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Optimization of fiber orientation and concentration in composites

by

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Linearly elastic fiber reinforced composite discs and laminates in plane stress with variable local orientation and concentration of one or two fiber fields embedded in the matrix material, are considered. The thickness and the domain of the discs or laminates are assumed to be given, together with prescribed boundary conditions and inplane loading along the edge.

The problem under study consists in determining throughout the structural domain the optimum orientations and concentrations of the fiber fields in such a way as to maximize the integral stiffness of the composite disc or laminate under the given loading. Minimization of the integral stiffness can also be carried out. The optimization is performed subject to a prescribed bound on the total cost or weight of the composite that for given unit cost factors or specific weights determines the amounts of fiber and matrix materials in the structure. Examples are presented by the end of the paper.

1. Introduction

This paper gives a brief account of recent research reported by the first author in [1] on optimization of fiber orientation and concentration in composite discs and laminates. The research is inspired by the initial work in the field by Rasmussen [2] (reported in Danish, account in English available in Niordson and Olhoff [3]) and by important recent developments of Pedersen [4–7]. Problems concerning optimization of fiber orientation have earlier been considered by Banichuk [8], and we refer to Sacchi Landriani and Rovati [9] for other current research activities in the area. Since the present research also comprises optimization of fiber concentration, and allows for more than one field of fibers, our development can be easily augmented with appropriate constitutive material models applicable for topology optimization, cf. Bendsøe [10–11], and results for problems of this type have already been obtained.

The motivation for the work described in this paper is that fiber reinforced composite materials are ideal for structural applications, where high stiffness and strength are required at low weight. Aircraft and spacecraft are typical weight sensitive structures, in which composite materials are cost effective. To obtain the full advantage of the fiber reinforcement, fibers must be distributed and oriented optimally with respect to the actual strain field. Hence, transfer of fiber material from initially lowly stressed parts of the body in order to strengthen the parts and directions that are subjected to large internal forces is the general idea of optimization of composite structures.

Thus, relative to refs. [4–9], we in this paper both use fiber orientations and – concentrations as design variables. Based on the strain field determined by finite element analysis we construct an iterative two–level optimization procedure that consists of an optimality criterion approach as described by Pedersen [4,5,7], and a mathematical programming technique. Here,

- in the first level, the local fiber orientations corresponding to a global optimum are determined using an optimality criterion for these design variables, and

- in the second level, the local distribution of the amounts of fiber and matrix materials available within a bound on total cost or weight, are determined on the basis of analytically derived design sensitivities. In this level, the optimization is carried out by means of a dual mathematical programming technique as implemented in the optimizer CONLIN by Fleury and Braibant [12].

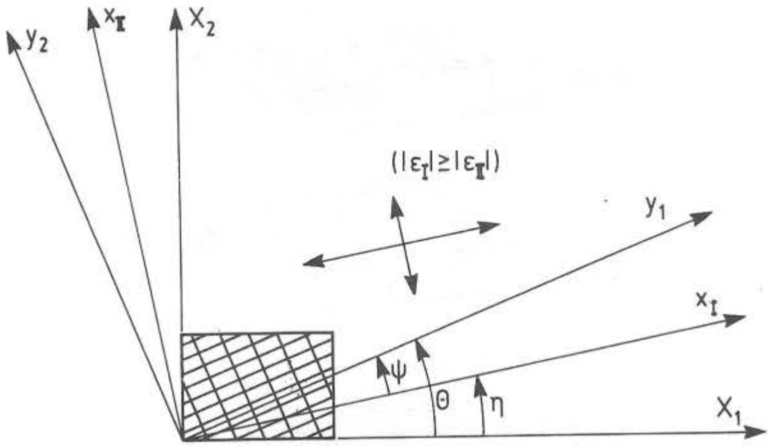

Figure 1. Definition of the angles ψ, θ and η for mutual rotations of the finite element coordinate system X_1, X_2, the principal strain coordinate system x_I, x_{II} and the material coordinate system y_1, y_2

2. Objective function

The integral stiffness of the composite structure will be selected as the objective function for optimization, and we will be primarily interested in maximization. The structure of maximum integral stiffness will be defined as the structure that has minimum total elastic strain energy subject to a given loading.

