Control
and Cybernetics
VOL. 19 (1990) No. 3-4

Shape Optimization of Contact Problems

by

A. Myśliński

System Research Institute,
Polish Academy of Sciences,
ul. Newelska 6,
01-447 Warsaw, Poland

A shape optimization problem of an elastic body being in contact with a rigid foundation is considered. The contact with given friction occurs at a portion of the boundary of the body. The problem consists in finding, in the contact region, the boundary of the domain occupied by the body in such a way that the normal contact stress is minimized. It is assumed that the volume of the body is constant. A dual variational method, where the function satisfying the state inequality and its gradient are selected as independent variables, is employed to solve this problem. Taking into account this dual approach ,the necessary optimality condition is formulated using the material derivative method. Numerical examples are provided.

1. Introduction

This paper is concerned with formulation of a necessary optimality condition for a shape optimization problem of an elastic body in unilateral contact with a rigid foundation as well as with numerical solution of this shape optimization problem. It is assumed that the contact with given friction [4,6,8,13] occurs at a portion of the boundary of the body. The equlibrium state of this contact problem is described by an elliptic variational inequality of the second order. The existence, uniqueness and regularity of solutions to this variational inequality were investigated in [4,6,8,13].

The shape optimization problem for the elastic body in contact consists in finding, in a contact region, such shape of the boundary of the domain occupied by the body that the normal contact stress is minimized. It is assumed that the volume of the body is constant.

Shape optimization of contact problems was considered, for instance in [6,7,11], where necessary optimality conditions were formulated and convergence of finite-dimensional approximation was shown. In literature this problem was solved by use of variational methods where the function satisfying the state inequality was selected as the state variable only.

Numerical experiments reported, e.g. in [10], indicate that we can solve numerically the elliptic problem more accurately using the decomposition approach. In this approach, based on Hellinger-Reissner variational principle [3], the function satisfying the state inequality and its gradient are chosen as independent variables.

In this paper we shall study this shape optimization problem for an elastic body in unilateral contact employing this dual variational approach. Taking into account this approach and using material derivative method [14] as well as the results of differentiability of solutions to the variational inequality [11,12] we calculate the directional derivative of the cost functional and we formulate necessary optimality condition for this problem. Note that in this paper we employ a different dual formulation of the contact problem than the one employed in [11]. The shape optimization problem for a punch pushed in a rigid foundation was solved numerically. Finite element method [3,8] was used as discretization method. Uzawa's algorithm [8] combined with SOR algorithm [8] were used to solve the state system. Shifted penalty function method combined with conjugate gradient method [6] were used as optimization method. Numerical examples are provided.

We shall use the following notation :

$\Omega \subset R^2$	is bounded domain with Lipschitz continuous boundary Γ
$L^2(\Omega)$	is the space of square integrable functions with inner product given by :

	(y,z) $= \int_\Omega y(x)z(x)dx$
$H^1(\Omega)$	is a Sobolev space of order 1 [1]
$H^{1/2}(\Gamma)$	is the space of traces of functions from the space $H^1(\Omega)$ on the boundary Γ [1]
$H^{-1/2}(\Gamma)$	is the space dual to $H^{1/2}(\Gamma)$ [1]
$n = (n_1, n_2)$	is the unit outward versor to the boundary Γ

$x = (x_1, x_2) \in R^2$, $y_{k,l} = \partial y_k/\partial x_l$, $y_{k,lj} = \partial^2 y_k/\partial x_l \partial x_j$, $k,l,j = 1,2$. $y = (y_1, y_2)$, $\nabla y = (\bar{y}^1, \bar{y}^2)$, $\bar{y}^i = col(y_{i,1}, y_{i,2})$ $\quad i = 1,2$ is a gradient of function y with respect to x, div y $= y_{1,1} + y_{2,2}$, ${}^*u$ (${}^*A$) is a transpose of a vector u (matrix A). We shall use the summation convention over repeated indices [3,4,8].

2. Contact problem formulation

Consider deformations of an elastic body occupying domain $\Omega = \Omega(v) \subset R^2$ depending on a function v. Domain $\Omega(v)$ has the following geometrical structure

$$\Omega = \Omega(v) = \{(x_1, x_2) \in R^2 : 0 < x_1 < a, \quad 0 \leq v(x_1) < x_2 \leq b\} \tag{2.1}$$

where a and b are given positive constants. The boundary Γ of domain Ω is Lipschitz continuous. The body is subjected to body forces $f = (f_1, f_2)$. Moreover, surface tractions $p = (p_1, p_2)$ are applied to a portion Γ_1 of the boundary Γ. We assume that the contact conditions are prescribed on the portion $\Gamma_2 = \Gamma_2(v)$ of the boundary Γ. Moreover $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j, i,j = 0,1,2$, $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$.

