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Mathematical aspects of optimal control problems for elliptic equations

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The optimal control problems for systems governed by the second order elliptic equations are discussed. The method of the extension of extremal problems is proposed in order to overcome the difficulties (nonexistence of optimal solutions, nonconvexity of sets of admissible controls etc.) which arise in the shape optimisation and other similar problems. The extension methods based on "convexification" or G–convergence are described and necessary conditions are given.

Optimal control problems for elliptic equations have some properties which are very different from those of the classical problems for ordinary differential equations. The essence is that the main part of elliptic operators can depend upon parameters - the case which corresponds to the optimal shape design and similar problems. In these situations the set U_{ad} of admissible controls u often consists of characteristic functions of sets related to materials or constructions.

We can distinguish here two cases. The first is narrow one where controls are characteristic functions of smooth domains, say uniformly Lipschitz with parameters from a given bounded closed set. The advantage of this case is that controls are easy to perform from the technical point of view and that the set U_{ad} is compact in some strong topologies (from the mathematical point of view). But such a narrowness of the set U_{ad} makes it very hard to carry out sensitivity analysis and practical optimization (we exclude here situations where the set of admissible controls depends ultimately only on some finite number of numerical parameters).

Sometimes such cases can be treated in the framework of the theory of potential where the unknown boundary of domain is given parametrically by control functions and the kernel of the corresponding integral equation depends on these control functions. In the case of piecewise Liapunov surfaces or curves it can be shown that the implicit function which gives the dependence of the solution of the integral equation on control is Frechet differentiable. Hence, the whole apparatus of differential calculus can be applied to it (see O.Lietuvietis [2])

The other case is the opposite one where the set U_{ad} contains characteristic functions of all measurable subsets of a given domain. Here the needle-like variations can be applied, but some other difficulties arise. First of all the set U_{ad} will not be sequentially compact in strong topologies and will not be closed in weak topologies of Lebesgue spaces. Therefore, we cannot assume existence of optimal controls (see, for example, F.Murat [4]). On the other hand, for the needle-like variations of controls the corresponding increment of the solution of the equation has the norm in H^1 which is equivalent to $\varepsilon^{1/2}$ wherein ε corresponds to the measure of the set where the control is varied if the eldest coefficients of the equation depend on control. Hence, if, for example, the cost functional depends nonlinearly on the gradient of the state then we have to take into account the second terms in the Taylor series. This results in lengthy formulas for the increment of functionals (see, for example, U.Raitums [5]).

And, after all, the set U_{ad} will not be convex. This disadvantage is very inconvenient from the point of view of numerical methods.

Therefore, we have to overcome two kinds of difficulties: nonexistence, in general, of optimal solutions (which leads to incorrectness of computional algorithms), and inconvenient structure (U_{ad} is not convex) of the set of admissible controls.

One of the possibilities of handling these problems is to use the extensions of original problems. Two approaches can be applied: the "convexification" of the origanal problem and the methods of the theory of G-convergence.

The "convexification" is nearly the same method as used in optimal control problems for ordinary differential equations. In other terms it is known as passing to the generalized controls (see, for example, G.Gamkrelidze [1], J.Varga [8]). The difference for the case of elliptic equations is that "convexification" does not ensure in general the existence of optimal controls. But nevertheless it gives convex sets of admissible operators and on the way allows to get the necessary conditions for optimality under assumptions weaker then in the case of using the needle-like variations.

The methods of the theory of G-convergence ensure as a rule the existence of optimal solutions in extended problems. But there is lack of effective (good for practical use) description of extended problems. And, additionally, passing to the G-closure of original sets of operators does not lead in general to convex sets.

Nevertheless, by combining both methods it is possible for practically important classes of problems to get extensions of original problems where the admissible sets of operators will be convex and an optimal solution will exist (see U.Raitums [5]).

To illustrate these rather abstract statements we shall consider the following optimal control problem.

Let Ω be a bounded domain in Euclidean space R^n, $n \geq 2$, with uniformly Lipschitz boundary $\partial\Omega$ and $x = (x_1, \ldots, x_n)$ be an arbitrary point in Ω.

