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Domain shape optimization in mechanics

by

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Introduction

Different mechanical objects are described by partial differential equations, and in many cases optimization of such objects is reduced to the domain shape optimization problems, that is to the optimal choice of the domain, in which these differential equations are solved (see [1–8]). In [7–8] the general formulations of such domain shape optimization problems were given for objects, described by systems of elliptical equations, and the question of solvability of these problems were studied. Some applications have also been made to the optimization problems in mechanics of deformable solid and in mechanics of viscous liquid.

Notice that in [7,8] optimization problems were studied in which weak solutions of state equations in the form of systems of elliptical equations were used. In many applications, though, only smooth solutions of the state equations are allowed. In particular this situation occurs in the optimizations problems for deformable solids, plates and shells with restrictions on strength. In such optimization problems solution of the state equation is the function of displacements, and this function as well as its derivatives of some orders should be continuous and bounded.

We do consider here the domain shape optimization problems for objects described by the system of equations elliptical in the sense of Douglis–Nirenberg [9–11]. These optimization problems are considered on the set of allowed domains and smooth solutions of elliptical equations in these domains. The general formulation of such optimization problem is given, and existence of the solution of that problem is stated. The Fréchet differentiability of the solution of the state equation and the cost and constraint functionals with respect to control, determinig the domain shape, is investigated. In terms of application the problem of the choice of optimal form for two–dimensional elastic solid is considered with which this solid has minimal weight (volume) and a restriction on strength is satisfied.

1. Systems of elliptical equations

Let the state of some object be described by the following system of differential equations:

$$A(x,D)u(x) = f(x) \quad x \in \Omega \tag{1.1}$$

$$B(x,D)u(x) = g(x) \quad x \in S \tag{1.2}$$

Here Ω is a bounded domain in $\mathbf{R}^n$ with the boundary S, $u = (u_1, \dots, u_m)$, $f = (f_1, \dots, f_m)$ are m–dimensional vector functions defined in Ω, $x = (x_1, \dots, x_n)$ are points of Ω, $A(x,D)$ is the square matrix $m \times m$ with elements $A_{ij}(x,D)$, $D = (D_1, \dots, D_n)$, $D_i = \frac{\partial}{\partial x_i}$, A_{ij} are polynomials in D with coefficients depending on x over Ω, $B(x,D)$ is a rectangular matrix, having r rows and m columns, with elements $B_{qj}(x,D)$, B_{qj} are polynomials in D with coefficients depending on x over S, $g = (g_1, \dots, g_r)$ is a r–dimensional vector function defined on S.

We assume that the problem (1.1), (1.2) is elliptical in the sense of Douglis–Nirenberg [9–12], that is the system of equations (1.1) is elliptical, the supplementary condition and the complementing boundary condition are satisfied. The problem (1.1), (1.2) is considered in Hölder spaces $C^l(\Omega)$ where $l > 0$, l not being an integer. $C^l(\Omega)$ is provided with the norm

$$\|u\|_{l,\Omega} = \|u\|_{C^{[l]}(\Omega)} + \sum_{|k|=[l]} \sup_{x,x'\in\Omega} \frac{|D^k u(x) - D^k u(x')|}{|x-x'|^{l-[l]}}, \tag{1.3}$$

where $[l]$ is such integer that $l-[l] \in (0,1)$,

$$\|u\|_{C^{[l]}(\Omega)} = \sum_{|k|\le[l]} \sup_{x\in\Omega} |D^k u(x)|. \tag{1.4}$$

Define the spaces V_l and H_l in the forms

$$V_l = \prod_{j=1}^{m} C^{l+t_j}(\Omega), \quad H_l = \prod_{i=1}^{m} C^{l-s_i}(\Omega) \times \prod_{q=1}^{r} C^{l-\sigma_q}(S), \tag{1.5}$$

Here t_j, s_i and σ_q are integers, $\max_i s_i = 0$. With this the order of the operator $A_{ij}(x,D)$ does not exceed $s_i + t_j$, $\sigma_q = \max_{j=1,\dots,m}(\beta_{qj} - t_j)$ and β_{qj} is the order of the operator $B_{qj}(x,D)$. We also define the operator $L \in \mathcal{L}(V_l, H_l)$ in the form $L : u \to Lu = (A(x,D)u, B(x,D)u)$ and let $\hat{V}_l = \ker L$, $\check{H}_l = L(V_l)$. Then the following representations are valid, [12],

$$V_l = \hat{V}_l \oplus \check{V}_l, \quad H_l = \hat{H}_l \oplus \check{H}_l, \tag{1.6}$$

where $\oplus$ is the sign of the direct sum of subspaces. With this the dimensions of $\hat{V}_l$ and $\hat{H}_l$ are finite, and $\hat{V}_l$ and $\hat{H}_l$ do not depend on l if the coefficients of the operators $A(x,D)$, $B(x,D)$ and the boundary S are of the classes C^∞.

THEOREM 1.1 (SOLONNIKOV'S THEOREM) *Let (1.1), (1.2) be an elliptical problem, l not an integer, $l > \max(0, \sigma_1, \dots, \sigma_r)$. Let also the boundary S be of $C^{l+t_{\max}}$, where $t_{\max} = \max(t_1, \dots, t_m)$, and the coefficients of the operators $A_{ij}(x,D)$, $B_{qj}(x,D)$ belong to $C^{l-s_i}(\Omega)$ and $C^{l-\sigma_q}(S)$, accordingly. Then the operator L is the isomorphism of $\check{V}_l$ onto $\check{H}_l$.*

The proof of Theorem 1.1 can be found in [12]. Taking into account Theorem 1.1 it is easy to obtain the following statement.

