Control and Cybernetics

VOL. **20** (1991) No. 1

Explicit solutions of coupled Riccati equations occuring in Nash games — the open–loop case.

by

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In this paper we present explicit closed form solutions of systems of coupled Riccati matrix differential equations appearing in open–loop Nash games. By means of appropriate algebraic transformations the problem is decoupled so that an explicit solution of the problem is available.

1. Introduction

When noncooperative problems are tackled, a game theoretic approach is necessary: each control agent (decision maker or player) tries to optimize his own cost function which conflicts more or less with the ones of the others. An equilibrium solution must be sought, and the Nash strategy is a normal choice. In this case, a player cannot improve his playoff by deviating unilaterally from his Nash strategy. Due to this noncooperation, the optimization problems of various players are strongly coupled and necessary conditions for Nash strategy lead to complex two-point boundary value problems (TPBVP). On the other hand when all the decision makers cooperate and associated TPBVP becomes fairly easy to solve.

Consider a two-player linear quadratic diffrential game defined by

$$x' = Ax + B_1 u_1 + B_2 u_2; \quad x(0) = x_0 \tag{1.1}$$

with the cost functionals associated with the players

$$\begin{aligned} J_i = \tfrac{1}{2}\left\{x_f^T K_{if} x_f + \int_0^{t_f} (x^T Q_1 x + u_1^T R_{i1} u_1 + u_2^T R_{i2} u_2) dt\right\}; \\ x_f = x(t_f) \end{aligned} \tag{1.2}$$

where all matrices are $n \times n$ symmetric with R_{ii}, $i = 1, 2$, positive definite. It is well known that the open-loop Nash controls must satisfy [7]:

$$u_i = -R_{ii}^{-1} B_i^T \Psi_i; \; \Psi_i' = -Q_i x - A^T \Psi_i, \; \Psi_i(t_f) = K_{if} x_f, \quad i = 1, 2 \tag{1.3}$$

Where Ψ_i is the costate vector associated with player "i". When the transformation $\Psi_i = K_i X$, is introduced, for $i = 1, 2$, the open loop Nash strategy (u_1^*, u_2^*) is given by

$$u_i^* = -R_{ii}^{-1} B_i^T K_i(t) \Phi(t, 0) x_0, \quad i = 1, 2 \tag{1.4}$$

where $K_1(t)$ and $K_2(t)$ are the solutions of the coupled Riccati matrix equations

$$\begin{aligned} K_1' &= -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2, \; K_1(t_f) = K_{1f} \\ K_2' &= -A^T K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1, \; K_2(t_f) = K_{2f} \end{aligned} \tag{1.5}$$

with

$$S_i = B_i R_{ii}^{-1} B_i^T, \quad i = 1, 2 \tag{1.6}$$

and $\Phi(t, 0)$ is the system's transition matrix satisfying

$$\Phi'(t, 0) = (A - S_1 K_1 - S_2 K_2)\Phi(t, 0), \quad \Phi(t, t) = I \tag{1.7}$$

Note that matrix R_{12} nad R_{21} do not appear in the necessary conditions. This is due to the fact that under the open-loop strategy assumptions, each decision maker optimizes his criterion knowing that $\partial u_1 / \partial x = \partial u_2 / \partial x = 0$.

The solution of system (1.5) is generally difficult to obtain due to the permanent coupling between the player's strategies. Numerical techniques are widely used to obtain an approximate or series solution [3]. An iterative algorithm for solving coupled Riccati systems of the type (1.5) have been given in [4]. For the case $Q_2 = \alpha Q_1$, where α is a scalar, an analytic solution of system (1.5) was pointed out in [1].

