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A method for finding bifurcation points.

by

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In the paper a method for finding bifurcation points along solution curves in free boundary problems is introduced. In this method, a point along a solution curve is determined as a bifurcation point where the smallest eigenvalue of a linearized problem is equal to zero. In order to verify the proposed method, numerical computations are carried out.

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1. Introduction

Free boundary problems recently received much attention. This is so because of their nonlinearity and of frequent appearance of bifurcation phenomena. If a solution curve is found, it is interesting to find both bifurcation points along this curve and the type of bifurcation. This, however, is not an easy task.

There are mainly two ways to investigate bifurcation phenomena. One is to use methods of global analysis and another is the local one. Let the global behavior of solution curves be only partially known. Then, the global analysis technique using the invariance of the topological degree by homotopy is useful [13]. Let solution curves be parametrized smoothly and its local behavior be known. Then, local analysis enables us to discuss the occurrence and types of bifurcation. It also enables us to discuss the number of branches at a bifurcation point [1, 2, 10, 12]. However, these methods are not suitable for numerical computations, so they are of no practical use.

In the present paper a practical method for finding bifurcation points in a free boundary problem is introduced. Let the free boundary problem introduce an additional boundary condition. Then it can be rewritten in the implicit form

$$\Psi(a, \gamma) = 0$$

where γ is a free boundary, and a is a bifurcation parameter. By the implicit theorem, bifurcation occurs when Jacobian $\frac{\partial}{\partial\gamma}\Psi$ does not have an inverse. Using domain dependence technique [6, 7, 8, 11], this Jacobian can be determined. Thus, a linearized problem is obtained. The regularity of the Jacobian corresponds to the smallest eigenvalue of a problem which is obtained by extending the linearized problem. This is the key point of our method. Therefore, in our method this smallest eigenvalue is used for specifying bifurcation points. It is not so difficult to obtain the smallest eigenvalue numerically, so our method is of practical use.

In order to verify our method, we have applied it to a free boundary problem related to two-dimensional plasma equilibrium subject to a surface current. The problem is given as follows (see Fig. 1).

Problem 1 *Find a closed Jordan curve γ and a function $u(x, y)$ such that*

$$\Delta u = 0 \quad \text{in} \quad \Omega_{\gamma,a}, \tag{1}$$

$$u = 0 \quad \text{on} \quad \gamma, \tag{2}$$

$$u = \kappa \quad \text{on} \quad \Gamma_a, \tag{3}$$

$$\frac{\partial u}{\partial \nu} = \frac{4}{\ell_\gamma} \quad \text{on} \quad \gamma \tag{4}$$

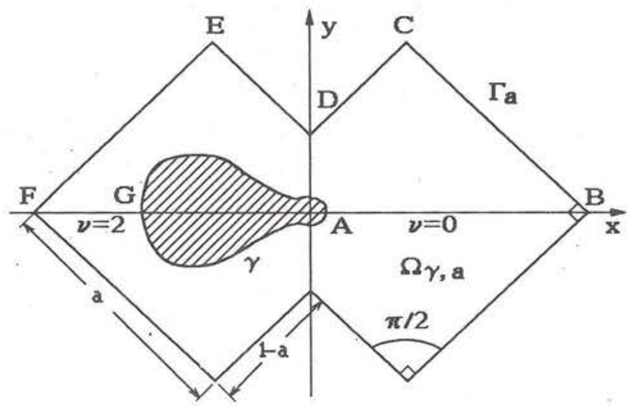

Figure 1. One–component asymmetric plasma

Here Δ is the Laplace operator, κ is a given positive constant, ℓ_γ is the length of γ, $\Omega_{\gamma,a}$ is the region between Γ_a and γ, ν is the unit inward normal vector to γ. The boundary Γ_a is fixed, parametrized by a $(1/2 < a \leq 1)$, piecewise smooth and symmetric with respect to both the x–axis and the y–axis. γ represents a free boundary which is assumed to be located inside Γ_a and symmetric with respect to the x–axis.
Concerning this problem, the following terminology is used.

(i) If γ consists of a simple closed Jordan curve, then plasma is called *one–component.* If γ consists of two isolated closed Jordan curves, then plasma is called *two–component.*

(ii) If γ is symmetric with respect to the y–axis, then the plasma is called *symmetric.* If γ is asymmetric, then plasma is called *asymmetric.*

Demidov obtained bifurcation diagram of both one– and two–component symmetric plasma [3, 4, 5]. Imai and Kawarada obtained bifurcation diagrams of both one–component symmetric and asymmetric plasmas [9]. Recently these authors have also obtained the whole bifurcation diagram of both one– and two–component plasmas. Under the limitations of the development of domain dependence techniques, our method is applicable to one–component plasma. So, numerical computations of the smallest eigenvalue are carried out along solution curves of one–component plasma.

