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Energy bounds by material rotation [1]

by

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Using recent results from sensitivity analysis, the energy bounds for anisotropic materials can be found. When the material is used according to the obtained criteria, this means that the most stiff or most flexible solid or structure is found. The paper includes extensions from orthotropic to general anisotropic and discusses the selection of an unambiguous material coordinate system.

1. Introduction

Design with advanced materials, such as anisotropic laminates, is a challenging area for optimization. We shall here restrict ourselves to plane problems, as in the early work of Banichuk [1] (which includes further early references). Recent work by the author [2], [3] was conducted independently and it is nice to note that the formulations are rather parallel. Similar research is carried out by

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Sacchi Landriani & Rovati [4]. The area as a whole is still very active and extended results are expected in the near future.

In a solid subjected to a non–uniform strain field, the extrema of elastic energy will be obtained with non–uniform material orientation. Thus, a complete evaluation of the possible elastic energies $U = U(\theta = \theta(x))$ is impossible. We shall describe an iterative technique to obtain the bounds for this more practical case. The results from [2] are applied in order to avoid solutions with local extrema, but like a gradient technique, every design change is based on an actual stress/strain field which itself changes with the design. Iteration will therefore still be necessary, but the number of necessary iterations are normally few (5–10).

The energy bounds and corresponding optimal fields of orientation depend on the specific problem, i.e. on the given plane domain – on the support condition – on the load condition – and on the orthotropic material. Therefore, a number of parameter studies can be performed. We have chosen to study in more detail the influence of the material parameters, limiting these only by the fact that the material constitutive law must be positive definite.

When the principal axes of an orthotropic material are equal to, say, the principal strain axes, it follows directly that principal stress axes also equal those of material and strain. However, optimal orientations exist for which the principal axes of material differ from those of the principal strains. Even for this case it is proved in [3] that the principal axes of stress equal those of the principal strains.

The sensitivity analysis that proves local gradient determination relative to a fixed strain field is presented. The physical understanding of these results have many aspects outside the scope of the present paper. The early paper by Masur [5] includes valuable information about this sensitivity analysis.

For orthotropic materials, a single optimization parameter controls the design angle. This parameter includes information about material as well as about the state of strain. It is used as an optimization criterion and in principle, the optimization procedure is a non–gradient technique. In this way local extrema are avoided. For non–orthotropic materials, analytical solutions are difficult to obtain, but Newton–Raphson iterations can be applied.

2. Sensitivity analysis for energy in non–linear elasticity

With reference to the chapter on energy methods in Przemieniecki [6] let us start with the *work equation*

$$W + W^C = U + U^C \tag{2.1}$$

where W, W^C are physical and complementary work of the external forces, and U, U^C are physical and complementary elastic energy, also named strain and stress energy, respectively. In fact (2.1) is just an identity for any part of the solid/structure and therefore for the total solid/structure.

The work equation (2.1) holds for any *design* h and therefore for the total differential quotient with respect to h

$$\frac{dW}{dh} + \frac{dW^C}{dh} = \frac{dU}{dh} + \frac{dU^C}{dh} \tag{2.2}$$

Now in the same way as h represents the design field generally, ϵ represents the strain field and σ represents the stress field. Remembering that as a function of h, ϵ we have W, U, while the complementary quantities W^C, U^C are functions of h, σ. Then we get (2.2) more detailed by

$$\begin{aligned} &\frac{\partial W}{\partial h} + \frac{\partial W}{\partial \epsilon}\frac{\partial \epsilon}{\partial h} + \frac{\partial W^C}{\partial h} + \frac{\partial W^C}{\partial \sigma}\frac{\partial \sigma}{\partial h} = \\ &\qquad \frac{\partial U}{\partial h} + \frac{\partial U}{\partial \epsilon}\frac{\partial \epsilon}{\partial h} + \frac{\partial U^C}{\partial h} + \frac{\partial U^C}{\partial \sigma}\frac{\partial \sigma}{\partial h} \end{aligned} \tag{2.3}$$

The *principles of virtual work* which holds for solids/structures in equilibrium are

$$\frac{\partial W}{\partial \epsilon} = \frac{\partial U}{\partial \epsilon} \tag{2.4}$$

for the physical quantities with strain variation and for the complementary quantities with stress variation we have

$$\frac{\partial W^C}{\partial \sigma} = \frac{\partial U^C}{\partial \sigma} \tag{2.5}$$

Inserting (2.4) and (2.5) in (2.3) we get

$$\frac{\partial U^C}{\partial h} - \frac{\partial W^C}{\partial h} = -\left(\frac{\partial U}{\partial h} - \frac{\partial W}{\partial h}\right) \tag{2.6}$$

and *for design independent loads*

$$\boxed{\left(\frac{\partial U^C}{\partial h}\right)_{\substack{\text{fixed}\\ \text{stresses}}} = -\left(\frac{\partial U}{\partial h}\right)_{\substack{\text{fixed}\\ \text{strains}}}} \tag{2.7}$$

as stated by Masur [5]. Note that the only assumption behind this is the design independent loads $\frac{\partial W}{\partial h} = 0$, $\frac{\partial W^C}{\partial h} = 0$.