Let U_{ad} be a set of admissible controls and we have to minimize the functional

$$I_0(u,z) = \int_\Omega g_0(x,u,z,z_x)dx \tag{1}$$

over all pairs

$$(u,z) \in U_{ad} \times H_0^1 \tag{2}$$

which satisfy the state equation

$$A(u)z \equiv -\frac{d}{dx_i}a_{ij}(x,u)z_{x_j} + b_0(x,u,z,z_x) = 0, \tag{3}$$

$$x \in \Omega, \quad z\,|_{\partial\Omega} = 0,$$

where the functions g_0, b_0 and the matrix-function $A(x,u) = (a_{ij}(x,u)), i,j = 1,\ldots,n$ are fixed.

Here, and in what follows, repeated indices i, j mean summation from 1 to n, equation (3) is understood in the sense of distributions, H_0^1 is the Sobolev space of functions whose first derivatives belong to $L_2(\Omega)$, and which vanish on the boundary $\partial\Omega$.

The Dirichlet boundary value problem and the absence of additional constraints are assumed only for the sake of brevity. The presence of the "youngest" term b_0 in equation (3) is mainly meant for a better description of the strong connection between perturbations which overcome the "youngest" term in equation and the integrand of the functional in the process of extension.

To begin with we recall a technical result (see U.Raitums [5]) which is very useful for the "convexification" approach to the problem (1)-(3).

Lemma 1 Let

$$Q \subset L_2^{(n)}(\Omega) \equiv \underbrace{L_2(\Omega) \times \ldots \times L_2(\Omega)}_{n}$$

be a set of elements $f = (f_1, \ldots, f_n)$ such that for every $f^1, f^2 \in Q$ and a measurable set $E \subset \Omega$ the element f,

$$f(x) = \begin{cases} f^1(x), & x \in E \\ f^2(x), & x \in \Omega \setminus E \end{cases}$$

belongs to Q too.

Then for every element $f^0 \in \overline{\text{co}}\ Q$ there exists a sequence $\{f^K\} \subset Q$ such that

1. $\{f^K\}$ converges weakly in $L_2^{(n)}(\Omega)$ to f^0 as $K \to \infty$.
2. The solutions z_k of boundary value problems

$$div \bigtriangledown z = div\ f, \quad x \in \Omega, \ z \mid_{\partial\Omega} = 0 \tag{4}$$

with $f = f^K$, $K = 0, 1, 2, \ldots$, converges strongly in H_0^1 to the z_0 as $K \to \infty$.

Let us introduce the following hypotheses and definitions.

H1. Functions $b_0(x, u, z, p)$, $g_0 = g_0(x, u, z, p)$, $a_{ij} = a_{ij}(x, u)$, $i, j = 1, \ldots, n$, $x \in \Omega$, $u \in R^m$, , $z \in R$, $p \in R^n$, satisfy the Caratheodory condition.

H2. There exist positive constants $0 < \nu < \mu$ such that for all $x \in \Omega$, $u \in R^m$, $\xi = (\xi_1, \ldots, \xi_n) \in R^n$

$$\nu \mid \xi \mid^2 \leq a_{ij}(x, u)\xi_i\xi_j \leq \mu \mid \xi \mid^2,$$

$$a_{ij}(x,u) = a_{ji}(x,u),\ i,j = 1,\dots,n.$$

H3. The set $U_{ad} \subset L_2^{(m)}(\Omega)$ is bounded in $L_\infty^{(m)}(\Omega)$ and satisfies the condition of decomposition, i.e. if $u_1, u_2 \in U_{ad}$ then for every measurable set $E \subset \Omega$ the function u,

$$u(x) = \begin{cases} u_1(x), & x \in E \\ u_2(x), & x \in \Omega \setminus E \end{cases}$$

belongs to U_{ad} too.

H4. There exists a neighbourhood ω of the zero element in $L_2^{(n)}(\Omega)$ such that for every element $f \in \omega$ and $u \in U_{ad}$ the equation

$$A(u)z = div f, x \in \Omega$$

has a unique solution $z = z(u,f) \in H_0^1$ and this implicit function is continuous in $f \in \omega$ uniformly with respect to $u \in U_{ad}$.

