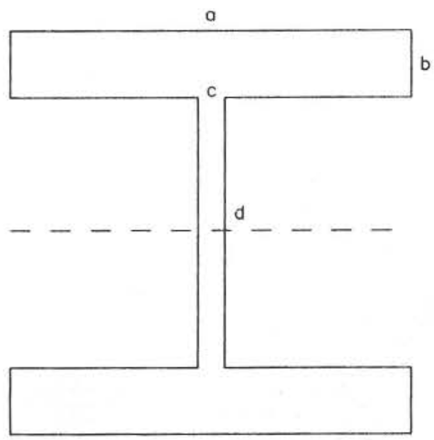

Figure 4: Cross-section of optimized vibrating beam.

density ρ, Young modulus E and the damping parameter is denoted by c, see eq.(28). Assuming the parameters describing the geometry of cross-section as the design parameters one can find, by the propper choice of them, the optimal solution providing, for assumed constant value of cross-sectional area A_0, the minimal value of the lowest eigenfrequency $\omega = Im(\lambda)$. The area of cross-section is given by the relation, Fig. 4,

$$A = 2ab + cd \tag{40}$$

but the moment of inertia J can be introduced as,

$$J = \frac{1}{12}cd^3 + ab(b+d)^2 \tag{41}$$

According to the relations presented before, one can consider the pure elastic case and transform obtained solution to the viscoelastic problem. The eigenvalue of elastic beam is given by the well known relation,

$$\omega_{e_k}^2 = (\frac{k\pi}{L})^4 EJ/\rho A \quad k = 1, \ldots, \infty \tag{42}$$

Let us assume that the area of cross-section is constant and equal to the given value A_0 $(A = A_0)$ during the optimization process. Referring to the discussion

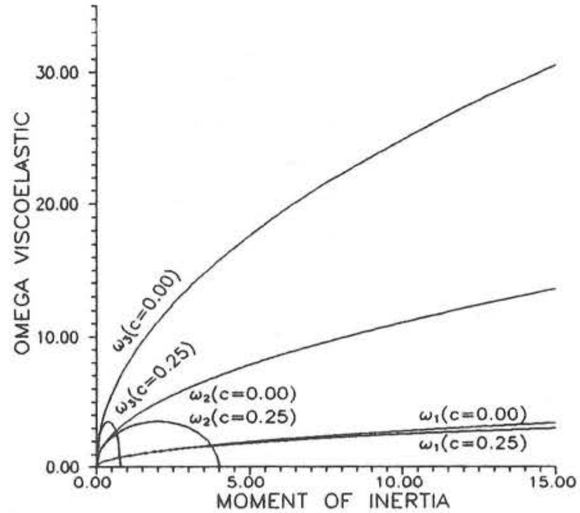

Figure 5: The lowest eigenfrequencies of system with daming.

in the previous sections one can derive the dependence between non-dimensional eigenfrequency of viscoelastic beam ω_k and non-dimensional moment of inertia B,

$$\omega_k = \frac{1}{2}k^2\sqrt{B(4 - k^4Bc^2)} \tag{43}$$

where $\omega_k = Im(\lambda_k)/\omega_{sk}$, $B = J/J_s$ and ω_{sk} denotes the eigenvalue of elastic beam that has square cross-section of area equal to A_0 and cross-sectional moment of inertia J_s. Fig. 5 presents three of the lowest non-dimensional eigenfrequencies $\omega_1, \omega_2, \omega_3$ of viscoelastic beam as the functions of non-dimensional moment of inertia B for damping parameter $c = 0$ and $c = 0.25$. One can observe that for the non-constrained problem the optimal solution is related to maximal value of ω_1 because all of the higher eigenmodes are overdamped.
The situation may be completely different when some restrictions are imposed on the geometry of the cross-section. As the cost function is not smooth, for some situations, depending on the value of material damping and the constraints imosed on the design parameters, the optimum may be placed in the points related to bimodal solutions, see Fig. 5.

4 Conclusions.

The problem of optimization of eigenfrequencies of viscoelastic systems is discussed in the present paper. The aim of this paper is to show that properties of structures with damping cause in some situations that the new, modified formulation of the problem is necessary. Even very simple example shows that the results of optimization strongly depend on the value of material damping. More investigations are necessary. In particular shape optimization, and comparison with classical solutions obtained for elastic structures would be very interesting.

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