To get further into a *physical interpretation* of $(\frac{\partial U}{\partial h})_{\text{fixed strains}}$ (and by (2.7) of $(\frac{\partial U^C}{\partial h})_{\text{fixed stresses}}$) we need the relation between external work W and strain energy U. Let us assume that this relation is given by the constant c

$$W = cU \tag{2.8}$$

For linear elasticity and dead loads we have $c = 2$ and in general we will have $c > 1$.

Parallel to the analysis from (2.1) to (2.3) we based on (2.8) get

$$\frac{\partial W}{\partial h} + \frac{\partial W}{\partial \epsilon}\frac{\partial \epsilon}{\partial h} = c\frac{\partial U}{\partial h} + c\frac{\partial U}{\partial \epsilon}\frac{\partial \epsilon}{\partial h} \tag{2.9}$$

that for design independent loads $\frac{\partial W}{\partial h} = 0$ with virtual work (2.4) gives

$$\frac{\partial W}{\partial \epsilon}\frac{\partial \epsilon}{\partial h} = \frac{\partial U}{\partial \epsilon}\frac{\partial \epsilon}{\partial h} = \frac{c}{1-c}\frac{\partial U}{\partial h} \tag{2.10}$$

and thereby

$$\frac{dU}{dh} = \frac{\partial U}{\partial h} + \frac{\partial U}{\partial \epsilon}\frac{\partial \epsilon}{\partial h} = \frac{1}{1-c}\left(\frac{\partial U}{\partial h}\right)_{\substack{\text{fixed}\\ \text{strains}}} \tag{2.11}$$

Note, in this important result that with $c > 1$ we have different signs for $\frac{dU}{dh}$ and $(\frac{\partial U}{\partial h})_{\text{fixed strains}}$.

For the case of *linear elasticity and dead loads* we have $c = 2$ and adding (2.7)

$$\frac{dU}{dh} = -\left(\frac{\partial U}{\partial h}\right)_{\substack{\text{fixed}\\ \text{strains}}} = \left(\frac{\partial U}{\partial h}\right)_{\substack{\text{fixed}\\ \text{stresses}}} \tag{2.12}$$

For the case of *non-linear elasticity* by

$$\sigma = E\epsilon^n \tag{2.13}$$

and still dead loads ($W^C = 0$) we get $c = 1 + n$ and thereby

$$\boxed{\frac{dU}{dh} = -\frac{1}{n}\left(\frac{\partial U}{\partial h}\right)_{\substack{\text{fixed}\\ \text{strains}}} = \frac{1}{n}\left(\frac{\partial U}{\partial h}\right)_{\substack{\text{fixed}\\ \text{stresses}}}} \tag{2.14}$$

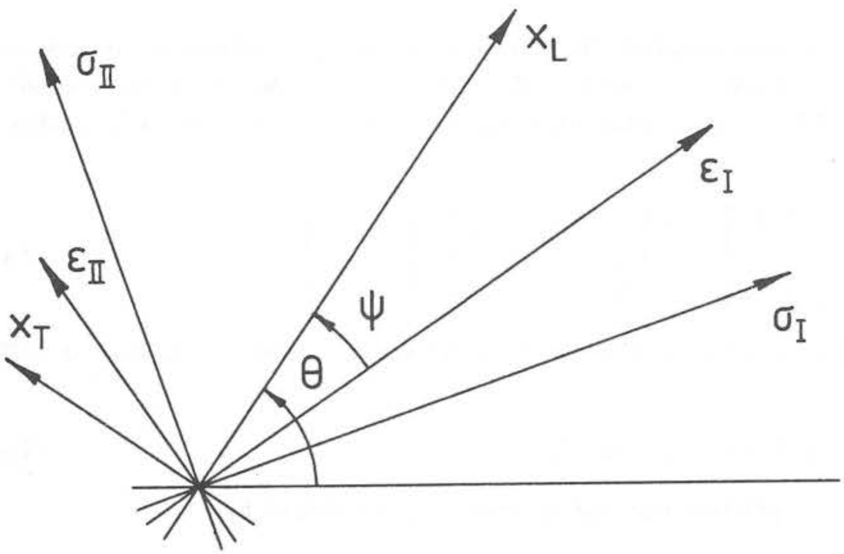


Figure 3.1. The Cartesian coordinate systems for – principal stress, principal strain and material.