THEOREM 1.2 *Assume that the conditions of Theorem 1.1 hold and the dimensions of $\hat{V}_l$ and $\hat{H}_l$ are equal. Let $\{\phi_i\}_{i=1}^k$ be a basis in $\hat{V}_l$ and $\{\psi_i\}_{i=1}^k$ be a basis in $\hat{H}_l$. Determine the operator $G \in \mathcal{L}(V_l, H_l)$ in the form*

$$G\phi_i = \psi_i \quad i = 1, \dots, k, \qquad Gu = 0 \quad \forall u \in \check{V}_l.$$

Then the operator $L_1 : u \to L_1 u = Lu + Gu$ is the isomorphism of V_l onto H_l.

2. Elliptical problems in domains and a fixed domain

Let M be a space of controls. We assume that M is an open set in an affine normed space X and M is provided with the topology generated by the topology of X. We suppose that the domain Ω_q in $\mathbb{R}^n$ with the boundary S_q of the class $C^{l+t_{\max}}$ and the diffeomorphism P_q of $\bar{\Omega}_q$ onto $\bar{\Omega}$ of the class $C^{[l]+1+t_{\max}}$ are given for every $q \in M$, that is

$$P_q \in C^{[l]+1+t_{\max}}(\Omega_q, \Omega), \qquad P_q^{-1} \in C^{[l]+1+t_{\max}}(\Omega, \Omega_q). \tag{2.1}$$

Then the mapping $u \to u \circ P_q$ is an isomorphism of $C^p(\Omega)$ onto $C^p(\Omega_q)$ and $C^p(S)$ onto $C^p(S_q)$ for any $p \in [0, l + t_{\max}]$. Such elliptical problem in Ω_q of the type pointed above is also given for every $q \in M$.

$$\begin{array}{rcll} A_q(y,D)u(y) & = & f_q(y) & y \in \Omega_q \\ B_q(y,D)u(y) & = & g_q(y) & y \in S_q \end{array} \tag{2.2}$$

We designate by V_{lq} and H_{lq} the spaces V_l and H_l (see (1.5)) in which Ω and S are substituted for Ω_q and S_q, accordingly. We suppose also that

$$\left.\begin{array}{l} \text{for any } q \in M \\ \text{the operator } L_q : u \to L_q u = (A_q(y,D)u, B_q(y,D)u) \\ \text{is an isomorphism of } V_{lq} \text{ onto } H_{lq}. \end{array}\right\} \tag{2.3}$$

For every $q \in M$ we define the operator $\tilde{L}_q = (\tilde{A}_q, \tilde{B}_q) \in \mathcal{L}(V_l, H_l)$ in the form

$$\begin{array}{rcl} \tilde{L}_q u & = & (\tilde{A}_q u, \tilde{B}_q u), \\ \tilde{A}_q u & = & (A_q(u \circ P_q)) \circ P_q^{-1}, \\ \tilde{B}_q u & = & (B_q(u \circ P_q)) \circ P_q^{-1}, \end{array} \tag{2.4}$$

Here A_q and B_q are the operators from (2.2), and we write A_q, B_q instead of $A_q(y,D)$, $B_q(y,D)$ here and below. The operator $\tilde{L}_q$ is obtained from the operator L_q under the replacement of variables, corresponding to the mapping P_q. Owing to (2.1) and (2.3) we have

$$\left.\begin{array}{l} \text{the operator } \tilde{L}_q u = (\tilde{A}_q, \tilde{B}_q) \\ \text{is an isomorphism of } V_l \text{ onto } H_l \end{array}\right\} \tag{2.5}$$

and if the function $\tilde{u} \in V_l$ is the solution of the problem

$$\begin{aligned} \tilde{A}_q \tilde{u} &= f_q \circ P_q^{-1} \quad \text{in } \Omega, \\ \tilde{B}_q \tilde{u} &= g_q \circ P_q^{-1} \quad \text{on } S, \end{aligned} \tag{2.6}$$

where $(f_q, g_q) \in H_{lq}$, then the function $u = \tilde{u} \circ P_q$ is the solution of the problem (2.2). On the contrary, if u is the solution of the problem (2.2), then $\tilde{u} = u \circ P_q^{-1}$ is the solution of the problem (2.6). Suppose that

$$\left.\begin{array}{l} q \to \tilde{L}_q u = (\tilde{A}_q, \tilde{B}_q) \text{ is continuous mapping} \\ \text{from } M \text{ into } \mathcal{L}(V_l, H_l). \end{array}\right\} \tag{2.7}$$

$$\left.\begin{array}{l} q \to (f_q \circ P_q^{-1}, g_q \circ P_q^{-1}) \text{ is continuous mapping} \\ \text{from } M \text{ into } H_l. \end{array}\right\}. \tag{2.8}$$

THEOREM 2.1 *Let the conditions (2.1), (2.3), (2.7), (2.8) hold. Then for every* $q \in M$ *there exists a unique solution* $\tilde{u}$ *of the problem (2.6), and the function* $\lambda : q \to \lambda(q) = \tilde{u}$, *determined by this solution, is a continuous mapping from* M *into* V_l.