We denote by $u = (u_1, u_2)$ the displacement of the body and by $\sigma = \{\sigma_{ij}(u(x))\}$,$i,j = 1,2$, the stress field in the body. We shall consider elastic bodies obeying Hook's law [3,8,13] :

$$\sigma_{ij}(x) = c_{ijkl}(x)e_{kl}(u(x)) \quad x \in \Omega \tag{2.2}$$

$i,j,k,l = 1,2$. We use here the summation convention over repeated indices [8]. The strain $e_{kl} = e_{kl}(u)$, $k,l = 1,2$, is defined by :

$$e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}) \tag{2.3}$$

$c_{ijkl}(x)$, $i,j,k,l = 1,2$ are components of Hook's tensor. It is assumed that elasticity coefficients c_{ijkl} satisfy usual symmetry and ellipticity conditions [3,8]. In an equlibrium state a stress field σ satisfies the system [3,6,8,13] :

$$\sigma_{ij}(x)_{,j} = -f_i(x),\ x \in \Omega\ \ i,j = 1,2 \tag{2.4}$$

where $\sigma_{ij}(x)_{,j} = \partial \sigma_{ij}(x)/\partial x_j,\ i,j = 1,2$. The following boundary conditions are given :

$$u_i(x) = 0 \ \text{ on } \Gamma_0\ i = 1,2 \tag{2.5}$$

i.e.,the body is clamped along the boundary Γ_0. The surface traction p applied on Γ_1 produces on Γ_1 the stress satisfying :

$$\sigma_{ij}(x)n_j = p_i \text{ on } \Gamma_1\ i,j = 1,2 \tag{2.6}$$

We shall consider deformation of an elastic body in contact with a rigid foundation along the boundary Γ_2 :

$$u_N \leq 0,\ \sigma_N \leq 0,\ u_N\sigma_N = 0 \text{ on } \Gamma_2 \tag{2.7}$$

We assume that unknown apriori contact region is inside a prescribed area. We shall consider contact problem on Γ_2 with a prescribed friction, i.e.[8,12] :

$$\sigma_T u_T + \mid u_T \mid = 0,\ \mid \sigma_T \mid \leq 1 \text{ on } \Gamma_2 \tag{2.8}$$

Here we denote :

$$u_N = u_i n_i\ \ \sigma_N = \sigma_{ij} n_i n_j\ \ i,j = 1,2 \tag{2.9}$$

$$(u_T)_i = u_i - u_N n_i\ \ (\sigma_N)_i = \sigma_{ij} n_j - \sigma_N n_i\ \ i,j = 1,2 \tag{2.10}$$

Taking into account (2.2)-(2.4) we obtain that for given body forces $f = (f_1, f_2)$ and surface tractions $p = (p_1, p_2)$ the displacement $u = (u_1, u_2)$ satisfies in Ω the system of equations [9,12] :

$$[c_{ijkl}(x)e_{kl}(u(x)]_{,j} = -f_i\ \ x \in \Omega \tag{2.11}$$

Moreover, the displacement u and its derivatives satisfy boundary conditions (2.5)-(2.8).

We shall consider problem (2.11) with boundary conditions (2.5)-(2.8) in variational form. Let $f \in [L^2(\Omega)]^2$ and $p \in [H^{-1/2}(\Gamma_1)]^2$. Let us introduce :

$$F = \{z \in [H^1(\Omega)]^2\ :\ z_i = 0 \text{ on } \Gamma_0\ , i = 1,2\} \tag{2.12}$$

$$K = \{z \in F\ :\ z_i n_i \leq 0 \text{ on } \Gamma_2\ , i = 1,2\} \tag{2.13}$$

By $a(.,.) : F \times F \to R$ we denote the bilinear form given by :

$$a(u,v) = \int_\Omega c_{ijkl} u_{i,j} v_{k,l} dx \tag{2.14}$$

By $l(.) \,:\, F \rightarrow R$ we denote the linear form :

$$l(v) = \int_\Omega f v dx + \int_{\Gamma_1} p v ds \tag{2.15}$$

By $j(.) :\, F \rightarrow R$ we denote the nondifferentiable functional :

$$j(v) = \int_{\Gamma_2} \mid v_T \mid ds \tag{2.16}$$

The system (2.13) with boundary conditions (2.5)-(2.8) can be written in an equivalent variational form [3,4,8,13] :

Find an element $u \in K$ such that :

$$a(u, v-u) + j(v) - j(u) \geq l(v-u) \; \forall v \in K \tag{2.17}$$

It was shown in [4,8] that problem (2.17) has a unique solution. Moreover, problem (2.17) is equivalent to the following optimization problem [3,8] :

Find an element $u \in K$ minimizing the cost functional :

$$J(v) = \frac{1}{2} a(v,v) - l(v) + j(v) \tag{2.18}$$

over the set K.