3. Sensitivity to material orientation

We shall here deal with three Cartesian coordinate systems, as shown in Fig. 3.1. The coordinate system of *principal strain* ϵ_I; ϵ_{II} directions has the major axis referring to the numerically larger strain

$$|\epsilon_I| > |\epsilon_{II}| \tag{3.1}$$

For simplicity we omit the case of $|\epsilon_I| = |\epsilon_{II}|$. Parallel to this, the major axis of *principal stress* is σ_I with

$$|\sigma_I| > |\sigma_{II}| \tag{3.2}$$

We shall see that for optimal solutions, the principal strain directions coincide with principal stress directions. However, it is not yet known how to decide whether ϵ_I coincide with σ_I or with σ_{II}.

For the *material coordinate system* we have used the notation x_L; x_T, often applied in laminate analysis, with L for the direction of fibers and T for the transverse direction. For an orthotropic material we let x_L correspond to the most stiff orthotropic direction, i.e. for the moduli of elasticity we have

$$E_L > E_T \tag{3.3}$$

For anisotropic materials, the literature gives very poor advice on unambiguous selection of material directions. We shall therefore go deeper into the discussion of the different possibilities with relation to the constitutive matrix $[C]_x$ defined by

$$\left\{ \begin{array}{c} \sigma_{LL} \\ \sigma_{TT} \\ \sigma_{LT} \end{array} \right\} = \left[\begin{array}{ccc} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{array} \right]_x \left\{ \begin{array}{c} \epsilon_{LL} \\ \epsilon_{TT} \\ 2\epsilon_{LT} \end{array} \right\} \tag{3.4}$$

We first state the criterion for the material to be orthtropic. As derived in [7] this is

$$C_7 C_2^2 - 4C_7 C_6^2 - 4C_6 C_3 C_2 = 0 \tag{3.5}$$

where the practical material parameters C_i, are defined by

$$\begin{array}{rcl} C_2 & := & \frac{1}{2}(C_{11} - C_{22})_x \\ C_3 & := & \frac{1}{8}((C_{11} + C_{22}) - 2(C_{12} + 2C_{33}))_x \\ C_6 & := & \frac{1}{2}(C_{13} + C_{23})_x \\ C_7 & := & \frac{1}{2}(C_{13} - C_{23})_x \end{array} \tag{3.6}$$

If the material is non–orthotropic, a natural choice of the x_L–axis would be the axis that maximizes C_{11}, like we did for the direction of an orthotropic material. Necessary conditions for a global maximum of $C_{11} = C_{11}(\theta)$ is $C_{11} > C_{22}$; $\frac{dC_{11}}{d\theta} = 0$ and $\frac{d^2 C_{11}}{d\theta^2} < 0$ which gives

$$C_2 > 0; \quad C_6 = -C_7 \text{ (i.e. } C_{13} = 0) \text{ and } C_2 > -4C_3 \tag{3.7}$$

This is not always a sufficient condition, and the most practical solution could be to plot the function $C_{11}(\theta)$ and choose according to this.

Now let us apply the general result (2.11), and a physical interpretation may be useful. When the material orientation is changed in a part of the solid, then in general the stress/strain field and thus the energy density is changed all over the solid. Furthermore, the constitutive relation changes in the part in question and this gives rise to further change in density, but only in that part. The two changes are related by the factor $\frac{c}{1-c}$. Thus, we can not only perform variations with fixed strains/displacements, but also concentrate on the actual part of the solid where the material orientation is subject to change. This means that the partial derivative $(\frac{\partial u}{\partial \psi})_{\text{fixed strain}}$ give all the necessary information. Here, u is the strain energy density at the point/element where angle ψ between material and principal strain is changed.

Extremum angle ψ	Low shear stiffness $C_3 > 0$		High shear stiffness $C_3 < 0$	
	$0 < \gamma < 1$	$1 < \gamma$	$\gamma < -1$	$-1 < \gamma < 0$
$0°$	GLOBAL MIN.	GLOBAL MIN.	GLOBAL MIN.	LOCAL MAX.
$\pm 90°$	LOCAL MIN.	GLOBAL MAX.	GLOBAL MAX.	GLOBAL MAX.
$\pm\frac{1}{2}\arccos(-\gamma)$	GLOBAL MAX.			GLOBAL MIN.

Table 3.1. Angles of global/local maximum/minimum for U.