We shall assume that our composite disc or laminate can be locally considered as a macroscopically homogeneous, orthotropic material. The strain energy density u will then be given by the following formula for an orthotropic laminate, see e.g. Jones [13],

$$u = \frac{1}{2}\{\epsilon\}^T[A]\{\epsilon\} = \frac{1}{2}A_{11}\epsilon_{11}^2 + \frac{1}{2}A_{22}\epsilon_{22}^2 + A_{12}\epsilon_{11}\epsilon_{22} + 2A_{66}\epsilon_{12}^2 \qquad (2.1)$$

where $\{\epsilon\} = \{\epsilon_{11}; \epsilon_{22}; 2\epsilon_{12}\}$ is the strain vector, and $[A]$ the stiffness matrix.

We now use well–known formulas to express the strain components in (2.1) by the principal strains, ϵ_I and ϵ_{II}, and the angle ψ from the direction corresponding to the numerically largest principal strain ϵ_I ($|\epsilon_I| \geq |\epsilon_{II}|$) to the direction associated with the largest stiffness A_{11} ($A_{11} \geq A_{22}$), see Fig. 1.

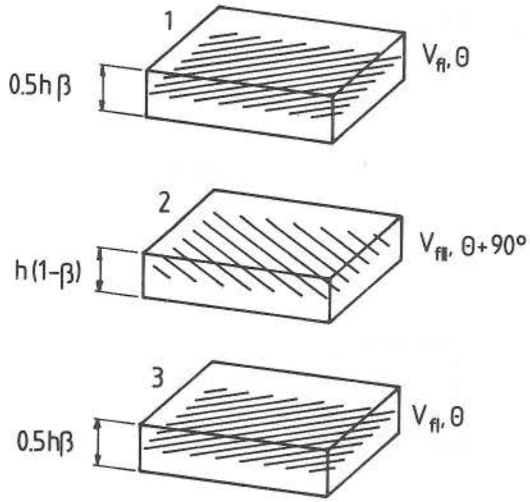

Figure 2. Design variables of an element consisting of 3 orthogonal plies

Since in the finite element analysis the structure is discretized into n elements with individual, constant laminate stiffness matrices $[A]_i$, the total elastic strain energy U for the structure is given by

$$\begin{aligned} U = & \sum_{i=1}^{n}((\frac{1}{8}A_{11}((\epsilon_I+\epsilon_{II})+(\epsilon_I-\epsilon_{II})\cos 2\psi)^2 \\ & +\frac{1}{8}A_{22}((\epsilon_I+\epsilon_{II})-(\epsilon_I-\epsilon_{II})\cos 2\psi)^2 \\ & +\frac{1}{4}A_{12}((\epsilon_I+\epsilon_{II})^2-(\epsilon_I-\epsilon_{II})^2\cos^2 2\psi) \\ & +\frac{1}{2}A_{66}(\epsilon_I-\epsilon_{II})^2\sin^2 2\psi)S)_i \end{aligned} \tag{2.2}$$

where S_i is the area of the i–th finite element.

3. Design model and cost function

The fiber orientation and concentration within each element of the discretized structure are adopted as design variables.

Our design model is made up of elements that consist of 3 fiber plies with the fiber orientations θ, $\theta+90°$ and θ, and the volumetric fiber concentrations

V_{fI}, V_{fII} and V_{fI}, see Fig. 2. Introducing the variable ratio β between the thickness of fiber ply 2 (in the middle) and the total thickness h of the element, we get the symmetric and orthotropic laminate shown in Fig. 2, which can have both unidirectional ($\beta = 0 \vee \beta = 1$) and cross ply ($0 < \beta < 1$) character.

We have now defined 4 design variables for each element: V_{fI}, V_{fII}, θ and β. For these design variables, we prescribe lower and upper constraint values as follows:

$$\begin{aligned} &0 \le (V_{fI})_i \le \bar{V}_f, \quad 0 \le (V_{fII})_i \le \bar{V}_f, \quad 0 \le \theta_i \le 180^\circ, \quad 0 \le \beta_i \le 1, \\ &i = 1, \dots, n \end{aligned} \tag{3.1}$$

Here the given upper constraint value $\bar{V}_f$ for the fiber concentrations depends on how densely the fibers can be packed in the matrix material in view of their cross–sectional shape.