PROOF. The existence and uniqueness of the solution of the problem (2.6) follow from the (2.1) and (2.3). Let $q \in M$, $\{q_k\}_{k=1}^{\infty} \subset M$ and $q_k \to q$ in M. Owing to (2.7) and (2.8) we have

$$\tilde{L}_{q_k} \to \tilde{L}_q \text{ in } \mathcal{L}(V_l, H_l), \tag{2.9}$$

$$(f_{q_k} \circ P_{q_k}^{-1}, g_{q_k} \circ P_{q_k}^{-1}) \to (f_q \circ P_q^{-1}, g_q \circ P_q^{-1}) \text{ in } H_l. \tag{2.10}$$

From (2.9) and from the reversibility of the operators $\tilde{L}_{q_k}$ and $\tilde{L}_q$ there follows the convergence of the inverse operators (see [14]), that is

$$\tilde{L}_{q_k}^{-1} \to \tilde{L}_q^{-1} \text{ in } \mathcal{L}(H_l, V_l). \tag{2.11}$$

Now from (2.10) and (2.11) we obtain $\lambda(q_k) \to \lambda(q)$ in V_l. ■

Determine the mapping $\mathcal{T} : M \times V_l \to H_l$ by the expression

$$\begin{array}{l} q \in M, \qquad u \in V_l \\ \mathcal{T}(q, u) = (\tilde{A}_q u - f_q \circ P_q^{-1}, \tilde{B}_q u - g_q \circ P_q^{-1}) \end{array} \tag{2.12}$$

It is obvious that the function $\lambda : M \to V_l$, introduced in the formulation of Theorem 2.1, is the implicit function, determined by the mapping $\mathcal{T}$, that is

$$\mathcal{T}(q, \lambda(q)) = 0, \tag{2.13}$$

and $\lambda(q) = \tilde{u}$, where $\tilde{u}$ is the solution of the problem (2.6). The existence and continuity of the implicit function λ follow from Theorem 2.1. Consider now the question of differentiability of the function λ. We would remind that M is the open set in the affine normed space X.

THEOREM 2.2 *Let the conditions (2.1), (2.3) hold and* $q \to \tilde{L}_q = (\tilde{A}_q, \tilde{B}_q)$, $q \to (f_q \circ P_q^{-1}, g_q \circ P_q^{-1})$ *are Fréchet continuously differentiable mappings from* M *into* $\mathcal{L}(V_l, H_l)$ *and into* H_l *accordingly. Then the function* λ*, determined by equation (2.13), is Fréchet continuously differentiable mapping from* M *into* V_l*, and Fréchet derivative* λ' *at a point* $q \in M$ *of the function* λ *is given by the expression*

$$\lambda'(q)h = -\tilde{L}_q^{-1} \circ \frac{\partial \mathcal{T}}{\partial q}(q, \lambda(q))h \qquad h \in X, \tag{2.14}$$

where the operator $\frac{\partial \mathcal{T}}{\partial q}(q, u)$ *is determined by the formula*

$$\begin{aligned} &\frac{\partial \mathcal{T}}{\partial q}(q, u)h = \\ &= \{(\tilde{A}_q'h)u - (f_q \circ P_q^{-1})'h, (\tilde{B}_q'h)u - (g_q \circ P_q^{-1})'h\} \quad h \in X \end{aligned} \tag{2.15}$$

Here $\tilde{A}_q'$, $\tilde{B}_q'$, $(f_q \circ P_q^{-1})'$, $(g_q \circ P_q^{-1})'$ *are Fréchet derivatives at a point* q *of mappings* $q \to \tilde{A}_q$, $q \to \tilde{B}_q$, $q \to (f_q \circ P_q^{-1})$, $q \to (g_q \circ P_q^{-1})$.

PROOF. It is obvious that $\mathcal{T}$ is a Fréchet continuously differentiable mapping from $M \times V_l$ into H_l. With this, $\frac{\partial \mathcal{T}}{\partial q}(q, u)$ is determined by formula (2.15) and $\frac{\partial \mathcal{T}}{\partial u}(q, u) = \tilde{L}_q = (\tilde{A}_q, \tilde{B}_q)$. From here, taking in account (2.3), we obtain that the operator $\frac{\partial \mathcal{T}}{\partial u}(q, u)$ is an isomorphism of V_l onto H_l. Now Theorem 2.2 follows from the Theorem of differentiability of the implicit function [14]. ■

REMARK. We assumed above that condition (2.3) holds. Let condition (2.3) be not satisfied now. In that case, owing to (1.6) the following representations are valid

$$V_{lq} = \hat{V}_{lq} \oplus \check{V}_{lq}, \qquad H_{lq} = \hat{H}_{lq} \oplus \check{H}_{lq}, \tag{2.16}$$

where $\hat{V}_{lq} = \ker L_q$, $\check{H}_{lq} = L_q(V_{lq})$. Suppose that

$$\dim \hat{V}_{lq} = \dim \hat{H}_{lq} = k_q \qquad \forall q \in M, \tag{2.17}$$

where k_q is a positive integer. By analogy to operator G from Theorem 1.2 we determine operator G_q in the form

$$G_q\phi_{qi} = \psi_{qi} \quad i = 1, \ldots, k_q, \qquad G_q u = 0 \quad \forall u \in \check{V}_{lq}, \tag{2.18}$$

where ϕ_{qi} and ψ_{qi} are basis functions in $\hat{V}_{lq}$ and $\hat{H}_{lq}$. Then, the operator $L_{q1} = L_q + G_q$ is an isomorphism of V_{lq} onto H_{lq}. Therefore, with all the suitable conditions stated above the results remain true if we substitute the operator L_q for the operator L_{q1}.