The results in [2] are presented in terms of a single optimization parameter γ defined by

$$\gamma := \frac{(C_{11} - C_{22})(1 + \frac{\epsilon_{II}}{\epsilon_I})}{((C_{11} + C_{22}) - 2(C_{12} + 2C_{33}))(1 - \frac{\epsilon_{II}}{\epsilon_I})} = \frac{C_2}{4C_3}\frac{1+\eta}{1-\eta} \tag{3.8}$$

where η is the principal strain ratio, $\eta = \frac{\epsilon_{II}}{\epsilon_I}$. By the definition of the coordinate systems, we have $C_2 > 0$ and $|\eta| < 1$. Thus the optimization parameter γ, which reflects material as well as strain state, has the same sign as the material parameter C_3.

By the definition of C_3 in (3.6) we see that a material with high shear stiffness C_{33} can make C_3 negative, but in most cases C_3 is positive. Note that $[C]$ must be positive definite, which gives $(C_{11} + C_{22} - 2C_{12}) > 0$.

Let us now treat only orthotropic materials, i.e. $C_6 = C_7 = 0$. For this class of materials we have

$$\left(\frac{\partial u}{\partial \psi}\right)_{\substack{\text{fixed}\\ \text{strains}}} = -4C_3(\epsilon_I - \epsilon_{II})^2 \sin 2\psi(\gamma + \cos 2\psi) \tag{3.9}$$

and

$$\left(\frac{\partial^2 u}{\partial \psi^2}\right)_{\substack{\text{fixed}\\ \text{strains}}} = -8C_3(\epsilon_I - \epsilon_{II})^2(\cos 2\psi(\gamma + \cos 2\psi) - \sin^2 2\psi) \tag{3.10}$$

With the sign change in (2.12), we obtain the table 3.1 of solutions for a global maximum or minimum.

For non–orthotropic materials, analytical solutions are difficult to obtain, and $(\frac{\partial u}{\partial \psi})_{\text{fixed strain}} = 0$ must be found numerically, say, with Newton–Raphson iterations. A more extended analysis than can be shown here, cf. Poulsen [8], can give information about appropriate starting points for such iterations.

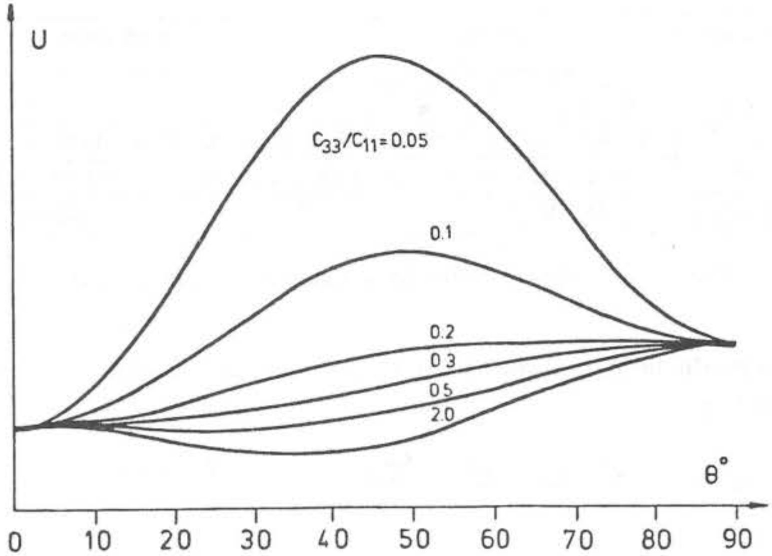


Figure 4.1. Strain energy as a function of material orientation, for $C_{22}/C_{11} = 1/2$; $C_{12}/C_{11} = 1/8$; C_{33}/C_{11} as shown.

4. Energy bounds for specific examples

We shall discuss the results for two specific examples, the first one with a uniaxial stress/strain state and a second more practical example. In the first example we shall study the influence of material parameters from a rather general point of view. Firstly the results in Table 3.1 are illustrated by the graphs in Fig. 4.1.

We note the possibilities of local minima as well as local maxima. Thus a procedure based on a gradient technique may fail to determine a global extreme.

Now, defining non–dimensional material parameters α_2, α_3 and α_4 by

$$\begin{aligned} \alpha_2 &:= 8\tfrac{C_2}{C_{11}}; \\ \alpha_3 &:= 8\tfrac{C_3}{C_{11}}; \\ \alpha_4 &:= 8\tfrac{C_4}{C_{11}} \quad \text{with } C_4 = \tfrac{1}{8}((C_{11}+C_{22}) + (3C_{12} - 2C_{33})) \end{aligned} \tag{4.1}$$

We show in Fig. 4.2 the results of a more complete study. The actual materials are orthotropic, i.e. $C_6 = C_7 = 0$, and uniaxial stress/strain state is assumed.