H5. Functions b_0 and g_0 have the first derivatives with respect to $z, p_1, \dots, p_n$; these derivatives are Caratheodory functions and there exist constants $r > n, \mu_0 > 0$, a positive function $h \in L_1(\Omega)$ and a continuous bounded function γ with $\gamma(0) = 0$ such that for all arguments

$$| b_0(x,u,z,p) |^{\frac{2r}{r+2}} + | g_0(x,u,z,p) \leq \mu_0[| h(x) |^{\frac{r}{2}} + | z |^{\frac{r}{r-2}} + | p |]^2$$

$$\begin{aligned}
| \frac{\partial}{\partial z} b_0(x,u,z,p) |^{\frac{r}{4}} + | \frac{\partial}{\partial p_i} b_0(x,u,z,p) |^{\frac{r}{2}} + | \frac{\partial}{\partial z} g_0(x,u,z,p) |^{\frac{r}{r+2}} + \\
+ | \frac{\partial}{\partial p_i} g_0(x,u,z,p) | \\
\leq \quad \mu_0[| h(x) |^{\frac{r}{2}} + | z |^{\frac{r}{r-2}} + | p |] \\
| \frac{\partial}{\partial z} b_0(x,u,z',p') - \frac{\partial}{\partial z} b_0(x,u,z'',p'') |^{\frac{r}{4}} + \\
+ | \frac{\partial}{\partial p_i} b_0(x,u,z',p') - \frac{\partial}{\partial p_i} b_0(x,u,z'',p'') |^{\frac{r}{2}} + \\
+ | \frac{\partial}{\partial z} g_0(x,u,z',p') - \frac{\partial}{\partial z} g_0(x,u,z'',p'') |^{\frac{r}{r+2}} + \\
+ | \frac{\partial}{\partial p_i} g_0(x,u,z',p') - \frac{\partial}{\partial p_i} g_0(x,u,z'',p'') | \leq \\
\leq \quad \mu_0[| h(x) |^{\frac{r}{2}} + | z' |^{\frac{r}{r-2}} + | z'' |^{\frac{r}{r-2}} + | p' | + | p'' |] * \\
* \gamma(| z' - z'' | + | p' - p'' |),
\end{aligned}$$

$$i = 1, \dots, n$$

DEFINITION 1 Class N consists of all the Nemitckii operators

$$D : X \to Y,$$

$$X \equiv L_{\frac{2r}{r-2}}(\Omega) \times \underbrace{L_2(\Omega) \times \ldots \times L_2(\Omega)}_{n}$$

$$Y \equiv L_{\frac{2r}{r+2}}(\Omega) \times \underbrace{L_2(\Omega) \times \ldots \times L_2(\Omega)}_{n} \times L_1(\Omega)$$

$$D \equiv (D_b, D_1, \ldots, D_n, D_g)$$

such that for $f = (f_0, f_1, \ldots, f_n) \in X$

$$(D_b f)(x) = b(x, f_0(x), \ldots, f_n(x)), \quad x \in \Omega, \tag{5}$$

$$(D_i f)(x) = a_{ij}(x) f_j(x), \quad i = 1, \ldots, n, \ x \in \Omega,$$

where functions $a_{ij}, i, j, = 1, \ldots, n$, are measurable and functions b and g are Caratheodory functions.

Class $M(\nu, \mu, \mu_0, \gamma, h, r)$ (or shortly M) consists of all the operators $D \in N$ such that functions $b, g, a_{ij}, i, j = 1, \ldots, n$, in the representation (5) satisfy hypotheses H1, H2, H5 as they stay for $b_0, g_0, a_{ij}, i, j = 1, \ldots, n$, respectively.

DEFINITION 2 A sequence $\{D^K\}$ of operators from the class N converges to the operator $D^0 : X \to Y$ iff for every $f \in X$ the sequence $D^K f$ converges strongly in Y to $D^0 f$.

