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Sensitivity analysis of shape optimization problems

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A method for the stability analysis of solutions to the shape optimization problems is proposed. A local solution is given in the form of a fixed point of the metric projection onto the set of admissible graphs. The results are derived for a shape optimization problem for the second order elliptic equation. The shape sensitivity analysis of the wave equation is performed.

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Introduction

The results on sensitivity analysis of solutions to a class of shape optimization problems for the systems governed by elliptic partial differential equations are presented in the paper. In [12] the differential stability of solutions for a class of the non-convex parametric optimization problems is established using the Hadamard derivative of a metric projection onto the set of admissible parameters. The analysis is based on the observation that any local minimizer of the regularized parametric optimization problem is given in the form of a fixed point of the metric projection onto the set of admissible parameters. In the present paper we use the method of sensitivity analysis proposed in [14] for convex problems, see also [15], [16].

The shape optimization problem considered in the present paper is non - convex. Let us consider an example of such problem in R^2.

We denote by $\Omega(v)$ the following domain

$$\Omega(v) = \{ \ (x_1, x_2) \in R^2 \mid 0 < x_1 < 1, \ 0 < x_2 < v(x_1) \ \}$$

where v is an element of the following set of admissible graphs (parameters)

$$\mathcal{U} = \{ \ v \in C^{0,1}[0,1] \mid 0 < a_1 \leq v(x_1) \leq a_2, \ |v'(x_1)| \leq a_3, \ x_1 \in (0,1) \ \}$$

where a_1, a_2, a_3 are positive constants such that the set $\mathcal{U}$ is nonempty.

Denote by $y(v) \in H_0^1(\Omega(v))$ the solution of the following state equation

$$-\Delta y(v) = f, \qquad \text{in } \Omega(v)$$

$$y(v) = 0, \qquad \text{on } \Gamma = \partial\Omega$$

where $f \in H^1(\Omega)$ is given.

Let $\alpha \geq 0$ be the regularization parameter and denote by $\mathcal{J}_\alpha(.)$ the following cost functional

$$\mathcal{J}_\alpha(.) = \frac{1}{2}\int_{\Omega(v)} (y(v) - z_d)^2 dx + \frac{\alpha}{2}\|v\|^2_{H^1(0,1)}$$

where $z_d \in H^1(R^2)$ is given.

Let us consider the following shape optimization problem

$$\min_{v \in \mathcal{U}} \mathcal{J}_\alpha(v)$$

and denote by $u_\alpha \in \mathcal{U}$ an optimal solution which exists for any $\alpha > 0$.

For $\alpha = 0$ to solve the optimization problem means to find the metric projection in L^2 of z_d onto the set

$$\{ \ y(v) \in H^1(\Omega(v)) \mid v \in \mathcal{U} \ \}$$

On the other hand for any $\alpha > 0$ the necessary optimality conditions read

$$u_\alpha = P_{\mathcal{U}}(-\frac{1}{\alpha}\mathcal{G}_\alpha(u_\alpha))$$

where $\mathcal{G}_\alpha(.)$ denotes the gradient of $\mathcal{J}_\alpha(.)$.

In the present paper we show that if the following condition is satisfied: there exists $\beta > 0$ such that $d^2\mathcal{J}_\alpha(u_\alpha; v, v) \geq \beta||v||^2_{H^1(0,1)}$, $\forall v \in \{S - S\}$, where S is a given cone in $H^1(0,1)$ associated to u_α, furthermore the set $\mathcal{U}$ satisfies an auxiliary condition at $u_\alpha \in \mathcal{U}$, then the local solution u_α is stable with respect to the perturbation of the data of the optimization problem under considerations.

The method of sensitivity analysis proposed here is general and can be used for the linear parabolic, hyperbolic, as well as some nonlinear partial differential equations. We shall consider a model problem with the set of admissible graphs which satisfy local inequality constraints. We refer the reader to [11] for related results on the differential stability of metric projection and some applications. The results on the shape sensitivity analysis of convex optimization problems for the distributed parameter systems are presented in [14], [15], [16] and on the shape sensitivity analysis of variational inequalities in [18]. The material derivative method in the shape optimization of the distributed parameter systems as well as the shape calculus for partial differential equations are described in [19]. The results on the shape sensitivity of Min - Max are provided in [3]. A shape estimation problem is considered in [2]. The second order optimality conditions are obtained in [6] for a shape optimization problem for an elliptic equation. The form of a distribution associated to the Hessian of a general integral shape functional is derived, using the material derivative method, by M. Delfour and J.-P. Zolesio [4]. The results and applications of the shape optimization in the structural mechanics are presented e.g. in [5], [8], [12]. We use standard notation throughout the paper [1],[15].