First of all, it should be noted that every point on every curve in Fig. 4.2 is a solution to an optimization problem, which is: Maximize stiffness or flexibility of a problem with specific material.

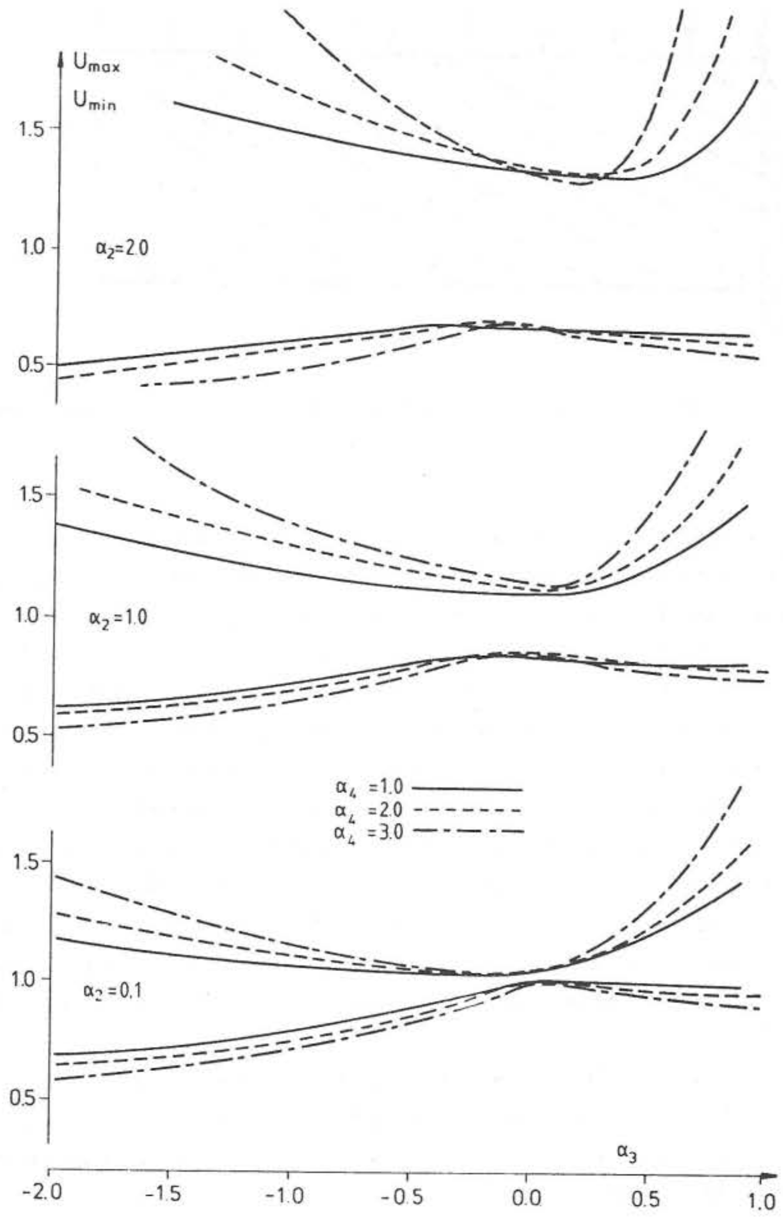


Figure 4.2. Bounds on elastic energy (density), normalized to $\frac{1}{2}(U_{\max}+U_{\min}) = 1$ for $\alpha_3 = 0$.

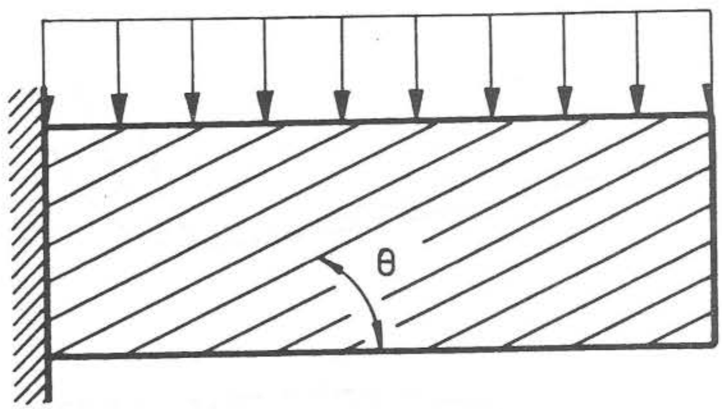


Figure 4.3. A short cantilever with uniformly distributed load. Ratio of length to height is three.