3. The problem of domain shape optimization

Let functionals Ψ_i over $M \times V_l$ be given such that

$$\left.\begin{array}{l}(q,u) \to \Psi_i(q,u) \text{ is continuous mapping} \\ \text{from } M \times V_l \text{ into } \mathbb{R}, \quad i = 0,1,\ldots,k\end{array}\right\} \tag{3.1}$$

We define the functionals Φ_i over M in the form

$$\Phi_i(q) = \Psi_i(q, \lambda(q)) \quad i = 0,1,\ldots,k, \tag{3.2}$$

where $\lambda(q)$ is determined by expression (2.13). Let M_1 be a compact set in M. We take set of admissible controls U in the form

$$U = \{q | q \in M_1, \Phi_i(q) \leq 0 \quad i = 1,2,\ldots,k\}. \tag{3.3}$$

The optimization problem consists in finding q_0 such that

$$q_0 \in U, \quad \Phi_0(q_0) = \inf_{q \in U} \Phi_0(q). \tag{3.4}$$

THEOREM 3.1 *Let conditions (2.1), (2.3), (2.7), (2.8), (3.1) hold, M_1 be a compact set in M and a non–empty set U determined by expression (3.3). Then there exists a solution of the problem (3.4).*

PROOF. As M_1 is a non–empty set, there exists a minimizing sequence $\{q_n\}$ such that

$$\{q_n\} \subset U, \quad \lim \Phi_0(q_n) = \inf_{q \in U} \Phi_0(q). \tag{3.5}$$

As M_1 is a compact set in M we can choose a subsequence $\{q_m\}$ such that $q_m \to z$ in M, $z \in M_1$. Owing to Theorem 2.1, $\lambda(q_m) \to \lambda(z)$ in V_l, where $q(z)$ is the solution of the problem (2.6) with $q = z$. Now it is easily seen that $q_0 = z$ is the solution of the problem (3.4). ■

In connection with finding a solution of a problem (3.4) there arises the question of differentiability of the functionals Φ_i. Using Theorem 2.2 and the theorem on differentiability of a composite function [14], we obtain such assertion.

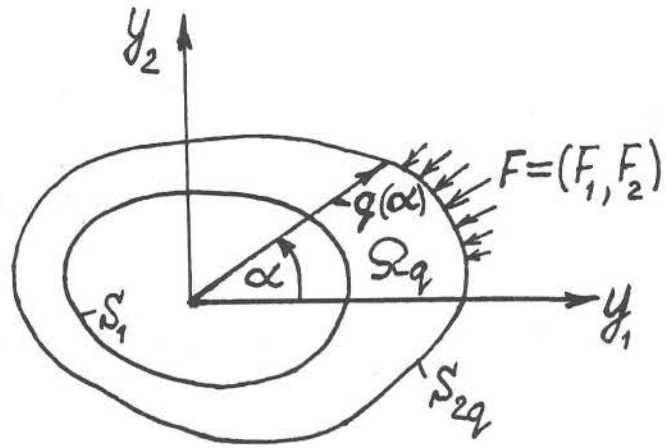

Figure 1.

THEOREM 3.2 *Let the conditions of Theorem 2.2 hold, and $\Psi_i : (q,u) \to \Psi_i(q,u)$ is a Fréchet continuously differentiable mapping from $M \times V_l$ into $\mathbf{R}$. Then the functional Φ_i, defined by formula (3.2), is a Fréchet continuously differentiable mapping from M into $\mathbf{R}$, and the Fréchet derivative Φ_i' of functional Φ_i at a point $q \in M$ is defined by the formula*

$$\Phi_i'(q)h = \frac{\partial \Psi_i}{\partial q}(q, \lambda(q))h + (\frac{\partial \Psi_i}{\partial u}(q, \lambda(q)) \circ \lambda'(q))h \quad h \in X.$$

4. Shape optimization of two–dimensional elastic body

4.1. Sets of controls and domains in the optimization problem

As before let M be a space of controls, which we shall define below. The domain Ω_q in $\mathbf{R}^2$, occupied by an elastic body, is given for every $q \in M$. The boundary S_q of Ω_q consists of two connected components S_1 and S_{2q} (see Fig.1). The points of S_1 are held fixed, and S_1 does not depend on control q. Surface forces $F = (F_1, F_2)$ are given on S_{21}, where S_{21} is an open set in S_{2q}, and these forces are continued onto all S_{2q} by zero. With this, S_{21} does not depend on a control q, and $S_{2q}^{(1)} = S_{2q} \setminus S_{21}$ is the controlled part of the S_{2q}, that is, $S_{2q}^{(1)}$ is the part of the boundary which should be chosen from the conditions of optimization.

Define the space of controls in the form

$$\begin{aligned} M &= \{q | q \in \tilde{C}^{[l]+3}(0, 2\pi), r_1 < q(\alpha) < r_2 \; \forall \alpha \in [0, 2\pi], \\ &\quad q(\alpha) = \beta(\alpha) \; \forall \alpha \in (\alpha_1, \alpha_2)\}. \end{aligned} \tag{4.1}$$

Here r_1, r_2 are positive constants, $r_1 < r_2$, β is given over the (α_1, α_2) function, which defines S_{21} in polar coordinates.

$\tilde{C}^{[l]+3}(0, 2\pi)$ is the subspace of periodical functions in $C^{[l]+3}(0, 2\pi)$. Periodicity of a function $q \in C^{[l]+3}(0, 2\pi)$ means that, if $\tilde{q}$ is periodical with the period $[0, 2\pi]$ continuation on $\mathbb{R}$ of q, then $\tilde{q} \in C^{[l]+3}(a, b)$ for arbitrary $[a, b] \subset \mathbb{R}$.

The set M is provided with the topology generated by the topology of $C^{[l]+3}(0, 2\pi)$. Now for each $q \in M$ we define the domain Ω_q such that the internal boundary S_1 of it is given in polar coordinates with the function $\gamma \in \tilde{C}^{[l]+3}(0, 2\pi)$ and the external boundary S_{2q} with the function q.