The outline of the paper is the following. In section 1 the method of sensitivity analysis is presented in an abstract setting. In section 2 the stability result for a shape optimization problem for the Laplace equation is obtained. In section 3 the shape sensitivity analysis for the wave equation is performed. Finally, section 4 is concerned with the directional differentiability of the metric projection in Sobolev spaces onto convex, closed subsets. We refer the reader to [17] for the proofs of results presented in the paper.

1. Sensitivity analysis

We briefly desribe, in an abstact setting, the method of sensitivity analysis proposed in [17]. Let K be a closed, convex subset of Hilbert space H, $J_{\alpha,\varepsilon}(u) = J_\varepsilon(u) + \frac{\alpha}{2}\|u\|_H^2$ a C^2 functional, $\alpha > 0$, $\varepsilon \in [0, \delta)$ are parameters. Set K denotes the set of admissible graphs for a shape optimization problem.

Denote by $\mathcal{G}_\varepsilon(u)$ the gradient, and by $\mathcal{H}_\varepsilon(u)$ the Hessian, of $J_\varepsilon(u)$. Consider the following optimization problem

$$\text{Minimize } J_{\alpha,\varepsilon}(u) \text{ over the set } K \subset H$$

The first order necessary optimality conditions takes the form of the following variational inequality:

$$u_\varepsilon \in K : (\mathcal{G}_\varepsilon(u_\varepsilon), v - u_\varepsilon)_H + \alpha(u_\varepsilon, v - u_\varepsilon)_H \geq 0, \ \forall v \in K \tag{1.1}$$

hence

$$u_\varepsilon = P_K(-\frac{1}{\alpha}\mathcal{G}_\varepsilon(u_\varepsilon)) \tag{1.2}$$

i.e. u_ε is the fixed point, P_K is the metric projection in H onto K. We shall consider the stablity of a local solution to (1.2) with respect to the parameter v. Denote $\mathcal{F}_\varepsilon(v) = -\frac{1}{\alpha}\mathcal{G}_\varepsilon(v)$ whence

$$u_\varepsilon = P_K(\mathcal{F}_\varepsilon(u_\varepsilon)) \tag{1.3}$$

If for a fixed $\alpha > 0$ the nonlinear C^1 mapping $\mathcal{F}_0$ is a contraction

$$|\mathcal{F}_0(v) - \mathcal{F}_0(u)| \leq L\|v - u\|_H \tag{1.4}$$

for some $0 < L < 1$, and the mapping $\varepsilon \to \mathcal{F}_\varepsilon(v)$ is locally Lipschitz continuous

$$|\mathcal{F}_\varepsilon(v) - \mathcal{F}_0(v)| \leq C_1\varepsilon\|v\|_H \tag{1.5}$$

it follows that the local solution to (1.1) is unique, and Lipschitz continuous with respect to ε

$$\|u_\varepsilon - u_0\|_H \leq \frac{C_1\varepsilon}{1-L}\|u_\varepsilon\|_H \leq C\varepsilon \tag{1.6}$$

since the norm $\|u_\varepsilon\|_H$ is bounded provided e.g. $J_\varepsilon(v) \geq 0$ for all $v \in K$.
It is clear that in order to obtain the differentiability of u_ε with respect to ε at 0^+, we should first establish the differentiability with respect to ε of $f_\varepsilon = \mathcal{G}_\varepsilon(u_\varepsilon)$. It follows [14] by the Lipschitz continuity (1.6) that there exists an element $q \in H$, in general non-unique, such that

$$u_\varepsilon = u_0 + \varepsilon q + r(\varepsilon) \tag{1.7}$$

assuming compactness of $\mathcal{G}_\varepsilon(.)$, $\varepsilon \geq 0$, it follows