For convenience, the necessary conditions to be satisfied, (1.1), (1.3), are rewritten in the matrix form as

$$\begin{bmatrix} x' \\ \Psi_1' \\ \Psi_2' \end{bmatrix} = \begin{bmatrix} A & -S_1 & -S_2 \\ -Q_1 & -A^T & 0 \\ -Q_2 & 0 & -A^T \end{bmatrix} \begin{bmatrix} x \\ \Psi_1 \\ \Psi_2 \end{bmatrix} = M \begin{bmatrix} x \\ \Psi_1 \\ \Psi_2 \end{bmatrix} \tag{1.8}$$

$$x(0) = x_0, \quad \Psi_1(t_f) = K_{1f} x_f, \quad \Psi_2(t_f) = K_{2f} x_f, \tag{1.9}$$

Now, let us introduce the change of basis defined by

$$\begin{bmatrix} x \\ \Psi_1 \\ \Psi_2 \end{bmatrix} = T \begin{bmatrix} x \\ \Psi_1 \\ w \end{bmatrix}; \quad T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & L & I \end{bmatrix} \tag{1.10}$$

for an approximate matrix L in $\mathbb{R}^{n\times n}$ to be determined. Thus, problem (1.8)–(1.9) is equivalent to the following one

$$\begin{bmatrix} x' \\ \Psi_1' \\ w' \end{bmatrix} = \begin{bmatrix} A & -S_1 - S_2 L & -S_2 \\ -Q_1 & -A^T & 0 \\ LQ_1 - Q_2 & LA^T - A^T L & -A^T \end{bmatrix} \begin{bmatrix} x \\ \Psi_1 \\ w \end{bmatrix} \tag{1.11}$$
$$x(0) = x_0, \quad \Psi_1(t_f) = K_{1f} x_f, \quad w(t_f) = (K_{2f} - LK_{1f}) x_f$$

The purpose of this transformation is to find under what conditions the players' optimization problem can be decoupled. In fact, note that if L satisfies the system

$$LQ_1 = Q_2, \quad LA^T = A^T L \tag{1.12}$$

the matrix $T^{-1}MT$ is reduced to a block triangular form and the costate vectors Ψ_1 and w are coupled only via the terminal condition (1.11).

2. Explicit solutions of coupled Riccati differential systems

Note that when matrices Q_1 and Q_2 are proportional, i.e. $Q_2 = \alpha Q_1$, for some scalar α, then taking $L = \alpha I$, one gets solutions of systems (1.12). In order to characterize the existence of solutions for the algebraic system (1.12), we recall the concept of tensor product of matrices. If A, B are matrices in $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{k\times s}$, respectively, then the tensor product of A and B, denoted $A\otimes B$, is defined as the partitioned matrix

$$A\otimes B=\begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ \vdots & & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}$$

If $A\in\mathbb{R}^{m\times n}$, we denote

$$A_{.j}=\begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj}\end{bmatrix},\quad 1\le j\le n;\qquad \text{vec } M=\begin{bmatrix} M_{.1} \\ \vdots \\ M_{.n}\end{bmatrix}$$

If M, N and P are matrices of suitable dimensions, then using the column lemma [5, p.410], we get

$$\text{vec }(MNP)=(P^T\otimes M)\text{vec } N \tag{2.1}$$

Taking into account (2.1), the algebraic system (1.12) may be rewritten in the form

$$C\text{vec } L=\text{vec }[0,Q_2] \tag{2.2}$$

where

$$C=\begin{bmatrix} I\otimes A^T-A\otimes I \\ Q_1^T\otimes I\end{bmatrix} \tag{2.3}$$

If we denote by C^+ the Moore–Penrose pseudoinverse of C, then from theorem 2.3.2 of [6], system (2.2) is compatible, if and only if

$$CC^+\text{vec }[0,Q_2]=\text{vec }[0,Q_2] \tag{2.4}$$

If condition (2.4) is satisfied, then the general solution of (2.2) is given by the expression

$$\text{vec } L=C^+\text{vec }[0,Q_2]+(I-C^+C)Z \tag{2.5}$$

where I denotes the identity matrix in $\mathbb{R}^{n^2\times h^2}$ and Z is an arbitrary vector in $\mathbb{R}^{n^2}$. Effective methods for computing C^+ may be found in [2, p. 12].