We finally formulate a constraint that enforces the total cost or weight C of the structure to be less than or equal to a given upper bound $\bar{R}$ if stiffness maximization is considered,

$$\begin{aligned} C \; = \; & \sum_{i=1}^{n} (c_f \{ (V_{fI})_i h \beta_i + (V_{fII})_i h (1 - \beta_i) \} \\ & + c_m \{ (1 - (V_{fI})_i) h \beta_i + (1 - (V_{fII})_i) h (1 - \beta_i) \}) S_i \quad \le \quad \bar{R} \end{aligned} \tag{3.2}$$

Here c_f and c_m are given so–called "unit cost factors". They denote the cost per unit volume of the fiber and matrix materials, respectively, for a cost constrained problem, whereas c_f and c_m denote the specific weights of the fiber and matrix materials, respectively, if the total weight is constrained.

4. Stiffness matrix in terms of design variables

The fiber and matrix materials will be assumed to be linearly elastic with given Young's moduli E_f and E_m and Poisson's ratios ν_f and ν_m. We now adopt the "rule of mixtures", see e.g. Jones [13], for determining the components of the tensor of elasticity for a lamina in our design model

$$\begin{aligned} &E_{Lj} = (1 - V_{fj}) E_m + V_{fj} E_f, \quad E_{Tj} = \frac{E_f E_m}{(1 - V_{fj}) E_f + V_{fj} E_m}, \\ &G_{LTj} = \frac{E_m E_f}{2(V_{fj}(E_m \nu_f - E_f \nu_m + E_m - E_f) + E_f \nu_m + E_f)}, \\ &\nu_{LTj} = (1 - V_{fj}) \nu_m + V_{fj} \nu_f, \quad j = I, II \end{aligned} \tag{4.1}$$

Here indexes L and T refer to the longitudinal and transverse directions of the fibers, respectively, and the index j will here and in the following take on the "values" I and II that refer to the fiber layers 1 and 2, respectively.

For a composite element as shown in Fig. 2 that consists of 3 lamina with the thicknesses $0.5h\beta$, $h(1-\beta)$ and $0.5h\beta$ and the fiber orientations θ, $\theta+90°$ and θ, we can easily obtain the laminate stiffness matrix $[A]$ by means of a formula given in Tsai & Pagano [14]. We get

$$\begin{aligned}[A] = \tfrac{E_{LI}h\beta}{8\alpha_{0I}} \Bigg(&\begin{bmatrix} 8 & 0 & 0 \\ & 8 & 0 \\ s & & 4 \end{bmatrix} + \alpha_{2I} \begin{bmatrix} \cos 2\theta - 1 & 0 & \sin 2\theta/2 \\ & -\cos 2\theta - 1 & \sin 2\theta/2 \\ s & & -1/2 \end{bmatrix} \\ &+ \alpha_{3I} \begin{bmatrix} \cos 4\theta - 1 & -\cos 4\theta & \sin 4\theta \\ & \cos 4\theta - 1 & -\sin 4\theta \\ s & & -\cos 4\theta - 1/2 \end{bmatrix} + \alpha_{4I} \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ s & & -1/2 \end{bmatrix} \Bigg) \\ + \tfrac{E_{LII}h(1-\beta)}{8\alpha_{0II}} \Bigg(&\begin{bmatrix} 8 & 0 & 0 \\ & 8 & 0 \\ s & & 4 \end{bmatrix} + \alpha_{2II} \begin{bmatrix} \cos 2\phi - 1 & 0 & \sin 2\phi/2 \\ & -\cos 2\phi - 1 & \sin 2\phi/2 \\ s & & -1/2 \end{bmatrix} \\ &+ \alpha_{3II} \begin{bmatrix} \cos 4\phi - 1 & -\cos 4\phi & \sin 4\phi \\ & \cos 4\phi - 1 & -\sin 4\phi \\ s & & -\cos 4\phi - 1/2 \end{bmatrix} + \alpha_{4II} \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ s & & -1/2 \end{bmatrix} \Bigg) \end{aligned} \tag{4.2}$$

where θ denotes the angle defining the fiber orientation, see Fig. 1, and the angle $\phi = \theta + 90°$ defines the orthogonal direction.