$$\mathcal{G}_\varepsilon(u_\varepsilon) = \mathcal{G}_0(u_0) + \varepsilon[\mathcal{H}_0(u_0)q + \partial\mathcal{G}(u_0)] + o(\varepsilon) \tag{1.8}$$

where $\partial\mathcal{G}$ denotes directional derivative of $\mathcal{G}_\varepsilon(.)$ with respect to ε at $\varepsilon = 0^+$. We can use (1.8) and differentiate (1.2) provided the metric projection P_K is directionally differentiable in the sense of Hadamard at $-\frac{1}{\alpha}\mathcal{G}_0(u_0)$. Suppose that there exists a mapping $\mathcal{Q} : H \to H$ such that for all $h \in H$ and $\tau > 0$, τ small enough

$$P_K(-\frac{1}{\alpha}\mathcal{G}_0(u_0) + \tau h) = P_K(-\frac{1}{\alpha}\mathcal{G}_0(u_0)) + \tau\mathcal{Q}(h) + o(\tau), \quad \text{in } H \tag{1.9}$$

where $\|o(\tau)\|_H/\tau \to 0$ with $\tau \downarrow 0$ uniformly, with respect to h, on compact subsets of H. Then

$$q = \mathcal{Q}(-\frac{1}{\alpha}\mathcal{H}_0(u_0)q - \frac{1}{\alpha}\partial\mathcal{G}(u_0)) \tag{1.10}$$

which means that the directional derivative q in (1.7) is given by the fixed point (1.10).

In references [11],[14],[17] we provide formula (1.9) for some specific convex sets in the Sobolev spaces. In particular (1.9) holds for any ball in Hilbert space, see e.g. [17]. For the sets with local constraints in the Sobolev spaces, see (4.2) for an example, under additional assumptions [11],[14],[17], $\mathcal{Q} = P_S$ where S is the convex cone in H of the following form

$$S = T_K(u_0) \cap [\frac{1}{\alpha}\mathcal{G}_0(u_0) + u_0]^\perp \tag{1.11}$$

here $T_K(u_0)$ is the closure of the tangent cone:

$$T_K(u_0) = \mathrm{cl}\{\ v \in H \mid \exists\tau > 0, \ \text{ such that } u_0 + \tau v \in K\ \}$$

to K at $u_0 \in K$. It turns out that if $\mathcal{Q} = P_S$ then the following condition leads to Lipschitz continuity and directional differentiability of u_ε, the local solution to (1.1), with respect to ε at 0^+

$$\exists\beta > 0 : d^2J_\alpha(u_0; v, v) \geq \beta\|v\|_H^2, \quad \forall v \in \{S - S\} \tag{1.12}$$

- see PROPOSITION 1 in [17]. It can be shown that (1.12) implies that the fixed point q defined by (1.10) is unique. In order to apply the abstract scheme it is required [17] that the metric projection P_K in H onto K is directionally differentiable in the sense of Hadamard at $-\frac{1}{\alpha}\mathcal{G}(u_0)$.

2. Shape optimization problem for Laplace equation

Let $Q \subset R^{n-1}$, $n \geq 2$, be a given domain with smooth boundary ∂Q. We denote by $\dot{\Omega} = \Omega_f \subset R^n$ the domain of the following form

$$\Omega = \{(x', x_n) | \; 0 < x_n < f(x'), x' \in Q\}$$

where $x' = (x_1, ..., x_{n-1})$, and $f(.)$ is a given function which belongs to the set

$$K = \{f \in H_0^s(Q) \mid 0 < \psi_1(x') \leq f(x') \leq \psi_2(x') \; , \; \forall x' \in Q\}$$

here $\psi_i(.) \in H_o^s(Q)$, $i = 1, 2$ are given elements such that the set K is nonempty, $s > n - 1$. We denote by D the following domain in R^n

$$D = \{(x', x_n) \mid 0 < x_n < \psi_2(x') \; , \; x' \in Q\}$$

therefore $\Omega_f \subset D$, $\forall f \in K$.