If we assume the existence of a solution L of system (1.12) then from (1.11) it follows that

$$\begin{bmatrix} x' \\ \Psi_1' \\ w' \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{V}} & -S_2 \\ & & 0 \\ \hline 0 & 0 & -A^T \end{array}\right] \begin{bmatrix} x \\ \Psi_1 \\ w \end{bmatrix}; \quad \begin{array}{l} x(0) = x_0, \\ \Psi_1(t_f) = K_{1f} x_f \\ w(t_f) = (K_{2f} - LK_{1f}) x_f \end{array} \tag{2.6}$$

where

$$V = \begin{bmatrix} A & -S_1 - S_2 L \\ -Q_1 & -A^T \end{bmatrix}$$

Let us consider the change $t = t(s) = t_f - s$, $0 \le s \le t_f$, and let

$$\begin{array}{l} \hat{x}(s) = x(t_f - s) = x(t), \ \hat{\Psi}_1'(s) = \Psi_1(t_f - s) = \Psi_1(t), \\ \hat{w}(s) = w(t_f - s) = w(t) \end{array} \tag{2.7}$$

Hence, problem (2.6) may be rewritten in the form

$$\begin{array}{l} (d/ds) \begin{bmatrix} \hat{x} \\ \hat{\Psi}_1 \\ \hat{w} \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{V}} & S_2 \\ & & 0 \\ \hline 0 & 0 & A^T \end{array}\right] \begin{bmatrix} \hat{x} \\ \hat{\Psi}_1 \\ \hat{w} \end{bmatrix} \\ \hat{x}(t_f) = x_0, \quad \hat{\Psi}_1(o) = K_{1f} x_f = K_{1f} \hat{x}(0), \\ \hat{w}(0) = (K_{2f} - LK_{1f}) \hat{x}(0) \end{array} \tag{2.8}$$

Solving (2.8) we obtain

$$\hat{w}(s) = \exp(sA^T) \hat{w}(0) \tag{2.9}$$

$$\begin{aligned} \begin{bmatrix} \hat{x}(s) \\ \hat{\Psi}_1(s) \end{bmatrix} &= \exp(-sV) \left\{ \begin{bmatrix} \hat{x}(0) \\ \hat{\Psi}_1(0) \end{bmatrix} + \right. \\ & \left. + \int_0^s \exp(uV) \begin{bmatrix} S_2 \\ 0 \end{bmatrix} \exp(uA^T) \hat{w}(0) du \right\} \end{aligned} \tag{2.10}$$

From (2.8), (2.10) we have

$$\begin{bmatrix} \hat{x}(0) \\ \hat{\Psi}_1(0) \end{bmatrix} = \begin{bmatrix} I \\ K_1 f \end{bmatrix} \hat{x}(0) \tag{2.11}$$

From this and (2.8), (2.10) it follows that

$$\begin{bmatrix} \hat{x}(s) \\ \hat{\Psi}_1(s) \end{bmatrix} = G(s) \hat{x}(0) \tag{2.12}$$

where

$$\begin{aligned} G(s) &= \exp(-sV)\left\{\begin{bmatrix} I \\ K_{1f} \end{bmatrix} + \right. \\ &\left. + \int_0^s \exp(uV)\begin{bmatrix} S_2 \\ 0 \end{bmatrix}\exp(uA^T)du(K_{2f} - LK_{1f})\right\} \end{aligned} \tag{2.13}$$

Thus, we have

$$\hat{x}(s) = [I, 0]G(s)\hat{x}(0); \quad \hat{\Psi}_1(s) = [0, I]G(s)\hat{x}(0) \tag{2.14}$$

Note that $[I, 0]G(0) = I$, and from the continuity condition of G there exists an interval $0 \leq s \leq \delta$, such that

$$[I, 0]\ G(s) \quad \text{is invertible for all} \quad s \in [0, \delta] \tag{2.15}$$