The parameters $\alpha_{0j}, \alpha_{2j}, \ldots, \alpha_{4j}$, in (4.2) can all be expressed explicitly in terms of the fiber concentrations V_{fj}, $j = I, II$, and the given elastic constants of the fiber and matrix materials, i.e.,

$$\alpha_{mj} = \alpha_{mj}(V_{fI}, V_{fII}, E_f, E_m, \nu_f, \nu_m) \quad m = 0,2,3,4 \quad j = I, II \tag{4.3}$$

For reasons of brevity, the reader is referred to [1] for the specific expressions.

5. Optimization technique

The optimization problem is solved iteratively via a two-level procedure of redesign. The stress/strain field is initially determined by finite element analysis using MODULEF [15] in each loop of redesign, and improved orientations of the fibers are subsequently determined by means of an optimality criterion in the first level of redesign. In the second level of redesign the distributions of fibers are improved via a method of sensitivity analysis and mathematical programming.

A notable feature of the present problem is that a usual gradient method may fail in determining the optimal orientation of the fibers, because local optima normally exist, see e.g. Fig. 4.4 in [7]. To circumvent this inherent difficulty in the first level of redesign we follow Pedersen [4,7] and perform an analytical investigation of the first and second derivative in order to determine the global optimum of the total strain energy with respect to fiber orientation. From (2.2) and (4.2) we get the following expression for first order sensitivities, cf. Pedersen [4,7],

$$\frac{dU}{d\theta_i} = \frac{dU}{d\psi_i} = (4A\alpha_3(\epsilon_I - \epsilon_{II})^2 \sin 2\psi(\gamma + \cos 2\psi)S)_i, \; i = 1, \ldots, n \qquad (5.1)$$

where A is a constant, and the parameter γ_i is defined by

$$\gamma_i = \left(\frac{\alpha_2}{4\alpha_3} \frac{1 + \frac{\epsilon_{II}}{\epsilon_I}}{1 - \frac{\epsilon_{II}}{\epsilon_I}} \right)_i , \quad i = 1, \ldots, n \qquad (5.2)$$

The material parameters α_2 and α_3 are those appearing in (4.2) and they are dubbed as C_2 and C_3 in [7]. The results of a complete investigation of the extrema of U with respect to the key parameters ψ, α_3 and γ are summarized in a table in refs. [1], [4] and [7] (Table 3.1 in [7]).

As described in [1], [4] and [7], the fiber orientation θ_i for each element can be determined by means of this table and the formula

$$\theta_i = \psi_i + \eta_i, \quad i = 1, \ldots, n \qquad (5.3)$$

where η_i is the angle of rotation of the principal strain or stress direction of the i–th element relative to the X_1 axis of the finite element coordinate system, see Fig. 1.

In the above solution procedure for the first level of redesign it was found that the fibers should be oriented along the principal stress directions if stiffness maximization is performed, and along the principal strain directions in problems of stiffness minimization.

The second stage in the loop of redesign consists in determining an improved distribution of the amount of fiber material, i.e., to obtain improved values of the design variables β_i, $(V_{fI})_i$ and $(V_{fII})_i$, $(i = 1, \ldots, n)$. This is done by a dual method of mathematical programming using mixed variables as developed by Fleury and Braibant [12] and implemented in the computer code CONLIN.

To this end we need the sensitivities of the objective function and constraints with respect to the aforementioned design variables.

Now, it is shown by Pedersen in [4,7] that by means of Clayperon's theorem and the principle of virtual displacements for structures with design independent loads, the gradient of the total strain energy can be determined from the gradient of the strain energy density u for a given element, whose strain field is considered to be fixed,

$$\frac{dU}{da_i} = -\frac{\partial u_i}{\partial a_i} S_i, \quad i = 1, \ldots, n \tag{5.4}$$

Here a_i denotes any of the design variables β_i, $(V_{fI})_i$, or $(V_{fII})_i$, $i = 1, \ldots, n$.