We provide the results on the shape sensitivity analysis of an elliptic state equation and of an associated shape functional in the variable domain setting i.e. using the so - called shape derivative of a solution to the elliptic equation. Such approach is, in some sense, straightforward compared to the fixed domain formulation [17],[18],[19]. We refer the reader to [19] for the detailed description of the material derivative method in the shape sensitivity analysis of partial differential equations. Let us consider the following elliptic equation

$$-\Delta y = F \; , \quad \text{in } \Omega_f \tag{2.1}$$

$$y = 0 \; , \quad \text{on } \partial\Omega_f \tag{2.2}$$

Denote

$$\Omega_t = \{(x', x) \mid 0 < x < f(x') + th(x') \; , \; x' \in Q\} \; , \tag{2.3}$$

where $h(.) \in H_0^s(Q)$ is given, t is small, furthermore we denote by

$$h_\nu(x') = \frac{h(x')}{(1 + |\nabla f(x')|^2)^{\frac{1}{2}}}$$

the normal component of the vector field $(0, ..., 0, h(x')) \in R^n$ on $\Gamma(f)$. Let

$$-\Delta y_t = F\ , \quad \text{in } \Omega_t \tag{2.4}$$

$$y_t = 0\ , \quad \text{on } \partial\Omega_t \tag{2.5}$$

Define the domain derivative [19] $y' = y'(h_\nu)$

$$y' = \frac{\partial Y}{\partial t}(x,t)_{|t=0} \tag{2.6}$$

where

$$Y(x,t) = \begin{cases} y_t(x) & \text{for } x \in \Omega_t, t \geq 0 \\ 0 & \text{for } x \notin \Omega_t, t \geq 0 \end{cases}$$

The domain derivative is given by a unique solution of the following elliptic equation [19]

$$-\Delta y' = 0\ , \quad \text{in } \Omega_f \tag{2.7}$$

$$y' = -h_\nu \frac{\partial y}{\partial n}\ , \quad \text{on } \Gamma(f) \tag{2.8}$$

$$y' = 0\ , \quad \text{on } \partial\Omega \setminus \Gamma(f) \tag{2.9}$$

Let us consider the following shape functional

$$J_\alpha(f) = \frac{1}{2}\int_{\Omega_f}(y(f;.) - z(.))^2 dx + \frac{\alpha}{2}\|f\|^2_{H_0^s(Q)} \tag{2.10}$$

where $z(.) \in H^1(D)$ is a given element.

The directional derivative for shape functional (2.10) in a direction h takes the form [17]

$$dJ_\alpha(f;h) = (\mathcal{G}_0(f),h)_{H_0^s(Q)} + \alpha(f,h)_{H_0^s(Q)} \tag{2.11}$$

here we assume $z = 0$ on $\Gamma(f)$

$$= \int_{\Omega_f}(y(f;.) - z(.))y'(h_\nu)dx + \alpha(f,h)_{H_0^s(Q)}$$

In standard way we introduce the adjoint state [17]

$$-\Delta p = y - z\ , \quad \text{in } \Omega_f \tag{2.12}$$

$$p = 0\ , \quad \text{on } \partial\Omega_f \tag{2.13}$$

then

$$dJ_\alpha(f;h) = \int_{\Gamma(f)} h_\nu \frac{\partial p}{\partial n}\frac{\partial y}{\partial n} d\Gamma + \alpha(f,h)_{H_0^s(Q)} \tag{2.14}$$

Finally we evaluate [17] the second derivative

$$\begin{aligned} d^2 J_\alpha(f;h,v) &= \\ &= \lim_{t\downarrow 0} \frac{1}{t}(dJ_\alpha(f+tv;h) - dJ_\alpha(f;h)) \\ &= (\mathcal{H}_0(f)h, v)_{H_0^s(Q)} + \alpha(h,v)_{H_0^s(Q)} = \\ & -\int_{\Gamma(f)} \left(h_\nu \frac{(\nabla f(x'), \nabla v(x'))}{(1+|\nabla f(x')|^2)} + v_\nu \frac{(\nabla f(x'), \nabla h(x'))}{(1+|\nabla f(x')|^2)} \right) \frac{\partial p}{\partial n}\frac{\partial y}{\partial n} d\Gamma \\ & -2\int_{\Gamma(f)} \kappa_m v_\nu h_\nu \frac{\partial p}{\partial n}\frac{\partial y}{\partial n} d\Gamma + \int_{\Gamma(f)} h_\nu v_\nu \frac{\partial}{\partial n}\left(\frac{\partial y}{\partial n}\frac{\partial p}{\partial n}\right) d\Gamma \\ & + \int_{\Gamma(f)} h_\nu \left(\frac{\partial p'}{\partial n}\frac{\partial y}{\partial n} + \frac{\partial p}{\partial n}\frac{\partial y'}{\partial n}\right) d\Gamma + \alpha(h,v)_{H_0^s(Q)} \end{aligned} \tag{2.15}$$