The lower set of curves, corresponding to $\alpha_2 = 0.1$, relate to a material that acts almost identically in the two orthotropic directions, without being isotropic. The other sets of curves, corresponding to increasing value of $\alpha_2 = 1.0$, 2.0, then correspond to materials with increasingly different behaviour for the two orthotropic directions. To give a short description of the results in relation to the parameter α_2, we may say that by increasing the value of α_2, we increase the distance between the bounds of elastic energy. This rather natural result means that the importance of material orientation increases with α_2.

Next let us look at the influence of the parameter α_4, reflecting an indirect effect like Poisson's ratio. In general, there is very little effect on the lower bounds, i.e. on the maximum stiffness. However the upper bound, i.e. the maximum flexibility, is strongly influenced, and the general result is that increasing the value of α_4 gives us the possibility of designinig a much more flexible structure.

The parameter α_3 is taken as the independent variable in Fig. 4.2 because it controls the nature of the solutions according to Table 3.1. We note that the bounds are strongly influenced by the value of α_3, i.e. by the relative shear stiffness of the orthotropic material.

For an isotropic material we have $\alpha_3 = 0$, and it is therefore natural that around $\alpha_3 = 0$, we have the smallest distance between the bounds. Or, putting it in a different way, when the parameter α_3 is close to zero, the orientation

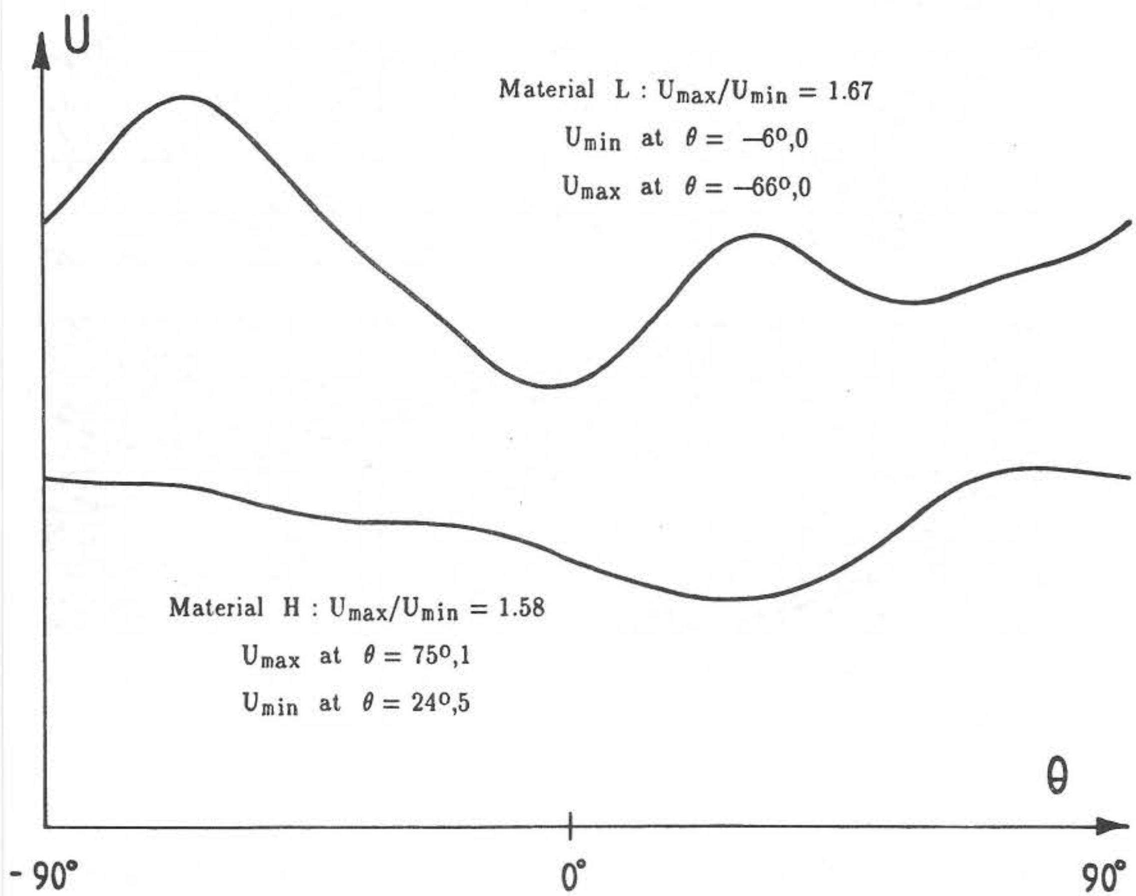
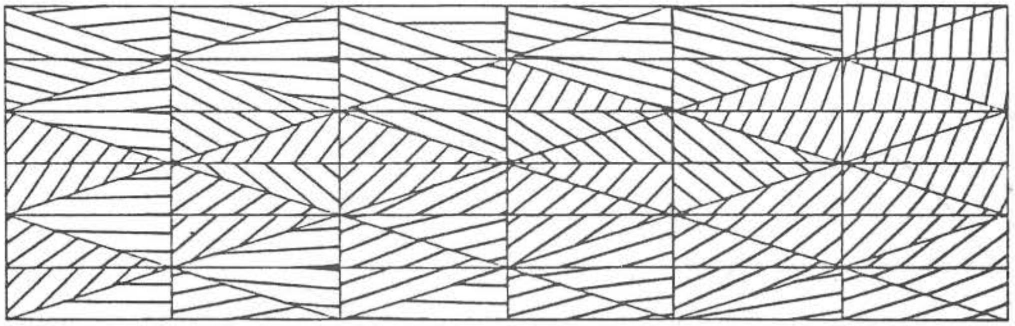