The sensitivities of the total strain energy U with respect to β_i, $(V_{fI})_i$ and $(V_{fII})_i$, can thus be determined by (2.2) and (5.4), assuming the strain field to be fixed, and restricting variation to the laminate stiffness matrix $[A]$. For the i–th element of the discretized geometry we then obtain the following expression for sensitivities w.r.t. the design variables a_i

$$\begin{aligned} U_{,a_i} \;=\; & -((\frac{1}{8}A'_{11}((\epsilon_I + \epsilon_{II}) + (\epsilon_I - \epsilon_{II})\cos 2\psi)^2 \\ & +\frac{1}{8}A'_{22}((\epsilon_I + \epsilon_{II}) - (\epsilon_I - \epsilon_{II})\cos 2\psi)^2 \\ & +\frac{1}{4}A'_{12}((\epsilon_I + \epsilon_{II})^2 - (\epsilon_I - \epsilon_{II})^2\cos^2 2\psi) \\ & +\frac{1}{2}A'_{66}(\epsilon_I - \epsilon_{II})^2\sin^2 2\psi)S)_i \end{aligned} \tag{5.5}$$

where a_i denotes any of the design variables β_i, $(V_{fI})_i$ or $(V_{fII})_i$, and A'_{kl} is a shorthand notation for the derivatives $\frac{\partial A_{kl}}{\partial a_i}$ of a component A_{kl} of the stiffness matrix $[A]$.

The derivatives of $[A]$ with respect to β_i are easily obtained from (4.2), with θ taken to be equal to zero to give the material orthotropic characteristics.

Since $[A]$ depends on α, E_L and V_f, i.e.,

$$\begin{aligned} [A_{kl}] \;=\; & [A_{kl}](\alpha_{0I}(V_{fI}), \alpha_{2I}(V_{fI}), \alpha_{3I}(V_{fI}), \alpha_{4I}(V_{fI}), \\ & E_{LI}(V_{fI}), \alpha_{0II}(V_{fII}), \alpha_{2II}(V_{fII}), \alpha_{3II}(V_{fII}), \\ & \alpha_{4II}(V_{fII}), E_{LII}(V_{fII})) \end{aligned} \tag{5.6}$$

the sensitivities of $[A]$ with respect to V_{fj} are found by means of the chain rule, and we get

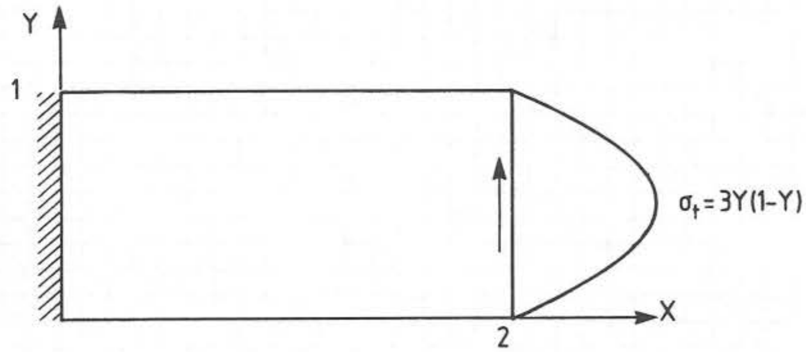

Figure 3. Example problem for optimization

$$\begin{aligned}\frac{\partial[A_{kl}]}{\partial V_{fI}} &= \sum_{m=0,2,3,4}\left(\frac{\partial[A_{kl}]}{\partial\alpha_{mI}}\frac{\partial\alpha_{mI}}{\partial V_{fI}}\right)+\frac{\partial[A_{kl}]}{\partial E_{LI}}\frac{\partial E_{LI}}{\partial V_{fI}}\\ \frac{\partial[A_{kl}]}{\partial V_{fII}} &= \sum_{m=0,2,3,4}\left(\frac{\partial[A_{kl}]}{\partial\alpha_{mII}}\frac{\partial\alpha_{mII}}{\partial V_{fII}}\right)+\frac{\partial[A_{kl}]}{\partial E_{LII}}\frac{\partial E_{LII}}{\partial V_{fII}}\end{aligned} \tag{5.7}$$

These sensitivities are derived analytically in [1], and the results are available therein. The sensitivities of the cost function (3.2) are readily derived analytically, and we thus have all the necessary sensitivity information that is required for the optimization in the second level of redesign.