where κ_m is the mean curvature on $\Gamma(f)$, and the domain derivatives $y'(v_\nu)$, $p'(v_\nu)$ satisfy the following elliptic equation

$$-\Delta y' = 0\,, \quad \text{in } \Omega_f \tag{2.16}$$

$$y' = -v_\nu \frac{\partial y}{\partial n}\,, \quad \text{on } \Gamma(f) \tag{2.17}$$

$$y' = 0\,, \quad \text{on } \partial\Omega \setminus \Gamma(f) \tag{2.18}$$

and

$$-\Delta p' = y' - z'\,, \quad \text{in } \Omega_f \tag{2.19}$$

$$p' = -v_\nu \frac{\partial y}{\partial n}\,, \quad \text{on } \Gamma(f) \tag{2.20}$$

$$p' = 0\,, \quad \text{on } \partial\Omega \setminus \Gamma(f) \tag{2.21}$$

here z' is the domain derivative of the observation $z(.)$, we can assume $z' = 0$.

REMARK 2.1 Since on $\Gamma(f)$ we have $y'(h_\nu) = -h_\nu \frac{\partial y}{\partial n}, p'(h_\nu) = -h_\nu \frac{\partial p}{\partial n}$, it follows that

$$\begin{aligned} & \int_{\Gamma(f)} h_\nu \left(\frac{\partial p'}{\partial n}\frac{\partial y}{\partial n} + \frac{\partial p}{\partial n}\frac{\partial y'}{\partial n}\right) d\Gamma \\ = & -\int_{\Gamma(f)} (y'(h_\nu)\frac{\partial p'}{\partial n}(v_\nu) + p'(h_\nu)\frac{\partial y'}{\partial n}(v_\nu)) d\Gamma \end{aligned} \tag{2.22}$$

We suppose that the element $z(.)$ in the cost functional (2.10) depends on the parameter $\varepsilon \in [0, \delta)$

$$z_\varepsilon = z + \varepsilon\vartheta \tag{2.23}$$

In the sequel we consider the sensitivity of a local solution to the shape optimization problem with respect to ε. It can be shown [17] that the derivative η of the adjoint state p with respect to ε is a unique solution of the following elliptic equation

$$\begin{aligned} -\Delta\eta &= -\vartheta\ , \quad \text{in } \Omega_f \\ \eta &= 0\ , \quad \text{on } \partial\Omega_f \end{aligned} \tag{2.24}$$

Furthermore [17] for any $\alpha > 0$ there exists an element $f_\varepsilon^\star$ which minimizes the perturbed cost functional

$$J_{\alpha,\varepsilon}(f) = \frac{1}{2}\int_{\Omega_f} (y(f;.) - z_\varepsilon(.))^2 dx + \frac{\alpha}{2}\|f\|^2_{H_0^s(Q)} \tag{2.25}$$

over the set $K \subset H_0^s(Q)$, for fixed $s > n-1$; we assume that $z = 0$ on $\Gamma(f_0^\star)$. Let $f_0^\star \in K$ be a local solution to the first order necessary optimality conditions for the shape optimization problem under consideration.