DEFINITION 3 A sequence D^K of operators from the class N SG - converges to the operator $D^0 : X \to Y$ iff for every sequence z_K convergent weakly in H_0^1, $z_k \rightharpoonup z_0$ as $K \to \infty$, such that

$$-\frac{d}{dx_i} D_i^K(z_K, z_{Kx}) + D_b^K(z_K, z_{Kx}) = \varphi \ in \ H^{-1} \tag{6}$$

$$K = 1, 2, \ldots,$$

there is

$$-\frac{d}{dx_i} D_i^0(z_0, z_{0x}) + D_b^0(z_0, z_{0x}) = \varphi \ in \ H^{-1} \tag{7}$$

and

$$D^K(z_K, z_{Kx}) \rightharpoonup D^0(z_0, z_{0x}) \tag{8}$$

weakly in Y as $K \to \infty$.

Here by (z, z_x) we denote the element $(z, z_{x_1}, \ldots, z_{x_n}) \in X$.

DEFINITION 4 By N_{ad} we shall denote the subset of the class N which contains all the operators $D \in N$ such that

$$(D_b f)(x) = b_0(x, u(x), f_0(x), \ldots, f_n(x)), \ x \in \Omega,$$

$$(D_i f)(x) = a_{ij}(x, u(x)) f_i(x), \ x \in \Omega, \ i = 1, \ldots, n,$$

$$(D_g f)(x) = g_0(x, u(x), f_0(x), \ldots, f_n(x)), \ \ x \in \Omega,$$

for some $u \in U_{ad}$.

The following properties of these convergences can be demonstrated (see U.Raitums [5]):

LEMMA 2 In every fixed class M convergence in the sense of definition 2 (pointwise convergence) coincides with some metric ρ_0 and in this metric M is a complete metric space.

LEMMA 3 In every class $M(\nu, \mu, \mu_0, \gamma, h, r)$ convergence in the sense of definition 3 (SG - convergence) coincides with some metric ρ_1, in this metric class M is precompact metric space and the clousure of the M in the metric ρ_1 belongs to some other class $M(\nu, \mu, \mu_0', \gamma', h', r)$ where μ_0', γ', h' depends only on $\Omega, n, r, \nu, \mu, \mu_0, \gamma, h$.

For a given set $M_0 \subset M$ in what follows we shall denote by $\overline{\text{co}}\ M$ the closed convex hull of the set M in the metric ρ_0 and by GM_0 the closure of the set M_0 in the metric ρ_1. By virtue of Lemmas 2,3 the set $\overline{\text{co}}\ M$ belongs to M but the set GM_0 belongs to some other class M and GM_0 is compact in metric ρ_1.

LEMMA 4 Every operator $D \in M$ is continuously Frechet differentiable.

Let M_0 be a subset of a given class M and let us denote by (M_0) the following optimal control problem:

$$\int_\Omega D_g(z, z_x) dx \to min \tag{9}$$

$$-\frac{d}{dx_i} D_i(z, z_x) + D_b(z, z_x) = 0 \quad in\ \Omega \tag{10}$$

$$D \in M_0, \quad z \in H_0^1 \tag{11}$$

It is obvious that the original problem (1) - (3) is equivalent to the problem (N_{ad}) if the set N_{ad} belongs to some class M. In this case all terms in (1) - (3) or in (9) - (11) are well defined.

For a given subset $M_0 \subset M$ let us denote by $Z(M_0)$ the set of all pairs $(z, c) \in H_0^1 \times R$ such that there exists an operator $D \in M_0$ for which

$$-\frac{d}{dx_i} D_i(z, z_x) + D_b(z, z_x) = 0 \quad in\ \Omega \tag{12}$$

and

$$c = \int_\Omega D_g(z, z_x) dx \tag{13}$$

On the basis of Lemma 1 the following theorem can be proved.