6. Examples

We first consider three example problems of optimization of the rectangular composite disc shown in Fig. 3. The disc has one of its sides fixed against displacements in the X and Y directions, while the opposite side is subjected to a parabolically distributed shear loading. The upper constraint value $\bar{V}_f$ for fiber concentration in (3.1) is taken to be $\bar{V}_f = 80$ pct, and we only consider cases of $c_m = 0$ and $c_f = 1$, which means that the fibers are dominating in the cost or weight function C in (3.2).

In the *first example* we consider maximization of the stiffness of the disc under the condition that only one fiber field is allowed in each element. This corresponds to the special case of $\beta = 0 \vee \beta = 1$, see Chapter 3. The structure is discretized into 20×40 4–node elements (type QUAD 2Q1D, see [15]). The result of the optimization is shown in Fig. 4, where the direction and density of the

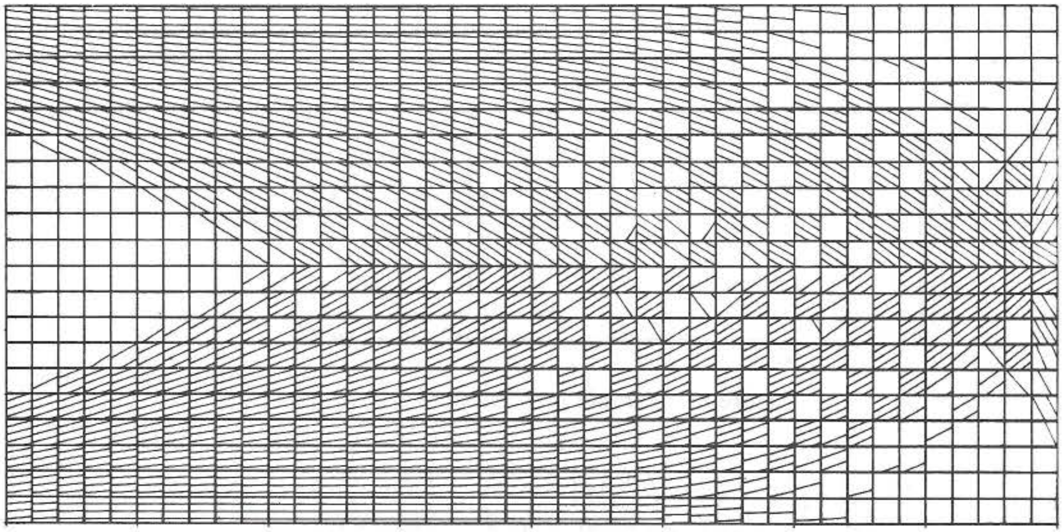

Figure 4. Optimal distribution and orientation of fibers in *first example:* One fiber field, n=800, maximization of stiffness

hatching within each element illustrate the fiber orientation and concentration, respectively. We see that the lowly stressed elements do not contain any fibers. It is also noteworthy that the design contains "holes" in the fiber reinforcement in the mid part of the structure, where shear forces are dominating.

No doubt this is due to the fact that only one fiber field is allowed to exist in each element. This is not favourable in shear dominated areas with almost equal principal stresses, and the pattern obtained in the mid part may be conceived as the best possible attempt of the structure to increase its "shear force stiffness" under the given design conditions. The design shown in Fig. 4 is associated with a reduction of the total elastic energy U by 51% relative to the initial design, where all the fibers were uniformly distributed and given the orientation $\theta_i = 0°$.