THEOREM 2.2 *Assume that there exists $\beta > 0$ such that*

$$d^2 J_\alpha(f_0^\star; v, v) \geq \beta\|v\|^2_{H_0^s(Q)}\ , \ \forall v \in \{S - S\} \tag{2.26}$$

and suppose that the following condition [17] is satisfied

$$T_K(f) \cap [f-g]^\perp = \mathrm{cl}(C_K(f) \cap [f-g]^\perp)$$

for $f = f_0^\star = P_K(-\frac{1}{\alpha}\mathcal{G}_0(f_0^\star))$, $g = -\frac{1}{\alpha}\mathcal{G}_0(f_0^\star)$.
Then for $\varepsilon > 0, \varepsilon$ small enough

$$f_\varepsilon^\star = f_0^\star + \varepsilon q + o(\varepsilon)\ , \quad in\ H_0^s(Q) \tag{2.27}$$

where $\|o(\varepsilon)\|_{H_0^s(Q)}/\varepsilon \downarrow 0$ with $\varepsilon \downarrow 0$ and the element q is given by a unique solution of the following optimality system, here we denote $f = f_0^\star$.

Find (y', p', q) such that the following system is satisfied

State equation :

$$-\Delta y' = 0\ , \quad in\ \Omega_f \tag{2.28}$$

$$y' = -q_\nu \frac{\partial y}{\partial n}\ , \quad on\ \Gamma(f) \tag{2.29}$$

$$y' = 0\ , \quad on\ \partial\Omega \setminus \Gamma(f) \tag{2.30}$$

Adjoint state equation :

$$-\Delta p' = y'\ , \quad in\ \Omega_f \tag{2.31}$$

$$p' = -q_\nu \frac{\partial p}{\partial n}\ , \quad on\ \Gamma(f)\ ,\ p' = 0\ , \quad on\ \partial\Omega_f \setminus \Gamma(f) \tag{2.32}$$

Optimality conditions :

$$q \in S = T_K(f_0^\star) \cap [f_0^\star + \frac{1}{\alpha}\mathcal{G}_0(f_0^\star)]^\perp\ ,\ f_0^\star = f\ , \tag{2.33}$$

$$\begin{aligned}
-\int_{\Gamma(f)} \left((h-q)_\nu \frac{(\nabla f(x'), \nabla q(x'))}{(1+|\nabla f(x')|^2)} + q_\nu \frac{(\nabla f(x'), \nabla (h-q)(x'))}{(1+|\nabla f(x')|^2)} \right) \frac{\partial p}{\partial n}\frac{\partial y}{\partial n} d\Gamma \\
-2\int_{\Gamma(f)} \kappa_m q_\nu (h-q)_\nu \frac{\partial p}{\partial n}\frac{\partial y}{\partial n} d\Gamma \\
+\int_{\Gamma(f)} (h-q)_\nu q_\nu \frac{\partial}{\partial n}\left(\frac{\partial y}{\partial n}\frac{\partial p}{\partial n}\right) d\Gamma \\
+\int_{\Gamma(f)} (h-q)_\nu \left(\frac{\partial p'}{\partial n}\frac{\partial y}{\partial n} + \frac{\partial p}{\partial n}\frac{\partial y'}{\partial n}\right) d\Gamma \\
+\alpha((h-q), q)_{H_0^s(Q)} + \int_{\Gamma(f)} (h-q)_\nu \frac{\partial \eta}{\partial n}\frac{\partial y}{\partial n} d\Gamma \ \geq 0\ ,\ \forall h \in S
\end{aligned} \tag{2.34}$$

Here $T_K(v)$ denotes the tangent cone to K at $v \in K$, $[f-v]^\perp$ is the hyperplane orthogonal in $H_0^s(Q)$ to $f-v$. Cone S takes the following form

$$S = \{\varphi \in H_0^s(Q)\ |\ \varphi \geq 0 \ \ on\ \Xi_1\ ,\ \varphi \leq 0 \ \ on\ \Xi_2\ ,$$
$$(f_0^\star + \frac{1}{\alpha}\mathcal{G}_0(f_0^\star), \varphi)_{H_0^s(Q)} = 0\}\ \ ,$$

where

$$\Xi_i = \{x \in Q\ |\ f_0^\star(x) = \psi_i(x)\}\ ,\ i = 1, 2$$

Proof of Theorem 2.2 is given in [17].