From (2.14) and (2.15) we obtain

$$\hat{x}(0) = \{[I, 0]G(s)\}^{-1}\hat{x}(s);\ \hat{\Psi}_1(s) = \{[0, I]G(s)\}\{[I, 0]G(s)\}^{-1}\hat{x}(s) \tag{2.16}$$

for $0 \leq s \leq \delta$. Now, from (1.10) and (2.7), it follows that

$$\hat{\Psi}_2(s) = L\hat{\Psi}_1(s) + \hat{w}(s); \quad \Psi_2(t) = L\Psi_1(t) + w(t) \tag{2.17}$$

From this and (2.9), (2.16) we obtain

$$\hat{\Psi}_2(s) = \{\exp(sA^T)[K_{2f} - LK_{1f}] + L[0, I]G(s)\}\{[I, 0]G(s)\}^{-1}\hat{x}(s) \tag{2.18}$$

for $0 \leq s \leq \delta$. From (2.7) and the relationship $\Psi_i(t) = K_i(t)x(t)$, for $i = 1, 2$, it follows that $\hat{\Psi}_i(s) = \hat{K}_i(s)\hat{x}(s)$, $i = 1, 2$. Taking also into acount (2.8), (2.16), we have

$$K_1(t) = [0, I]G(t_f - t)\{[I, 0]G(t_f - t)\}^{-1} \tag{2.19}$$

and

$$\begin{aligned} K_2(t) &= \{\exp(A^T(t_f - t))[K_{2f} - LK_{1f}] + \\ &\quad + L[0, I]G(t_f - t)\}\{[I, 0]G(t_f - t)\}^{-1} \end{aligned} \tag{2.20}$$

for all $t \in [t_f - \delta, t_f]$, where G is defined by (2.13). Thus the following result has been proved:

Theorem. *Let us assume that matrices A and Q_1 satisfy the condition (2.4) where C is defined by (2.3), and let L be a solution of the algebraic system (1.12). Then, there exists a positive number δ such that on the interval $[t_f - \delta, t_f]$, the unique solution of the coupled Riccati system (1.5) is given by (2.19), (2.20).*

Remark. *Note that the case $Q_2 = \alpha Q_1$, where α is a scalar, is a particular case of the previous theorem taking $L = \alpha I$. It is important to note that from (2.19) and (2.20), we have the following relationship between $K_1(t)$ and $K_2(t)$:*

$$K_2(t) = LK_1(t) + \exp(A^T(t_f - t))\{K_{2f} - LK_{1f}\}\{[I, 0]G(t_f - t)\}^{-1}$$

and as the function $\{[I, 0]G(t_f - t)\}^{-1}$ is involved in the computation of the K_1, the computational cost is reduced because $K_2(t)$ is expressed in terms of $K_1(t)$. Finally we recall that efficient methods for computing exponentials of matrices and integrals involving them that appear in the expression of $G(s)$, may be found in [8]. These procedures are extremely easy to implement and yield an estimation of the approximation error.

Acknowledgements

This research has been supported in part by the D. G. I. C. Y. T. Grant PS87-0064.

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Jawne rozwiązania sprzężonych równań Riccatiego występujących w grach Nasha z otwartą pętlą

W pracy przedstawiono jawną postać rozwiązań dla układów sprzeżonych macierzowych równań rożniczkowalnych Riccatiego pojawiających się w grach Nasha z otwartą pętlą. Za pomocą odpowiednich przekształceń algebraicznych doprowadzono do rozsprzęgnietych równań, co pozwala otrzymać jawną postać rozwiązania zadania.

Явное решение сопряженных уравнений Риккати, выступающих в играх Нэша с разомкнутой петлей

В работе представлен явный вид решений для систем сопряженных матричных дифференцируемых уравнений Риккати, появляющихся в играх Нэша с разомкнутой петлей. С помощью соответствующих алгебраических преобразований получены распряженные уравнения, что позволяет достичь решение задачи в явном виде.