Our goal is to consider problem (2.17) or (2.18) in the mixed formulation, convenient for numerical computation [9,10]. In order to do it we introduce the derivative of the function u as new independent variable. We use this variable to formulate a dual optimization problem to the problem (2.18). Let us introduce the following spaces :

$$S = S(\Omega) = \{\sigma_{ij} \in [L^2(\Omega)]^4 \;:\; \sigma_{ij} = \sigma_{ji} \; i,j = 1,2\}$$

$$Q = Q(\Omega) = \{\sigma \in S \;:\; \text{div}\, \sigma \in [L^2(\Omega)]^2\}$$

$$\begin{aligned} Z = Z(\Omega) = \{\sigma \in Q \quad &: \; \text{div}\sigma = f \text{ in } \Omega, \; \sigma_{ij} n_j = p_i \text{ on } \Gamma_1 \\ &\mid \sigma_T \mid \leq 1 \;\; \sigma_N \leq 0 \text{ on } \Gamma_2\} \end{aligned} \tag{2.19}$$

Note, that $\sigma_N \in H^{-1/2}(\Gamma_2)$ as well as $\sigma_T \in H^{-1/2}(\Gamma_2)$ [5,13]. $\sigma_N \leq 0$ denotes that $(\sigma_N, v) \leq 0$ for all $v \geq 0$, $v \in H^{-1/2}(\Gamma_2)$. $\mid \sigma_T \mid \leq 1$ on Γ_2 denotes that restriction of σ_T on Γ_2 belongs to $(L^\infty(\Gamma_2))^2$ and $\mid \sigma_T \mid \leq 1$, a.e. on Γ_2 [4,12].

Recall from [9] that the dual optimization problem to the problem (2.18) has the following form :

Find $\sigma \in Z$ minimizing the cost functional :

$$I(\tau) = \frac{1}{2} \int_\Omega b_{ijkl} \tau_{ij} \tau_{kl} dx \tag{2.20}$$

over the set Z given by (2.19) .

where $b_{ijkl}, i, j, k, l = 1, 2$ are the compliance coefficients satisfying usual symmetry and ellipticity conditions [4,5,13]. In order to formulate necessary optimality condition for problem (2.20) we introduce Lagrange multiplier $q \in K$ corresponding to the constraints (2.19). The necessary optimality condition to the problem (2.20) is equivalent to the conditions for a saddle point of a Lagrangian L of this problem [3,4,8]. The Lagrangian L is defined by :

$$L(.,.) \; : \; Q \times K \rightarrow R$$

$$\begin{aligned} L(\tau, q) \quad &= 1/2 \textstyle\int_\Omega b_{ijkl} \tau_{ij} \tau_{kl} dx + \int_\Omega \tau_{ij} e_{kl}(q) dx - \int_\Omega f q dx \\ &\qquad - \textstyle\int_{\Gamma_1} p q ds - \int_{\Gamma_2} \tau_T q_T ds \end{aligned} \tag{2.21}$$

Note that the Lagrange multiplier q corresponds to the displacement field [3,4,8].

The necessary optimality condition for the problem (2.20) can be written in the form [9] :

Find $(\sigma, q) \in Q \times K$ satisfying :

$$\int_\Omega b_{ijkl} \sigma_{ij} \tau_{kl} dx + \int_\Omega \tau_{ij} e_{kl}(q) dx - \int_{\Gamma_2} \tau_T q_T ds = 0 \;\; \forall \tau_{ij} \in Q \tag{2.22}$$

$$\int_\Omega \sigma_{ij} e_{kl}(\eta) dx - \int_\Omega f \eta dx \int_{\Gamma_1} p \eta ds - \int_{\Gamma_2} \sigma_T \eta_T ds \geq 0 \; \forall \eta \in K \tag{2.23}$$

Note that since there exists solution, to the primal problem (2.18) it follows that the set Z is nonempty convex set in Q [8]. Moreover the cost functional (2.20) is strongly convex. It implies existence of a unique solution to the dual optimization problem (2.20).