2. A method for finding bifurcation points

Our method for finding bifurcation points consists of the following steps. The free boundary problem is transformed into a problem in the implicit form. Using domain dependence techniques, the linearization concerning a free boundary is carried out. The linearized problem is extended to an eigenvalue problem. A point on a solution curve is determined as a bifurcation point if the smallest eigenvalue is equal to zero at this point.

In order to show the concrete implementation of our method and to verify it, let us refer to Problem 1. According to the algorithm mentioned above, Problem 1 is transformed into the following:

Problem 2 *Let κ be fixed. Then for given a find γ such that*

$$\Psi(a,\gamma) \equiv \left.\frac{\partial u_D}{\partial \nu}\right|_\gamma - \frac{4}{\ell_\gamma} = 0, \tag{5}$$

where a function $u_D(x,y)$ is a solution of the following Dirichlet problem:

$$\begin{aligned} \Delta u_D &= 0 \quad \text{in} \quad \Omega_{\gamma,a}, && (6)\\ u_D &= \kappa \quad \text{on} \quad \Gamma_a, && (7)\\ u_D &= 0 \quad \text{on} \quad \gamma. && (8) \end{aligned}$$

Using domain dependence techniques [6, 7, 8, 11], the linearization of Ψ concerning γ is calculated formally as follows. For given perturbation $\delta\gamma(\theta)$ $(0 \le \theta \le 2\pi)$,

$$\frac{\partial \Psi}{\partial \gamma}\delta\gamma = \left.\frac{\partial z}{\partial \nu}\right|_\gamma + 4 < \nu, r > \frac{H}{\ell_\gamma}\delta\gamma - \frac{4}{\ell_\gamma^2}\int_\gamma H < \nu, r > \delta\gamma d\ell, \tag{9}$$

where $r = (\cos\theta, \sin\theta)$, $< \cdot, \cdot >$ denotes the inner product and H is the curvature of γ (positive if the inner domain of γ is convex). Let $z = \frac{\partial u_D}{\partial \gamma}\delta\gamma$. Then, it satisfies

$$\begin{aligned} -\Delta z &= 0 \quad \text{in} \quad \Omega_{\gamma,a}, && (10)\\ z &= \kappa \quad \text{on} \quad \Gamma_a, && (11)\\ z + < \nu, r > \frac{4}{\ell_\gamma}\delta\gamma &= 0 \quad \text{on} \quad \gamma. && (12) \end{aligned}$$

In order to investigate the regularity of $\frac{\partial\Psi}{\partial\gamma}$, it is necessary to solve

$$\frac{\partial\Psi}{\partial\gamma}\delta\gamma = 0. \tag{13}$$

Then, from (9)–(13), the following linearized problem is obtained:

Problem 3 *Find a function $z(x,y)$ such that*

$$\begin{aligned} -\Delta z &= 0 \quad \text{in} \quad \Omega_{\gamma,a}, & (14)\\ z &= 0 \quad \text{on} \quad \Gamma_a, & (15)\\ \frac{\partial z}{\partial \nu} - Hz + \frac{1}{\ell_\gamma}\int_\gamma Hz d\ell &= 0 \quad \text{on} \quad \gamma. & (16) \end{aligned}$$

Remark 1 *Let a point of the solution curve be a bifurcation point. Then, from (13), $\delta\gamma \neq 0$. This means that a solution $z(x,y)$ of Problem 3 is not identically equal to zero. Conversely, let a point of the solution curve be a regular point. Then $\delta\gamma = 0$. This means that a solution $z(x,y)$ of Problem 3 is identically zero.*

From Remark 1 it is convenient to consider the following eigenvalue problem along a solution curve for finding bifurcation points.

Problem 4 *Find the smallest eigenvalue λ_0 such that*

$$\begin{aligned} -\Delta z &= \lambda z \quad \text{in} \quad \Omega_{\gamma,a}, & (17)\\ z &= 0 \quad \text{on} \quad \Gamma_a, & (18)\\ \frac{\partial z}{\partial \nu} - Hz + \frac{1}{\ell_\gamma}\int_\gamma Hz d\ell &= 0 \quad \text{on} \quad \gamma. & (19) \end{aligned}$$

Remark 2 *It follows from Remark 1, that if the smallest eigenvalue is zero at a point of the solution curve, then this point is a bifurcation point. It is not so difficult to obtain the smallest eigenvalue of the above problem numerically. So, this method is of practical use for finding bifurcation points.*

Remark 3 *As follows from the above procedure, our method is also applicable to other free boundary problems.*

3. A modification

In order to carry out more precise numerical computations, a modification is necessary. Note that we are only interested in knowing a solution $z(x,y)$ of

Problem 3 is identically equal to zero or not. Hence Problem 3 is transformed into Problem 3' using the following hodograph transformation [3, 4, 5, 9]. This technique is well–known in engineering as a grid generation using a body–fitted curvilinear coordinate system [14–17]. Then an eigenvalue problem is considered (Problem 4').