Figure 4.4. Elastic strain energy for different material orientations. Minimum energy measures maximum stiffness and maximum energy measures maximum flexibility.



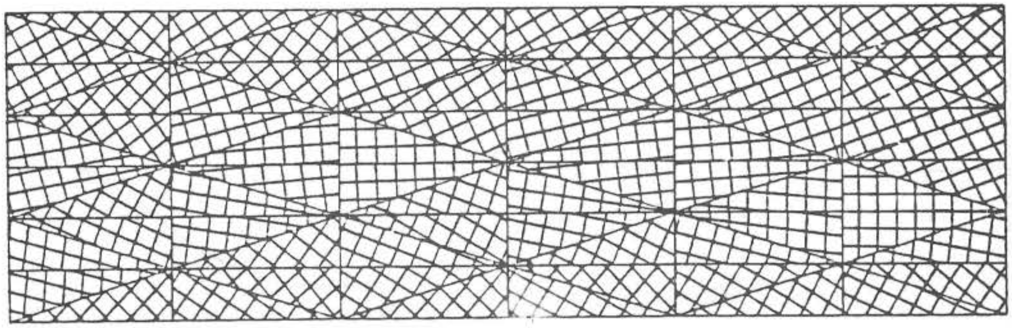


Figure 4.5. For a LOW shear stiffness material, the optimal fields of orientation for the cantilever problem in Fig. 4.3. Upper field for maximum stiffness and lower field for maximum flexibility.

of the material cannot change the stiffness/flexibility very much. For all the results, of this problem we see that for increasing numerical value of α_3, the maximum stiffness increases (lower bound decreases) and that the maximum flexibility also increases. Both these tendencies add to the increase in distance between the bounds, thus giving the designer good possibilities for choosing desirable material orientations.

The second example is shown in Fig. 4.3. The length of this short cantilever is three times the height and the load is uniformly distributed. It is modelled by 72 triangular elements of constant stress/strain, as illustrated in Fig. 4.5 and 4.6. Two different materials, say laminates, are applied, in order to demonstrate differences between solutions with a "low shear stiffness material" ($C_3 > 0$) compared to the solutions with a "high shear stiffness material" ($C_3 < 0$). The specific parameters are:

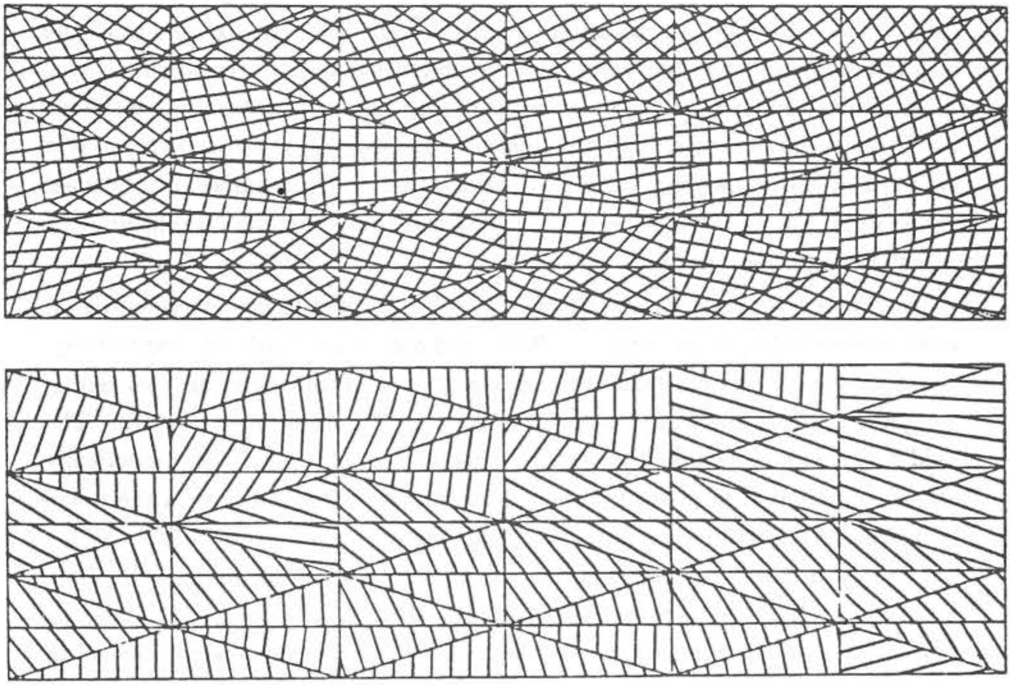


Figure 4.6. For a HIGH shear stiffness material, the optimal fields of orientation for the cantilever problem in Fig. 4.3. Upper field for maximum stiffness and lower field for maximum flexibility.

$$\begin{gathered}\text{Material L — low shear stiffness:}\\ C_{11}=8;\quad C_{22}=4;\quad C_{12}=1;\quad C_{33}=0.5;\quad (C_3=1)\\[1em] \text{Material H — high shear stiffness:}\\ C_{11}=8;\quad C_{22}=4;\quad C_{12}=3;\quad C_{33}=3.5;\quad (C_3=-1)\end{gathered} \tag{4.2}$$

First, we shall present the results corresponding to a uniform material orientation throughout the model. The angle θ of the material is shown in Fig. 4.3 and by means of a number of finite element solutions, we can obtain the results shown in Fig. 4.4. This will naturally change with a refinement of the model, but here we are mostly interested in the relative change in the elastic energy, i.e. $U = U(\theta)$. The minimum value and the maximum value are also given in Fig. 4.4 and we notice the importance of the material orientation when dealing with anisotropic materials.

Next, we shall extend the analysis to local (here element) material orientation. In a solid subject to a non–uniform strain field, the extremum of the elastic energy will be obtained with non–uniform material orientation. Thus, in the finite element model, we must deal with different material orientations in the different elements. The orientation of the individual element material is changed in the reanalyses. In total, it means that a large number of rotational transformations have to be performed, which is the reason for establishing the transformation reported in [7].

In Fig. 4.5 we show for material L the two fields of optimal material orientation, corresponding to maximum stiffness and maximum flexibility, respectively. Compared with the corresponding values with uniform material orientation, we get $U_{\min}/U_{\min}(\text{uniform}) = 0.506$ and $U_{\max}/U_{\max}(\text{uniform}) = 1.373$. The field for maximum stiffness is in the directions of the resulting, numerically larger, principal strains, while the field for maximum flexibility is in the range of $\pm(45° - 51°)$ relative to the principal strains.

Finally, in Fig. 4.6, we present the corresponding result for material H. Here, we get $U_{\min}/U_{\min}(\text{uniform}) = 0.859$ and $U_{\max}/U_{\max}(\text{uniform}) = 1.528$. For this material, the field of maximum flexibility is in the direction perpendicular to the resulting, numerically larger, principal strain, while the field of maximum stiffness is mostly in the range of $\pm(30° - 45°)$ relative to the principal strains.

5. Conclusion

Although optimization for minimum strain energy is generally of primary interest, we have also shown solutions which give maximum energy absorption for a structure/solid, where the design parameters are the individual orientations of anisotropic material in the structural elements.

The simplicity of gradient localization is stated, and also the relation between gradients of stress and strain energy is proven. From the practical point of view, it means that we can deal with fixed strain or stress fields.

For a given plane problem, i.e. given plane domain, supports, loads and material parameters, the iterative procedure finds the field of optimal orientations. A normal gradient technique will generally not work, because a spectrum of local optima exists. Therefore, design changes in each iteration must be based on a criterion that identifies the global minimum/maximum.

With coinciding principal directions for material properties, stresses and strains, we always obtain stationary energy solutions. An additional extremum often exists, where the principal strain and stress directions coincide, but are different from the material principal axes. With all these possibilities in each element (and therefore a very large number of combinations), we see the need for a criterion that can identify the globally best solutions.

Much of the analysis of the present paper and its references can be applied to bending of plates. The strains will then be replaced by bendings and the extensional stiffness by bending stiffnesses. The orthotropic nature of the material will then correspond to one without coupling between bending and torsion. In agreement with the early work by Masur [5], the results of present paper may also be valid in relation to strength optimization.

For non–orthotropic cases the domain of angle design must be extended from $0 \leq \theta \leq 90°$ to $0 \leq \theta \leq 180°$. Furthermore, the global extremum in each iteration must be obtained by local iterations to obtain zero gradient of the energy density.

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