3. Formulation of the shape optimization problem

Let $\tilde{\Omega} \subset R^2$ be a domain such that for all $v : \Omega(v) \subset \tilde{\Omega}$. Let M be a set :

$$M = \{\phi \in [H^1(\tilde{\Omega})]^2 \ : \ \phi \leq 0 \text{ on } \tilde{\Omega}, \ \| \phi \|_{H^1(\tilde{\Omega})} \leq 1\} \tag{3.1}$$

We shall consider domains $\Omega(v)$ depending on a function v from the set U_{ad} defined by :

$$\begin{aligned} U_{ad} = \ & \{v \in C^{1,1}([0,a]) \ : \mid \tfrac{dv}{dx_1} \mid \leq c_1, \ \forall x_1 \in [0,a] \\ & \mid \tfrac{d^2v}{dx_1^2} \mid \leq c_2 \text{ a.e. in } (0,a) \ , \ \textstyle\int_{\Omega(v)} dx = c_3\} \end{aligned} \tag{3.2}$$

where c_1,c_2 and c_3 are given positive constants. The set U_{ad} is assumed to be nonempty. $C^{1,1}([0,a])$ [6] denotes the set of Lipschitz continuous functions on $[0,a]$ having also Lipschitz continuous first derivatives. We shall consider the following shape optimization problem :

For given function $\phi \in M$, find function $v \in U_{ad}$

minimizing the cost functional

$$J_\phi(v) = \int_{\Gamma_2(v)} \sigma_N \phi_N ds \tag{3.3}$$

over the set U_{ad}. σ_N is a normal component of the

stress field σ satisfying (2.22)- (2.23).

The cost functional (3.3) approximates normal contact stress on the boundary Γ_2, i.e., original boundary flux cost functional [6]. Since the shape optimization problem with the boundary flux cost functional is difficult to solve we modify the original cost functional introducing an auxiliary function ϕ [6] and the form (3.3). The shape optimization problem (3.3) consists in finding such boundary $\Gamma_2(v)$ depending on function v that the normal contact stress is minimized. It is assumed that the volume of the body is constant and the function v is bounded and has bounded first and second derivatives.

4. Necessary optimality condition

LEMMA 1 *The directional derivative $dJ_\phi(v;k)$ of the cost functional (3.3) for given function $\phi \in M$ with respect to $v \in U_{ad}$ at a point $v \in U_{ad}$ in a direction $k \in U_{ad}$ has the form :*

$$\begin{aligned} dJ_\phi(v;k) = \textstyle\int_\Gamma [\sigma_{ij} e_{kl}(\phi + w) - f(\phi + w)] k(x_1) n_2 ds - \\ \textstyle\int_{\Gamma_1} [(\nabla p(\phi + w) + p \nabla(\phi + w)) k(x_1) + p(\phi + w) D] ds + \\ E_1(\lambda, u) - E_1(\sigma, \phi + w) - E_2(\sigma, \lambda, u) \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} E_1(\tau, q) = \quad & \textstyle\int_{\Gamma_2(v)} \{[(\nabla \tau k(x_1))_T - (\bar{N} + \bar{N}^*)\tau_N] q_T + \\ & [(\nabla q(k(x_1))_T - (\bar{N} + \bar{N}^*) q_N] \tau_T + \tau_T q_T D\} ds \end{aligned} \tag{4.2}$$

$$E_2(\sigma, \tau, q) = \int_\Gamma (b_{ijkl} \sigma_{ij} \tau_{kl} + e_{ij}(q) \tau_{kl}) k(x_1) n_2 ds \tag{4.3}$$

$$D = -n_1 n_2 \frac{dk}{dx_1}, \ \bar{N} = D(1 - n_1, -n_2), \quad \bar{N}^* \textit{ is a transpose of } \bar{N} \tag{4.4}$$

The stress field $\lambda \in Q$ and the displacement field $w \in F_1$ are the solution of the following adjoint system :

$$\int_\Omega (b_{ijkl} \lambda_{ij} \tau_{kl} + e_{ij}(\phi + w) \tau_{kl}) dx - \int_{\Gamma_2(v)} \tau_T (\phi + w)_T ds = 0 \ \ \forall \tau \in Q \tag{4.5}$$

$$\int_\Omega \lambda_{ij} e_{kl}(\eta) dx - \int_{\Gamma_2(v)} \lambda_T \eta_T ds = 0 \ \ \forall \eta \in F_1 \tag{4.6}$$

$$F_1 = \{z \in F : z_2 = v \ \ on \ B\}, \ \ B = \{x \in \Gamma_2(v) : u_2 = v\} \tag{4.7}$$