Let $v(x,y)$ be harmonically adjoint to a solution $u(x,y)$ of Problem 1. Choose it in such a way that $v=0$ on the segment AB in Fig. 1. Then $A(u,v)$ and $B(u,v)$ are introduced as

$$\frac{\partial z}{\partial w} = e^{A(u,v)+iB(u,v)}, \quad w(z) = u(x,y) + iv(x,y), \quad z = x+iy. \tag{20}$$

Functions $A(u,v)$ and $B(u,v)$ have already been obtained [9]. Using these representations, Problem 3 is transformed into the following problem in the rectangular domain.

Problem 3' *Find a function $z(u,v)$ such that*

$$\begin{aligned} -\Delta z &= 0 \quad \text{in} \quad G, &(21)\\ \frac{\partial z}{\partial v} &= 0 \quad \text{on} \quad \Gamma_1 \quad \text{and} \quad \Gamma_3, &(22)\\ z &= 0 \quad \text{on} \quad \Gamma_2, &(23)\\ \frac{\partial z}{\partial u} + \frac{\partial B}{\partial v}z - \frac{1}{e^{A_0}}\int_0^2 \left(\frac{\partial B}{\partial v}z\right)(0,v)dz &= 0 \quad \text{on} \quad \Gamma_0. &(24) \end{aligned}$$

Here $A_0 = A(0,v)$ (= constant), $G = \{(u,v)|0 < u < \kappa, 0 < v < 2\}$, $\Gamma_0 = \{(0,v)|0 < v < 2\}$, $\Gamma_1 = \{(u,0)|0 < u < \kappa\}$, $\Gamma_2 = \{(\kappa,v)|0 < v < 2\}$ and $\Gamma_3 = \{(u,2)|0 < u < \kappa\}$.

Then an eigenvalue problem corresponding to the above problem is given as follows.

Problem 4' *Find the smallest eigenvalue λ_0 such that*

$$\begin{aligned} -\Delta z &= \lambda z \quad \text{in} \quad G, &(25)\\ \frac{\partial z}{\partial v} &= 0 \quad \text{on} \quad \Gamma_1 \quad \text{and} \quad \Gamma_3, &(26)\\ z &= 0 \quad \text{on} \quad \Gamma_2, &(27)\\ \frac{\partial z}{\partial u} + \frac{\partial B}{\partial v}z - \frac{1}{e^{A_0}}\int_0^2 \left(\frac{\partial B}{\partial v}z\right)(0,v)dz &= 0 \quad \text{on} \quad \Gamma_0. &(28) \end{aligned}$$

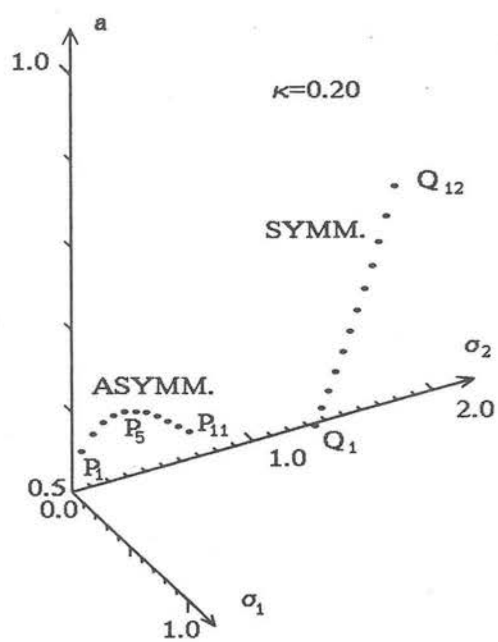

Figure 2. Bifurcation diagram for $\kappa = 0.2$.