THEOREM 1 *Let the hypotheses H1, H3-H5 be fulfilled. Then the set* $Z(\overline{\text{co}}\ N_{ad})$ *coincides with the closure of the set* $Z(N_{ad})$ *in the strong topology in* $H_0^1 \times R$*, i.e. if the element* z_0 *is a solution of the equation (12) with some* $D = D^0 \in \overline{\text{co}}\ N_{ad}$ *then there exists a sequence of operators* $D^K \subset N_{ad}$ *such that solutions* $z_K \in H_0^1$ *of equation (12) with* $D = D_K$*,* $K = 1, 2, \ldots$*, converges strongly in* H_0^1 *to* z_0 *and*

$$\int_\Omega D_b^K(z_K, z_{Kx}) dx \to \int_\Omega D_b^0(z_0, z_{0x}) dx \quad as\ K \to \infty.$$

This theorem shows that passage from the set N_{ad} to $\overline{\text{co}}\ N_{ad}$ is an extension of the original optimal control problem $(1) - (3)$ and this extension preserves the price of the problem.

Here and further on the notion of the price of an optimal control problem will refer to the infimum of the cost functional over all pairs (u, z) $(\text{or}(D, z))$ which satisfy all constraints of the problem.

The extended problem corresponding to the set $\overline{\text{co}}\ N_{ad}$ has in general no optimal solution. Only in the case where functions a_{ij}, $i, j = 1, \ldots, n$, do not depend on control (in the original statment of problem) and all solutions of the equation (10) with $D \in \overline{\text{co}}\ N_a d$ belong to a bounded set in H_0^1 the extended problem $(\overline{\text{co}}\ N_{ad})$ has an optimal solution (under hypotheses H1, H2, H5).

If the original problem has an optimal solution then the extended problem $(\overline{\text{co}}\ N_{ad})$ has the same solution (under hypotheses H1, H3-H5). Hence, the necessary conditions for optimality in the extended problem for this solution are valid for the original problem. Since the set $\overline{\text{co}}\ N_{ad}$ is convex, derivation of these conditions is easier for the extended problem than for the original one. One of the first who used such a method for optimal control problems for elliptic equations was L.Tartar [6] in the case when the methods of the G - convergence were applied.

In this way we obtain for our original problem (1) - (3) the following result (see U.Raitums [5]).

THEOREM 2 *Let the hypotheses H1-H5 be fulfilled, the pair (u_0, z_0) be an optimal solution of the problem (1) - (3) and the Frechet derivative of the operator $A(u_0)$ at the element z_0 be invertible as linear operator from H_0^1 to H^{-1}.*

Then the function

$$\begin{aligned} F(x,u) \quad \equiv \quad & g_0(x,u,z_0(x),z_{0x}(x)) - \\ & a_{ij}(x,u)z_{0x_j}(x)\Psi_{x_i}(x) - b_0(x,u,z_0(x),z_{0x}(x))\Psi(x) \qquad (14) \end{aligned}$$

attains its minimum with respect to $u \in U_{ad}$ at the element u_0 for a.e. $x \in \Omega$.

Here the function $\Psi \in H_0^1$ is the solution of the adjoint state equation

$$\begin{aligned} & -\frac{d}{dx_i}[a_{ji}(x,u_0)\Psi_{xj} + b_{0p_i}(x,u_0,z_0,z_{0x})\Psi] + b_{0z}(x,u_0,z_0,z_{0x}) = \\ & \qquad -\frac{d}{dx_i}g_{0p_i}(x,u_0,z_0,z_{0x}) + g_{0z}(x,u_0,z_0,z_{0x}) \text{ in } \Omega \qquad (15) \end{aligned}$$

REMARK 1 Analogous result with standard Lagrangian instead of F in (14) is valid for the case of systems of equations

$$\begin{aligned} & -\frac{d}{dx_i}a_i^l(x,u,z_1,\ldots,z_{l_0},z_{1x},\ldots,z_{l_0x}) + \\ & \qquad + b^l(x,u,z_1,\ldots,z_{l_0},z_{1x},\ldots,z_{l_0x}) = 0, \qquad (16) \\ & x \in \Omega, \quad z_l \in H_0^1, \quad l = 1,\ldots,l_0, \end{aligned}$$

where l_0 is less then n with additional constraints in the form of equalities and inequalities for integral functionals.

For Theorem 2 and for the case of system (16) it is essential that the set U_{ad} satisfy the condition of decomposition.