Proof. The material derivative $dJ_\phi(v : k)$ is defined by :

$$dJ_\phi(v;k) = \lim_{t \to 0} [J_\phi(v_t) - J_\phi(v)]/t \tag{4.8}$$

where $v_t = v + tk$. Taking into account that the cost functional (3.3) can be written in the form [4,8] :

$$J_\phi(v) = \int_\Omega \sigma_{ij} e_{kl}(\phi) dx - \int_\Omega f\phi dx - \int_{\Gamma_1} p\phi ds - \int_{\Gamma_2(v)} \sigma_T \phi_T ds \tag{4.9}$$

as well as applying the material derivative method [7,11,12,14] to transport the cost functional $J_\phi(v_t)$ into a fixed domain $\Omega(v)$ and taking into account the results of differentiability of solutions to a variational inequality [11,12] to calculate the limit (4.8) with $t \to 0$ we obtain (4.1). □

LEMMA 2 *There exists a Lagrange multiplier $\mu \in R^3$ such that for all functions $k(x_1) \in U_{ad}$ the following condition holds :*

$$dJ_\phi(v;k) + \mu_1 \int_\Gamma k(x_1) n_2 ds + \mu_2 sgn(\frac{dv}{dx_1}) \frac{dv}{dx_1} + \mu_3 sgn(\frac{d^2v}{dx_1^2}) \frac{d^2k}{dx_1^2} \geq 0 \tag{4.10}$$

where $dJ_\phi(v;k)$ is given by (4.1) and the function sgn is defined as follows : $sgn(x) = 1$ for $x > 0$, $sgn(x) = 0$ for $x = 0$, $sgn(x) = -1$ for $x < 0$.

Proof of Lemma 2 is standard [2].

5. Numerical results

The discretized optimization problem (3.3) was solved numerically. The boundary Γ of the domain (2.1) is divided into parts Γ_0, Γ_1 and $\Gamma_2 = \Gamma_2(v)$ given by :

$$\begin{aligned} \Gamma_0 = &\ \{x \in R^2 : \quad x_1 = 0, a \quad x_2 \in (v(x_1), b)\} \\ \Gamma_1 = &\ \{x \in R^2 : \quad x_1 \in (0,a) \quad x_2 = b\} \\ \Gamma_2 = &\ \{x \in R^2 : \quad x_2 = v(x_1)\} \end{aligned} \tag{5.1}$$

Function v belongs to the set of admissible design parameters U_{ad} given by (3.2).

The conforming finite element method [3,8] was used as an discretization method of the optimization problem (3.3). The boundary $\Gamma_2(v)$ was approximated by a piecewise linear function. The domain (5.1) was divided into triangles satisfying usual requirements concerning the mutual positions of two triangles [8]. The sets Q, Z, K were approximated by the sets of piecewise linear functions using superelement technique [3,8]. Since the considered optimization problem (3.3) with (5.1) is symmetric with respect to the line $x_1 = a/2$ in the computation one half of the domain Ω was used only.

Uzawa's method combined with SOR method [8] were used to solve the discretized systems (2.22), (2.23) and (4.5), (4.6). The conjugate gradient method combined with shifted penalty function method [2,5,7] were used to solve the discretized optimization problem (3.3).

The numerical data are as follows : $a = 8$ $b = 1$, $f = 0$ $p_1 = 0$ $p_2 = -5.610^6$, $\phi = 1$, on $\tilde{\Omega} \supset \Omega(v)$ for all v.

The results are presented in Table 1, where the initial shape S_i as well optimal shape S_o of one half of the contact boundary Γ_2 are given. The point $x_1 = 4.0$ corresponds to the middle point of the contact boundary. For the body with initial shape boundary S_i the normal contact stress has its peak in the middle of the contact boundary. For the body with optimal shape boundary S_o the normal contact stress is almost constant and shows very mild increase while approaching the middle point of the contact boundary.

The speed of the proposed algorithm is very slow. The speed of the algorithm depends on a proper choice of parameter values in Uzawa's and SOR algorithms as well as on accuracy of calculation of the Lagrange multipliers.

x_1	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
S_i	0.12	0.10	0.08	0.06	0.04	0.02	0.01	0.00	0.00
S_o	0.12	0.08	0.04	0.01	0.00	0.00	0.03	0.06	0.08

Table 1. Initial and optimal shapes of the contact boundary.

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