We define the domain Ω in the form

$$\Omega = \{x | x = (x_1, x_2), \quad 1 < x_1^2 + x_2^2 < 4\}. \tag{4.2}$$

Designate by E the function which maps polar coordinates onto Cartesian coordinates,

$$E : (r, \alpha) \to E(r, \alpha) = (y_1, y_2), \; y_1 = r \cos \alpha, \; y_2 = r \sin \alpha, \tag{4.3}$$

and let E^{-1} be the inverse of function E. Determine $P_q : \bar{\Omega}_q \to \bar{\Omega}$ by the formula

$$P_q = E \circ G_q \circ E^{-1}, \tag{4.4}$$

where $G_q : E^{-1}(\bar{\Omega}_q) \to E^{-1}(\bar{\Omega})$

$$\begin{aligned} &(r, \alpha) \to G_q(r, \alpha) = (\varrho, \phi), \\ &\varrho = \frac{r - 2\gamma(\alpha) + q(\alpha)}{q(\alpha) - \gamma(\alpha)}, \; \phi = \alpha. \end{aligned} \tag{4.5}$$

The mapping $G_q^{-1} : E^{-1}(\bar{\Omega}) \to E^{-1}(\bar{\Omega}_q)$ has the form

$$\begin{aligned} &(\varrho, \phi) \to G_q^{-1}(\varrho, \phi) = (r, \alpha), \\ &r = 2\gamma(\phi) - q(\phi) + [q(\phi) - \gamma(\phi)]\varrho, \; \alpha = \phi. \end{aligned} \tag{4.6}$$

It is easily seen that the mapping P_q defined by formulae (4.4), (4.5), satisfies conditions (2.1).

4.2. A theory for problems of elasticity in domains.

In the domain Ω_q the operator A_q of the theory of elasticity has the form

$$A_q u = \left\{ \begin{array}{l} \mu\Big(\dfrac{\partial^2 u_1}{\partial y_1^2} + \dfrac{\partial^2 u_1}{\partial y_2^2}\Big) + (\lambda+\mu)\Big(\dfrac{\partial^2 u_1}{\partial y_1^2} + \dfrac{\partial^2 u_2}{\partial y_1 \partial y_2}\Big) \\ \mu\Big(\dfrac{\partial^2 u_2}{\partial y_1^2} + \dfrac{\partial^2 u_2}{\partial y_2^2}\Big) + (\lambda+\mu)\Big(\dfrac{\partial^2 u_1}{\partial y_1 \partial y_2} + \dfrac{\partial^2 u_2}{\partial y_2^2}\Big) \end{array} \right\} \tag{4.7}$$

Where $u = (u_1, u_2)$ is a vector function of displacement, λ, μ are the positive constants. Designate by $\varepsilon_{ij}(u)$, $\sigma_{ij}(u)$ the components of strain and stress tensors

$$\begin{aligned} &\varepsilon_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i}\Big), \\ &\sigma_{ij}(u) = \lambda\Big(\frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_2}\Big)\delta_{ij} + 2\mu\varepsilon_{ij}(u) \quad i,j = 1,2, \end{aligned} \tag{4.8}$$

where $\delta_{ij} = \left\{ \begin{array}{ll} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{array} \right.$. Define the boundary operators B and B_q, respectively on S and S_{2q} by the expressions

$$Bu = \left\{ \begin{array}{c} u_1|_{S_1} \\ u_2|_{S_1} \end{array} \right\}, \qquad B_q u = \left\{ \begin{array}{c} (\sigma_{11}(u)\nu_{1q} + \sigma_{12}(u)\nu_{2q})|_{S_{2q}} \\ (\sigma_{21}(u)\nu_{1q} + \sigma_{22}(u)\nu_{2q})|_{S_{2q}} \end{array} \right\}, \tag{4.9}$$

where ν_{iq} are the components of the unit outward normal to S_{2q}, $i = 1, 2$.

THEOREM 4.1 *Let the set M be defined by expression (4.1). For each $q \in M$ determine a two-connected domain $\Omega_q \subset \mathbb{R}^2$ such that the internal and external boundaries of Ω_q are defined in polar coordinates with the functions $\gamma \in \tilde{C}^{[l]+3}(0, 2\pi)$ and q. Then the operator $L_q : u \to L_q u = (A_q u, Bu, B_q u)$, defined by formulae (4.7), (4.9), where λ, μ are the positive constants, is an isomorphism of the space*[1] *$V_{lq} = C^{l+2}(\Omega_q)^2$ onto the space $H_{lq} = C^l(\Omega_q)^2 \times C^{l+2}(S_1)^2 \times C^{l+1}(S_{2q})^2$.*

PROOF. Consider the problem

$$\begin{aligned} A_q u &= f \ \text{ in } \Omega_q, \\ Bu &= g_1 \text{ on } S_1, \\ B_q u &= g_2 \text{ on } S_{2q}, \end{aligned} \tag{4.10}$$

where $(f, g_1, g_2) \in H_{lq}$. The ellipticity of the operator A_q follows from Korn's inequality [15,16]. The ellipticity of problem (4.10) follows from the ellipticity

[1] Here and further on: $C^k(\theta)^2 = C^k(\theta) \times C^k(\theta)$

of the first and second problems of the theory of elasticity [17]. The kernel space of the operator A_q is the space of small rigid displacements, which has the form, [7,16],

$$\begin{aligned} Q \;=\; & \{u|u=(u_1,u_2),\ u_1=a_1+a_3y_2,\ u_2=a_2-a_3y_1, \\ & a_1,a_2,a_3\in\mathbb{R}\}. \end{aligned} \tag{4.11}$$

Let $y^{(1)}=(y_1^{(1)},y_2^{(1)})$, $y^{(2)}=(y_1^{(2)},y_2^{(2)})$ be two different points of S_1. From condition $Bu=0$ it follows that $u(y^{(1)})=u(y^{(2)})=0$, and if, in addition, $u\in Q$, then, owing to (4.11), we have $a_1=a_2=a_3=0$. Therefore the kernel space of the operator $L_q=(A_q,B,B_q)$ consists only of zero. For each $(f,g_1,g_2)\in H_{lq}$ there exists a solution of the problem (4.10). Now Theorem 4.1 follows from Theorem 1.1.