3. Shape optimization problem for wave equation

Let us consider the shape estimation problem for the wave equation

$$y_{tt} - \Delta y = F \ , \quad \text{in } \Omega_f \times (0,T) \tag{3.1}$$

$$y = 0 \ , \quad \text{on } \partial\Omega_f \times (0,T) \tag{3.2}$$

$$y(0) = y_0 \ , \ y_t(0) = y_1 \ , \quad \text{in } \Omega_f \tag{3.3}$$

we use here notation of section 2. We assume that F, z are defined in $D \times (0,T)$, y_0, y_1 are defined in D, the data are supposed to be smooth enough. For the shape functional

$$J_\alpha(f) = \frac{1}{2}\int_0^T \int_{\Omega_f} (y(f;.,.) - z(.,.))^2 dxdt + \frac{\alpha}{2}\|f\|^2_{H_0^s(Q)} \tag{3.4}$$

the directional derivative is given by

$$dJ_\alpha(f;h) = \int_0^T \int_{\Gamma(f)} h_\nu \frac{\partial p}{\partial n}\frac{\partial y}{\partial n} d\Gamma dt + \alpha(f,h)_{H_0^s(Q)} \tag{3.5}$$

here we assume $z = 0$ on $\Gamma(f) \times (0,T)$. The adjoint state p satisfies the following equation

$$p_{tt} - \Delta p = y - z \ , \quad \text{in } \Omega_f \times (0,T) \tag{3.6}$$

$$p = 0 \ , \quad \text{on } \partial\Omega_f \times (0,T) \tag{3.7}$$

$$p(T) = 0 \ , \ p_t(T) = 0 \ , \quad \text{in } \Omega_f \tag{3.8}$$

We can evaluate the shape derivative $y'(h_\nu)$ [19], which satisfies the following wave equation

$$y'_{tt} - \Delta y' = 0 \ , \quad \text{in } \Omega_f \times (0,T) \tag{3.9}$$

$$y' = -h_\nu \frac{\partial y}{\partial n} \ , \quad \text{on } \partial\Omega_f \times (0,T) \tag{3.10}$$

$$y'(0) = 0 \ , \ y_t(0) = 0 \ , \quad \text{in } \Omega_f \tag{3.11}$$

In the same way as before we can evaluate the second derivative of the cost functional

$$d^2 J_\alpha(f;h,v) = \tag{3.12}$$

$$-\int_0^T \int_{\Gamma(f)} (h_\nu \frac{(\nabla f(x'), \nabla v(x'))}{(1+|\nabla f(x')|^2)} + v_\nu \frac{(\nabla f(x'), \nabla h(x'))}{(1+|\nabla f(x')|^2)} + 2\kappa_m v_\nu h_\nu) \frac{\partial p}{\partial n}\frac{\partial y}{\partial n} d\Gamma dt$$
$$+\int_0^T \int_{\Gamma(f)} h_\nu v_\nu \frac{\partial}{\partial n}\left(\frac{\partial y}{\partial n}\frac{\partial p}{\partial n}\right) d\Gamma dt$$
$$+\int_0^T \int_{\Gamma(f)} h_\nu \left(\frac{\partial p'}{\partial n}\frac{\partial y}{\partial n} + \frac{\partial p}{\partial n}\frac{\partial y'}{\partial n}\right) d\Gamma dt + \alpha(h,v)_{H_0^s(Q)}$$

where κ_m is the mean curvature on $\Gamma(f)$, and the domain derivative $y'(v_\nu)$ satisfies to (3.9) - (3.11) with h_ν replaced by v_ν, the domain derivative $p'(v_\nu)$ satisfies the following wave equation

$$p'_{tt} - \Delta p' = y' - z' , \text{ in } \Omega_f \times (0,T) \tag{3.13}$$

$$p' = -v_\nu \frac{\partial p}{\partial n} , \text{ on } \partial\Omega_f \times (0,T) \tag{3.14}$$

$$p'(T) = 0 , \ p'_t(T) = 0 , \text{ in } \Omega_f \tag{3.15}$$

here $z'(v_\nu)$ denotes the shape derivative of the observation z. We can obtain the same stability results for the shape optimization problems for the wave equation as in section 3 for the Laplace equation. We refer the reader to [9] for the results on the sensitivity analysis of the convex control problems for the wave equation.