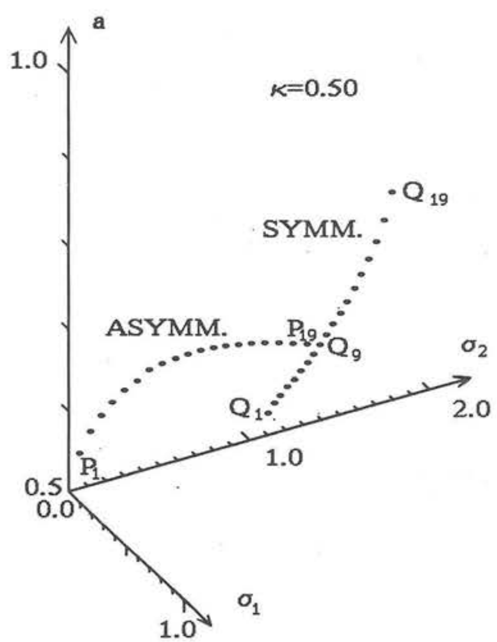

Figure 3. Bifurcation diagram for $\kappa = 0.5$.

4. Numerical results

Numerical computations are carried out by discretizing Problem 4' and using the finite difference method. The 20×20 grid points are used here. The power method is used to obtain the smallest eigenvalue. Numerical results are shown at Tables 1 and 2. The mark "–" in Tables means that the iteration does not end in finite time. However, to see behaviour of the smallest eigenvalue in iterations, maximum and minimum values are shown when iteration does not converge.

Bifurcation diagrams show that bifurcation occurs near the points P_5 for $\kappa = 0.2$, P_{19} and Q_9 for $\kappa = 0.5$ (Figs. 2 and 3, [9]). Tables 1 and 2 show that near these points the smallest eigenvalue λ_0 becomes close to zero. This means that our method is of practical value.

5. Conclusions

Asymm. ($\kappa = 0.2$)					Symm. ($\kappa = 0.2$)		
	a	λ_0	Max.	Min.		a	λ_0
P_1	0.546	43.36			Q_1	0.564	65.43
	0.565	43.49				0.591	57.30
	0.577	—	92.43	41.75		0.619	56.66
	0.583	34.06				0.648	55.95
P_5	0.585	24.76				0.677	76.17
	0.583	15.49				0.708	76.49
	0.579	7.76				0.739	76.18
	0.573	1.96				0.771	75.59
	0.566	−2.11				0.804	75.13
	0.558	−4.82				0.838	75.06
P_{11}	0.549	−6.55				0.873	75.48
					Q_{12}	0.913	72.54

Table 1. The smallest eigenvalue for $\kappa = 0.2$.

A method for finding bifurcation points in a class of free boundary problems is introduced. In our method bifurcation points are found according to the zero eigenvalue of a problem which is obtained by linearizing the problem. In order to verify our method, it has been applied to a two-dimensional free boundary

Asymm. ($\kappa = 0.5$)			Symm. ($\kappa = 0.5$)				
	a	λ_0		a	λ_0	Max.	Min.
P_1	0.550	3.86	Q_1	0.540	—	35.96	14.58
	0.573	3.87		0.559	—	31.55	14.33
	0.591	3.79		0.575	—	29.36	13.89
	0.606	3.63		0.591	—	27.47	13.95
	0.619	3.36		0.606	25.08		
	0.630	2.98		0.622	—	24.16	14.13
	0.639	2.50		0.638	—	22.83	14.00
	0.647	1.96		0.655	—	21.71	−3.01
	0.653	1.39	Q_9	0.672	−1.83		
	0.658	0.83		0.690	−0.68		
	0.663	0.30		0.709	0.34		
	0.666	−0.17		0.728	1.25		
	0.668	−0.58		0.749	2.04		
	0.670	−0.92		0.770	2.73		
	0.671	−1.19		0.792	13.26		
	0.672	−1.40		0.816	12.52		
	0.673	−1.55		0.842	18.11		
	0.673	−1.66		0.871	18.09		
P_{19}	0.673	−1.72		0.909	18.01		

Table 2. The smallest eigenvalue for $\kappa = 0.5$.

problem. Numerical results show that our method is of practical use. Under the limit of the present developement in the linearization techniques, our method is applied to two-dimensional free boundary problem. However, due to its generality, the method may be applied to higher-dimensional problems in the near future.

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Metoda znajdowania punktow bifurkacji

W artukule wprowadzono metodę znajdowania punktów bifurkacji dla zadań ze swobodną granicą. W metodzie tej punkt na krzywej rozwiązań jest traktowany jak punkt bifurkacji jeśli najmniejsza wartość własna zagadnienia zlinearyzowanego jest równa zeru. W celu weryfikacji zaproponowanej metody przeprowadzono odpowiednie obliczenia numeryczne.

Метод нахождения точек бифуркации

В статье представлен метод нахождения точек бифуркации для задач со свободным пределом. В этом методе точка на кривой решений воспринимается как точка бифуркации, если наименьшее собственное значение линеаризованной задачи равно нулю. С целью проверки предложенного метода проведены соответствующие численные эксперименты.