4.3. The problem of optimization

We would remind that for each $q\in M$ we determine the two–connected domain Ω_q such that the internal, S_1, and external, S_{2q}, boundaries of Ω_q are defined in polar coordinates with the functions γ and q. For each $q\in M$ we consider the problem

$$\begin{aligned} A_q u_q &= 0 \text{ in } \Omega_q, \\ B u_q &= 0 \text{ on } S_1, \\ B_q u_q &= F \text{ on } S_{2q}, \end{aligned} \tag{4.12}$$

where the operators A_q, B, B_q are defined by formulae (4.7), (4.9). We suppose that

$$F\in C^{l+1}(S_{2q})^2,\quad \text{supp } F\subset S_{21}\subset S_{2q}, \tag{4.13}$$

where (see (4.1) and (4.3))

$$S_{21}=\{s|s=E(\beta(\alpha),\alpha),\ \alpha\in(\alpha_1,\alpha_2)\}$$

Now we pass to the constraint on strength. For a vector function of displacement $u=(u_1,u_2)$ the components of the stress deviator (shear stress tensor) are defined by the formula

$$\tau_{ij}(u)=\sigma_{ij}(u)-\frac{1}{2}(\sigma_{11}(u)+\sigma_{22}(u))\delta_{ij}\quad i,j=1,2,$$

and the second invariant of the stress deviator has the form of

$$\mathcal{T}(u) = \sum_{i,j=1}^{2} (\tau_{ij}(u))^2 = \frac{1}{2}(\sigma_{11}(u) - \sigma_{22}(u))^2 + 2(\sigma_{12}(u))^2. \tag{4.14}$$

Define the functional G_1 over M by the formula

$$G_1(q) = \max_{y \in \bar{\Omega}_q} [(\mathcal{T}(u_q))(y) - b], \tag{4.15}$$

where u_q is the solution of problem (4.12), b is a positive constant. For an isotropic material the restriction on strength may be taken in the form $G_1(q) \leq 0$. The volume of the material is defined by the expression

$$G_0(q) = \int_{\Omega_q} dy. \tag{4.16}$$

Define the set M_1 in the form

$$\begin{aligned} M_1 \quad = \quad & \{q | q \in M,\ q \in \tilde{C}^{l+3}(0, 2\pi),\ \|q\|_{\tilde{C}^{l+3}(0,2\pi)} \leq c_1, \\ & r_1 + \delta \leq q(\alpha) \leq r_2 - \delta \quad \forall \alpha \in [0, 2\pi]\}, \end{aligned} \tag{4.17}$$

where M is determined by (4.1), c_1, δ are positive constants and δ is small. We take the set of admissible controls U in the form

$$U = \{q | q \in M_1, G_1(q) \leq 0\}. \tag{4.18}$$

The optimization problem consists in finding q_0 satisfying

$$q_0 \in U \quad G_0(q_0) = \inf_{q \in U} G_0(q). \tag{4.19}$$

THEOREM 4.2 *Let the operators A_q, B, B_q be defined by formulae (4.7), (4.9), and λ, μ be positive constants. Suppose that condition (4.13) hold, and the functionals G_1 and G_0 are defined over M by formulae (4.15), (4.16) where u_q is the solution of problem (4.12). Let also a non-empty set U be given by expressions (4.1), (4.17), (4.18). Then, for any $l > 0$, l not being an integer, there exists a solution of the problem (4.19).*

PROOF. Define the spaces V_l and H_l by expressions

$$V_l = C^{l+2}(\Omega)^2, \quad H_l = C^l(\Omega)^2 \times C^{l+2}(S_{01})^2 \times C^{l+1}(S_{02})^2.$$

Here Ω is the domain defined by (4.2), S_{01} and S_{02} are the internal and external boundaries of Ω. In the same way as it was made in Section 2, the problem (4.12) is reduced to such problem: find the function $\tilde{u}_q \in V_l$, satisfying

$$\begin{aligned} \tilde{A}_q \tilde{u}_q \quad &= \quad 0 \text{ in } \Omega, \\ \tilde{B} \tilde{u}_q \quad &= \quad 0 \text{ on } S_{01}, \\ \tilde{B}_q \tilde{u}_q \quad &= \quad F \circ P_q^{-1} \text{ on } S_{02}, \end{aligned} \tag{4.20}$$

Here operators $\tilde{A}_q$, $\tilde{B}$, $\tilde{B}_q$ are defined by expressions

$$\begin{aligned}\tilde{A}_q u &= (A_q(u \circ P_q)) \circ P_q^{-1},\\ \tilde{B}u &= (B(u \circ P_q)) \circ P_q^{-1},\\ \tilde{B}_q u &= (B_q(u \circ P_q)) \circ P_q^{-1}.\end{aligned} \tag{4.21}$$

From (4.12), (4.20) and (4.21) it follows that $u_q = \tilde{u}_q \circ P_q$. It is then easily seen that

$$\left.\begin{array}{l} q \longrightarrow \tilde{L}_q = (\tilde{A}_q, \tilde{B}, \tilde{B}_q) \text{ is continuous mapping} \\ \text{from } M \text{ into } \mathcal{L}(V_l, H_l). \end{array}\right\} \tag{4.22}$$