REMARK 2 As was mentioned above, using of the needle-like variations demands stronger assumptions. Up to this time the necessary conditions for optimality for the problem (1) - (3) are proved with this approach only under assumptions that the derivatives g_{op_i}, b_{op_i} of the functions g_0 and b_0 satisfy the Lipschitz condition with respect to p (see, for example, U.Raitums [5]). Of course, the situation with convex set can be easily treated on the basis of the implicit function theorem if the functions a_i^l, b^l in the system (16) and integrands of functionals are smooth enough with respect to (u, z, p).

Although the "convexification" approach allows us to obtain the necessary conditions for optimality and "improve" properties of the admissible set of operators the question of the existence of optimal solutions remains open. Here the extension on the basis of G - convergence can be applied.

By virtue of Lemma 3 we have the following result (see U.Raitums [5]).

THEOREM 3 *Let the hypotheses H1,H2,H5 be fulfilled, at least one of the equations (12) with $D \in N_{ad}$ have a solution and all solutions of the equation (12) with $D \in GN_{ad}$ belong to a bounded set in H_0^1.*

Then the extended optimal control problem (GN_{ad}) has an optimal solution. If, additionally every equation (12) with $D \in GN_{ad}$ has exactly one solution then the price of the extended problem is equal to the price of the original problem.

In this approach it remains unsolved how effectively (for example - in analytical formulas) the set GN_{ad} can be described. Only for some special cases the effective description is known (see, for example, K.A.Lurie and A.V.Cherkhaev [3], L.Tartar [7]).

Another shortcoming is that the set GN_{ad} can be, in general, nonconvex. Here are two possibilities of proceeding. The first one is to pass at first from N_{ad} to $\overline{\text{co}}\ N_{ad}$ and after that to $G\overline{\text{co}}\ N_{ad}$. If functions b_0 and g_0 have the following representation:

$$b_0(x, u, z, p) = b_{0i}(x, u)p_i + b_{00}(x, u)b_{0*}(x, z),$$

$$g_0(x, u, z, p) = g_{0i}(x, u)p_i + g_{00}(x, u)g_{0*}(x, z)$$

then it can be shown that the set $G\overline{\text{co}}\ N_{ad}$ is convex and closed (in the metrics ρ_0 and ρ_1). In this case the set $G\overline{\text{co}}\ N_{ad}$ can be fully described in terms of $(n+3) \times (n+3)$ symmetric matrix functions whose elements correspond to coefficients $a_{ij}, b_{0i}, g_{0i}, b_{00}, g_{00}$.

The other approach can be used in the case when the functions b_0 and g_0 are affine with respect to p and do not depend on u. Such situation often appears in the optimal shape design problems.

In such cases the original problem and the extended problem can both be fully described by the sets Q_0 and Q_*, respectively, of the coefficient matrices $A = (a_{ij}(x)), i, j = 1, \dots, n$ of the equation (12). Let us illustrate it by the following simple example.

Let us have the problem

$$\int_\Omega g_0(x, z)dx \to min \tag{17}$$

$$divA(x)\nabla z = b_0, \quad x \in \Omega \tag{18}$$

$$A \in Q_0, \quad z \in H_0^1 \tag{19}$$

where the set Q_0 satisfies the condition of decomposition, consists of measurable symmetric matrix-functions A with eigenvalues $\lambda_1(A)(x), \dots, \lambda_n(A)(x)$ from some fixed interval $[\nu, \mu]$ with $\nu > 0$ and if the matrix $A_0 \in Q_0$ then the set Q_0 contains all measurable symmetric matrix-functions A such that $\lambda_i(A)(x) = \lambda_i(A_0)(x), x \in \Omega, i = 1, \dots, n$. Of course the set Q_0 does in general not contain all symmetric matrix-functions with eigenvalues from the interval $[\nu, \mu]$. For example, the set Q_0 consists from all measurable symmetric matrix-functions A with $\lambda_i(A)(x) = \nu$ or $\mu, x \in \Omega, i = 1, \dots, n$.

The set Q_* is constructed in the following way.