4. APPENDIX

Finally we provide some results on the directional differentiability of the metric projection onto the set K. We assume for simplicity that the set of admissible graphs takes the following form

$$K = \{f \in H_0^m(Q) \mid f(x) \geq \psi(x), x \in Q\} \tag{4.1}$$

here we denote x for x', $\psi(.) = \psi_1(.) \in H_0^m(Q)$, and $m \geq n-1$ is an integer. Exactly the same results can be derived [17] in the case of convex set

$$K = \{f \in H_0^m(Q) \mid \psi_2(x) \geq f(x) \geq \psi_1(x), x \in Q\} \tag{4.2}$$

where $\psi_2(x) \geq \psi_1(x) \geq c > 0$ are given in $H_0^m(Q)$. Tangent cone $T_K(f)$ is the closure in $H_0^m(Q)$ of the convex cone

$$C_K(f) = \{v \in H_0^m(Q) \mid \exists t > 0 \quad f(x) + tv(x) \geq \psi(x), \quad \text{in } \Omega\} \tag{4.3}$$

For $g \in H_0^m(Q)$, such that $f = P_K(g)$ let us define in $H_0^m(Q)$ the following convex cone

$$S = T_K(f) \cap [g - P(g)]^{\perp} = T_K(f) \cap [f - g]^{\perp} \tag{4.4}$$

LEMMA 4.1 *Assume that for $f = P_K g$, $g \in H_0^m(Q)$ the following condition is satisfied*

$$S = \mathrm{cl}(C_K(f) \cap [f - g]^{\perp}) \tag{4.5}$$

here cl stands for the closure. Then for any $h \in H_0^m(Q)$ and $\tau > 0$, τ small enough

$$P_K(g + \tau h) = P_K(g) + \tau P_S h + o(\tau)$$

where $\|o(\tau)\|_{H_0^m(Q)}/\tau \to 0$ with $\tau \downarrow 0$ uniformly with respect to h on compact subsets of $H_0^m(Q)$.

PROOF OF LEMMA 4.1 is given in [7],[10].

REMARK 4.2 It can be shown [11] that in general

$$S \neq \mathrm{cl}(C_K(f) \cap [f - g]^{\perp}) \tag{4.6}$$

in the Sobolev spaces $H_0^m(\Omega)$, $m = 2, 3, \ldots$

Denote

$$\Xi = \{x \in Q \mid f(x) = \psi(x)\} \tag{4.7}$$

hence Ξ is compact, and let the non - negative Radon measure μ be defined by

$$(g - f, \varphi)_{H_0^m(Q)} = \int \varphi d\mu \,,\ 0 \leq \varphi \in C_0^{\infty}(Q) \tag{4.8}$$

We denote by $\mathrm{spt}\mu$ the support of measure μ, $\mathrm{spt}\mu$ is compact. It can be shown [17], that μ integrates all elements of the Sobolev space $H_0^m(Q)$.

DEFINITION 4.3 Compact $F = \mathrm{spt}\mu$ is admissible provided condition (3.5) is satisfied.

THEOREM 4.4 *Assume that the compact $F = \mathrm{spt}\mu$ satisfies the following condition:*
$\varphi \in H_0^m(Q)$, $\varphi = 0$ on F implies $\varphi \in H_0^m(Q \setminus F)$,
then $\mathrm{spt}\mu$ is admissible.

COROLLARY 4.5 *Assume that condition (4.5) is satisfied for* $f = P_K(g)$, $g = g(0)$ *and* $g(.) : [0, \delta) \to H_0^m(Q)$, $2m > n$, *is given mapping strongly differentiable at* 0^+, *then for* $\varepsilon > 0$, ε *small enough*

$$P_K(g(\varepsilon)) = P_K(g(0)) + \varepsilon P_S(g'(0^+)) + o(\varepsilon) \tag{4.9}$$

where we denote

$$S = T_K(f) \cap [g(0) - f]^{\perp} \tag{4.10}$$

$$= \{\varphi \in H_0^m(Q) \mid \varphi(x) \geq 0, \text{ on } \Xi , \int \varphi d\mu = 0\} \tag{4.11}$$

Finally we have the following remarks

For the observation $z \in H^1(D')$, where $D' \subset D$, we replace Ω_f in (2.10) by $\Omega_f \cap D'$ and we can proceed in the same way.

The form of second derivative (2.15) can be obtained using the classical formula for the time derivative of the surface integral over a moving manifold. We refer the reader to [17] for the details.

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