Owing to (4.13) $q \longrightarrow F \circ P_q^{-1}$ is the constant mapping, and from Theorem 2.1 we now obtain

$$\left.\begin{array}{l} q \longrightarrow \tilde{u}_q \text{ is continuous mapping} \\ \text{from } M \text{ into } V_l, \end{array}\right\} \tag{4.23}$$

where $\tilde{u}_q$ is the solution of problem (4.20). Under the replacement of variables corresponding to the mapping P_q the functionals G_1 and G_0 from (4.15), (4.16) take on the form

$$G_1(q) = \max_{x \in \bar{\Omega}} (\mathcal{T}(\tilde{u}_q \circ P_q))(P_q^{-1}(x)) - b, \tag{4.24}$$

$$G_0(q) = \int_\Omega \det |(P_q^{-1})'(x)| dx. \tag{4.25}$$

Here $(\mathcal{T}(\tilde{u}_q \circ P_q))(P_q^{-1}(x))$ is the value of the function $\mathcal{T}(\tilde{u}_q \circ P_q)$ at a point $P_q^{-1}(x)$ and $(P_q^{-1})'(x)$ is the value of Fréchet derivative of the mapping P_q^{-1} at a point x. Taking into account (4.23)–(4.25) and (4.4)–(4.6) we obtain that, G_1 and G_0 are continuous functionals over M. As the imbedding of $C^{l+3}(0, 2\pi)$ into $C^{[l]+3}(0, 2\pi)$ is compact, we have that M_1 is a compact set in M. Therefore there exists a solution of the problem (4.19).

REMARK. In the considered case the function $q \longrightarrow (\tilde{A}_q, \tilde{B}, \tilde{B}_q)$ is a Fréchet continuously differentiable mapping from M into $\mathcal{L}(V_l, H_l)$, and using Theorem 2.2 we obtain that $q \longrightarrow \lambda(q) = \tilde{u}_q$ is a Fréchet continuously differentiable mapping from M into V_l. However, the functional G_1 from (4.24) is not differentiable as the functional $v \longrightarrow \max_{x \in \bar{\Omega}} v(x), v \in C(\bar{\Omega})$ is not differentiable. On the set $Q = \{v | v \in C(\bar{\Omega}), v(x) \geq 0, \ \forall x \in \bar{\Omega}\}$, however, the last functional may be approximated by a Fréchet continuously differentiable functional $v \longrightarrow \|v\|_{L_p(\Omega)}$ if p is sufficiently large.

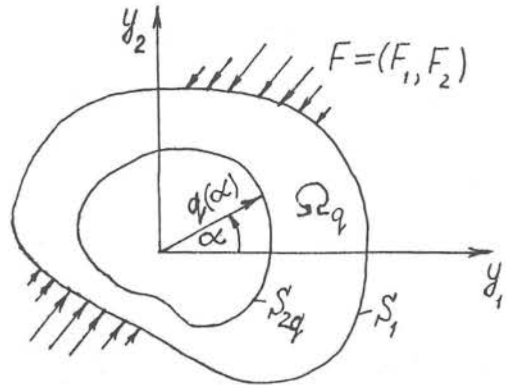

Figure 2.

5. Optimization of internal boundary of a two–dimensional elastic body

In Section 4 we considered the optimization problem for two–connected elastic body, in which we give displacement on the internal boundary and surface forces on the external boundary. In that case there exists a unique solution of the problem (4.10) for any $(f, g_1, g_2) \in H_{lq}$, and owing to Theorem 4.1 condition (2.3) holds. Now we consider the optimization problem for two–connected elastic body in which we gave surface forces on the internal and external boundaries of the body. In this case condition (2.3) is not satisfied.

Let us pass over to the formulation of the problem. Let M be the space of controls, and the two-dimensional domain Ω_q, occupied by elastic body be defined for every $q \in M$. The boundary S_q of Ω_q consists of two-connected components, S_{2q} and S_1 are the internal and external boundaries of Ω_q, respectively (see Fig.2.) We give "self–balanced" forces F on S_1 and zero forces on S_{2q}. With this S_1 and F do not depend on a control q and S_{2q} should be chosen from the conditions of optimization.

Define the space of controls M in the form

$$M = \{q | q \in \tilde{C}^{[l]+3}(0, 2\pi), r_1 < q(\alpha) < r_2 \ \forall \alpha \in [0, 2\pi]\}, \tag{5.1}$$

where r_1, r_2 are positive constants. For each $q \in M$ determine the two–connected domain Ω_q such that internal boundary S_{2q} is defined in polar coordinates with q and the external boundary S_1 with the fixed function $\gamma \in$

$\tilde{C}^{[l]+3}(0,2\pi)$. The domain Ω is defined by (4.2) and the mapping P_q by (4.4), where $G_q : E^{-1}(\bar{\Omega}_q) \to E^{-1}(\bar{\Omega})$ has the form

$$\begin{aligned} &(r,\alpha) \to G_q(r,\alpha) = (\varrho,\phi), \\ &\varrho = \frac{r - 2q(\alpha) + \gamma(\alpha)}{\gamma(\alpha) - q(\alpha)}, \ \phi = \alpha. \end{aligned} \tag{5.2}$$

The invers mapping $G_q^{-1} : E^{-1}(\bar{\Omega}) \to E^{-1}(\bar{\Omega}_q)$ has the form

$$\begin{aligned} &(\varrho,\phi) \to G_q^{-1}(\varrho,\phi) = (r,\alpha), \\ &r = 2q(\phi) - \gamma(\phi) + [\gamma(\phi) - q(\phi)]\varrho, \ \alpha = \phi. \end{aligned} \tag{5.3}$$