However, the convergence is very slow, and different designs may be obtained as a result of the optimization. In particular, the designs depend on the size of the applied FE–mesh, and it is not possible to obtain a limiting, numerically stable design by consecutively decreasing the mesh size. These features, along with the generation of "holes" in the design, indicate the necessity of a regularization of the formulation of the optimization problem (see, e.g., the survey by

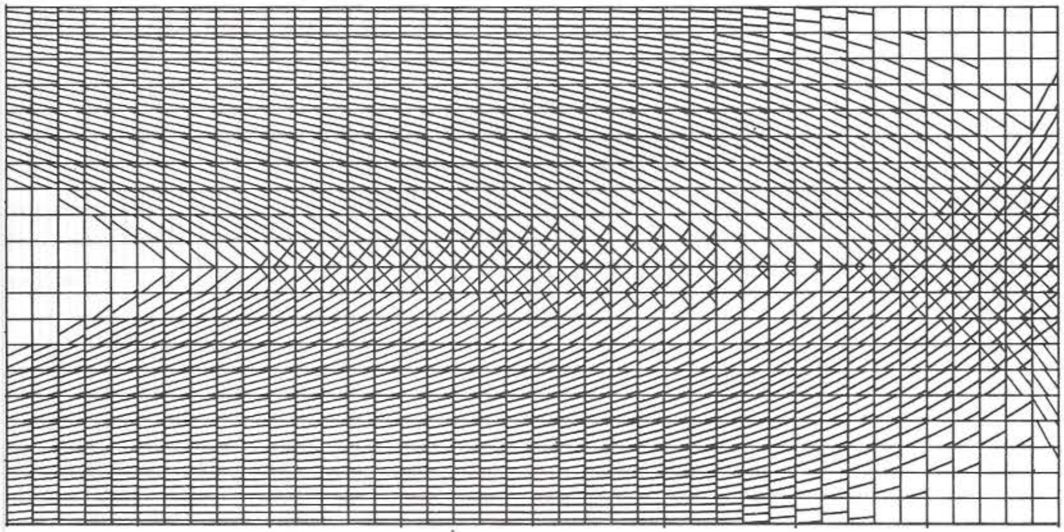

Figure 5. Optimal distribution and orientation of fibers in *second example*: Two fiber fields, n=800, maximization of stiffness

Olhoff and Taylor [16]).

This leads to our *second example*: Regularization of the formulation of the type of problem just considered is simply obtained by extending the design space such as to allow for formation of two orthogonal fiber fields everywhere in the disc (which is actually covered in the preceding chapters). Introducing two fiber fields, the design in Fig. 4 is replaced by the solution shown in Fig. 5, where the "shear force reinforcement" appears along the horizontal center line in agreement with the boundary and symmetry conditions. Optimizing the structure, U is reduced by 55%. Now the convergence is rapid and the design is found to be independent of the discretization, which confirms that regularization has been achieved.

In our *third example*, we consider the somewhat abnormal, but theoretically interesting problem of *minimization* of the integral stiffness, assuming two fiber fields and the total amount of fiber material to be larger than a given lower bound. The result of this problem is shown in Fig. 6, where we see that the design only uses one of the fiber fields, and clearly distributes and orientates this field in such a way as to avoid its properties as stiffness reinforcement. In this example U is increased by 270%, compared with an initial design where the