At the first step we construct the sets

$$K_1(x) \equiv \overline{\text{co}}\ \{(\lambda_1, \lambda_n) \in R^2 : (\lambda_1, \lambda_n) = (\lambda_1(A)(x), \lambda_n(A)(x)), A \in Q_0\},$$

$$K_2(x) \equiv \{(\alpha, \beta) \in R^2 : \lambda_1 \leq \alpha \leq \beta \leq \lambda_n, (\lambda_1, \lambda_n) \in K_1(x)\},$$

$$K_3(x) \equiv \{(t, \tau) \in R^2 : (t^{-1}, \tau) \in K_2(x)\},$$

$$K_4(x) \equiv \{(\xi, \zeta) \in R^2 : (\xi^{-1}, \zeta) \in \overline{\text{co}}\ K_3(x)\}.$$

After that the set Q_* consists of all measurable symmetric matrix-functions $A = (a_{ij}(x)), i, j = 1, \dots, n$, such that $(\lambda_1(A)(x), \lambda_n(A)(x)) \in K_4(x)$ for a.e.$x \in \Omega$.

Then, the extended problem is formulated via (17) - (19) where the set Q_0 is replaced by the set Q_*.

The main property of this extension (see U.Raitums [5]) is that the set $Z(Q_*)$ of all solutions of the equation (18) with $A \in Q_*$ coincides with the set of all weak limit elements in H_0^1 of sequences of solutions of the equation (18) with $A \in Q_0$. Hence, the extended problem has an optimal solution and the prices of both problems are equal.

All these constructions are local, and therefore, if in the original problem (17) - (19) matrices A from Q_0 are fixed in some subset $\Omega_0 \subset \Omega$ then in the extended problem matrices from Q_* are fixed too.

Of course, other boundary value problems can be treated analogously.

The case with additional constraints has some specific features. When we pass to the wider sets of controls (or operators) then in general there can appear new components of the set of pairs (u, z) which satisfy all constraints of the problem. Therefore, the question of maintaining the price of the problem is open. In this connection we can point out one condition which allows us to handle this question.

Let us denote by $I(\overline{\epsilon})$ the price of the problem (1) - (3) with additional constraints

$$I_i(u,z) = \epsilon_i, \quad i = 1, \dots, m_1, \; I_i(u,z) \leq \epsilon_0, \quad i = m_1 + 1, \dots, m_0, \tag{20}$$

$$I_i(u,z) \equiv \int_\Omega g_i(x,u,z,z_x)dx, \; i = 1, \dots, m_0, \; \overline{\epsilon} \equiv (\epsilon_0, \epsilon_1, \dots, \epsilon_{m_1}).$$

If there does not exist a pair $(u, z) \in U_{ad} \times H_0^1$ which satisfies all constraints of this problem then, by definition, $I(\overline{\epsilon}) = \infty$.

Definition 5 The optimal control problem (1) - (3), (20) satisfies at $\overline{\epsilon} = 0$ the first condition of approximation if the function $I = I(\overline{\epsilon})$ is continuous at $\overline{\epsilon} = 0$.

If functions g_i in (20) satisfy the same H1, H5 as formulated for function g_0 then we can in a natural way extend the definitions of classes N and M and the corresponding definition of the problem (M_0) (relationships (9) - (11) - by means of additional components of operators D responsible for constraints (20).

Then it can be shown that if

1. The first condition of approximation at $\overline{\epsilon} = 0$ is satisfied, then with transition from N_{ad} to $\overline{\mathrm{co}}\ N_{ad}$ or GN_{ad} the price of the problem is maintained.

Finally, we shall point out that the first condition of approximation characterizes in some way the well-posedness of the original problem. That is because in practical problems the values of right hand sides in constraints (20) are not strictly defined (exceptions ocour when these constants coincide with some physical constants, for example, the melting temperature). Hence, the discontinuity of $I(\overline{\epsilon})$ at $\overline{\epsilon} = 0$ shows that the possible optimal solutions are not stable and small perturbations in the statement of the problem can lead to a very different situation, for example, another price of the problem.

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