The operator A_q is defined by (4.7) and the boundary operators B_q and B are given by expressions

$$B_q u = \left\{ \begin{array}{c} (\sigma_{11}(u)\nu_{1q} + \sigma_{12}(u)\nu_{2q})|_{S_{2q}} \\ (\sigma_{21}(u)\nu_{1q} + \sigma_{22}(u)\nu_{2q})|_{S_{2q}} \end{array} \right\}, \tag{5.4}$$

$$Bu = \left\{ \begin{array}{c} (\sigma_{11}(u)\nu_{1} + \sigma_{12}(u)\nu_{2})|_{S_{1}} \\ (\sigma_{21}(u)\nu_{1} + \sigma_{22}(u)\nu_{2})|_{S_{1}} \end{array} \right\}. \tag{5.5}$$

Here ν_{iq} and ν_i are the components of the unit outward normal to S_{2q} and S_1 accordingly, $i = 1,2$. Define spaces V_{lq} and H_{lq} in the form

$$V_{lq} = C^{l+2}(\Omega_q)^2, \quad H_{lq} = C^l(\Omega_q)^2 \times C^{l+1}(S_{2q})^2 \times C^{l+1}(S_1)^2. \tag{5.6}$$

where $l > 0$, l not being an integer. It is obvious that

$$L_q = (A_q, B_q, B) \in \mathcal{L}(V_{lq}, H_{lq}).$$

The kernel space of the operator L_q is the three–dimensional space $\hat{V}_{lq} = Q$, where Q is defined by (4.11). The following functions are the basis in $\hat{V}_{lq}$

$$\phi_{q1} = (1,0), \quad \phi_{q2} = (0,1), \quad \phi_{q3} = (y_2, -y_1). \tag{5.7}$$

For given functions of volume and surface forces (f, R, F), defined on Ω_q, S_{2q} and S_1 accordingly, consider the problem of finding the function of displacement u such that

$$\begin{aligned} A_q u &= f \text{ in } \Omega_q, \\ B_q u &= R \text{ on } S_{2q}, \\ Bu &= F \text{ on } S_1. \end{aligned} \tag{5.8}$$

We assume the volume and surface forces (f, R, F) to be "self–balanced", that is these forces are orthogonal to the space $\hat{V}_{lq}$ in the sence that

$$
\begin{aligned}
&\int_{\Omega_q} f_i dy + \int_{S_{2q}} R_i ds + \int_{S_1} F_i ds = 0 \quad i = 1,2, \\
&\int_{\Omega_q} (f_1 y_2 - f_2 y_1) dy + \int_{S_{2q}} (R_1 y_2 - R_2 y_1) ds + \\
&\qquad + \int_{S_1} (F_1 y_2 - F_2 y_1) ds = 0
\end{aligned}
\tag{5.9}
$$

Conditions (5.9) are necessary and sufficient for the existance of a solution of problem (5.8) (see [16]). From (5.9) it follows that the space $\hat{H}_{lq} = H_{lq} \setminus L_q(V_{lq})$, where $L_q = (A_q, B_q, B)$, is three dimensional, and the following functions are the basis in $\hat{H}_{lq}$:

$$
\begin{aligned}
\psi_{q1} &= ((1,0),(1,0),(1,0)), \\
\psi_{q2} &= ((0,1),(0,1),(0,1)), \\
\psi_{q3} &= ((y_2,-y_1),(y_2,-y_1),(y_2,-y_1)),
\end{aligned}
\tag{5.10}
$$

Here in the expressions for the functions ψ_{qi} the first pair belongs to $C^l(\Omega_q)^2$, the second to $C^{l+1}(S_{2q})^2$, and the third to $C^{l+1}(S_1)^2$. In the case considered equality (2.17) holds with $k_q = 3$, and we deifne the operator G_q by (2.18). Owing to the Remark of Section 2 we obtain the following assertion.

THEOREM 5.1 *Let the spaces V_{lq} and H_{lq} be given by expressions (5.6), the operator $L_q = (A_q, B_q, B)$ be defined by the formulae (4.7), (5.4), (5.5) and the operator G_q by (2.18) with $k_q = 3$, where ϕ_{qi}, ψ_{qi} are determined by (5.7) and (5.10). Then the operator $L_{q1} = L_q + G_q$ is an isomorphism of V_{lq} onto H_{lq}.*

Suppose now that the surface forces F, given on the external boundary, satisfy conditions

$$
\begin{aligned}
&F \in C^{l+1}(S_1)^2, \quad \int_{S_1} F_i ds = 0 \quad i = 1,2, \\
&\int_{S_1} (F_1 y_2 - F_2 y_1) ds = 0.
\end{aligned}
\tag{5.11}
$$

Then, owing to Theorem 5.1, there exists a unique function $u_q \in C^{l+2}(\Omega_q)^2$ such that

$$
\begin{aligned}
&A_q u_q = 0 \text{ in } \Omega_q, \quad B_q u_q = 0 \text{ on } S_{2q}, \\
&B u_q = F \text{ on } S_1, \quad G_q u_q = 0 \text{ in } \hat{H}_{lq} (\text{in } \mathbb{R}^3).
\end{aligned}
\tag{5.12}
$$

In the same way as above, using Theorem 5.1 we prove the subsequent assertion.

THEOREM 5.2 *Let the conditions of Theorem 5.1 hold and the sets M, M_1 be defined by (5.1) and (4.17), the functionals G_1 and G_0 given by (4.15), (4.16), where u_q is the solution of the problem (5.12). Let also function F satisfies conditions (5.11), and a non-empty set U be given by (4.18). Then there exists a solution of problem (4.19).*

It is obvious that in the considered case the optimization problem consists in finding the shape of internal boundary with which the elastic body has minimal weight (volume) and the constraint on strength is satisfied.

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(all positions, exept [9], [14–16], are in Russian)

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