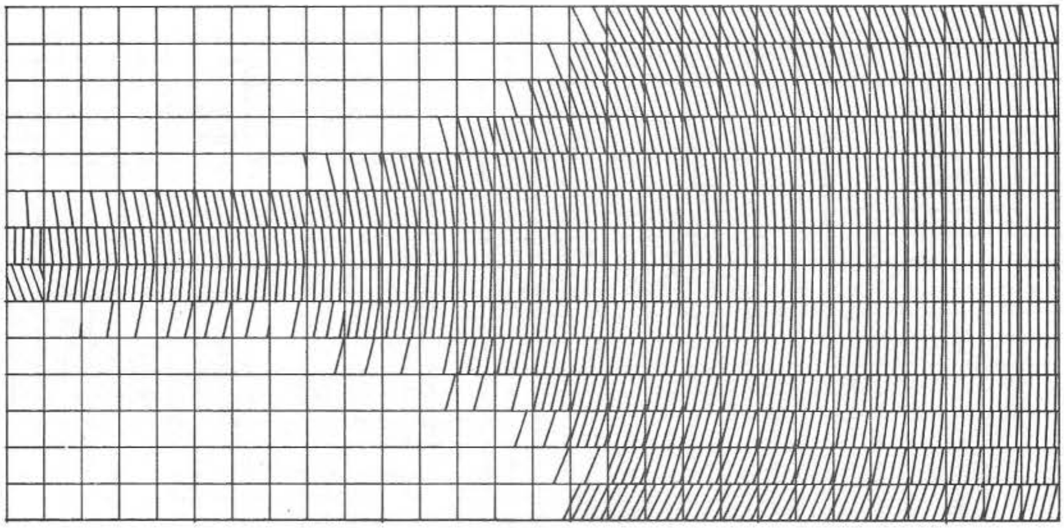

Figure 6. Optimal fiber "reinforcement" in *third example*: Two fiber fields, n=392, minimization of stiffness

fibers are uniformly distributed in two fiber fields with the orientations $\theta = 0°$ and $\theta = 90°$.

In a *final example*, we wish to demonstrate that the method and software developed here for optimization of composites has been extended to cover topology optimization as well. In the latter type of problem the structure is considered as a domain of space with a high concentration of material, see Bendsøe [10–11]. The present extension is made by introducing relationships between stiffness components and concentration of layered, second rank microstructures that replace the two fiber fields, and assigning the matrix material vanishing stiffness. The reader is referred to Bendsøe [10–11]. Fig. 7 shows the result of optimizing the topology of a structure for which the left hand side of the rectangular domain in Fig. 7 offers full fixation, and where the structure is required to carry a vertical force at the lower right hand corner. The result is clearly seen to become a truss–like structure.

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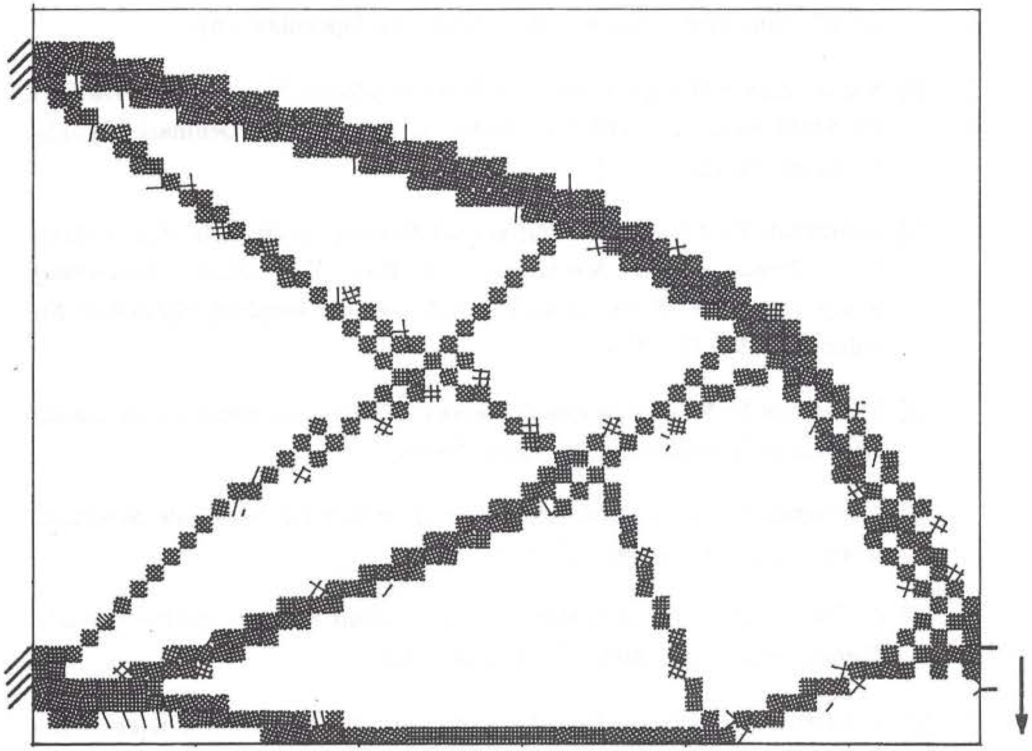

Figure 7. Example of topology optimization (see text)

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