Control
and Cybernetics
VOL. 20 (1991) No. 1

Communication games.

by

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The paper considers the class of n-person games with side payments in which cooperation among palyers is communication-wise limited through imposition of certain graph structures. It is shown that there is a unique allocation rule, both efficient and fair, essentially given by the Shapley value of a restricted game. Moreover, if the game is superadditive then the allocation rule is stable.

Introduction

In the analysis of n-person games with side payments it is often assumed that all players will cooperate with each other and therefore the formation of the grand coalition is taken for granted. This of course is not the case in many practical situations where only partial cooperation may be sought by some of the players. In several cicumstances this may be due to a lack of communication among several participants. These situations were first considered by Myerson [7] who studied in particular the problem of how the reward (or the cost) resulting from the corresponding games should depend on which players cooperate with each other. As far as this article is concerned, the method of research has been to impose various communication graphs on groups of players in order to describe the communication properties and the economic possibilities. By borrowing from the theory of cooperative games with coalition structures as established by Aumann and Drèze [1], a unique allocation rule which is both efficient and fair can be derived and shown to be essentially given by Shapley value of a restricted game. Furthermore, it turns out that if the game is superadditive, then the proposed allocation rule is stable, that is to say two players can always benefit from reaching bilateral agreements.

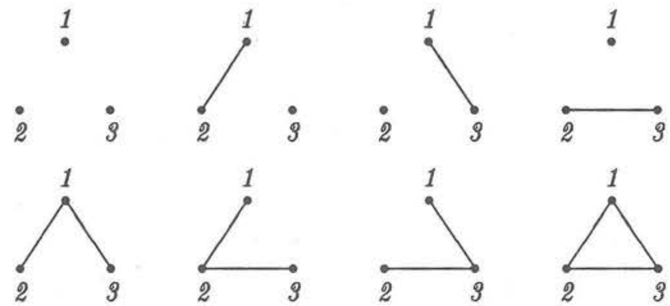
A more general problem dealing with conference structures and fair allocation rules has been investigated by Myerson [8]. His contribution extends the results of this paper to games without side payments and generalizes the fair allocation rule discussed here.

Most of the definitions and results pertaining to the n-person cooperative games with transferable utility are available in [9], and for mathematical models in social sciences a standard reference is the book of Kemeny and Snell [5].

1. Communication graphs

Let $N = \{1, 2, \ldots, n\}$ be a non-empty finite set, referred to as the set of players. An indirect graph A on N is a subset of unordered (and unequal) pairs of elements in N: we will call these pairs arcs. If $\{i, j\} \in A$, then the interpretation is that the players i and j can communicate directly with each other. Furthermore, i and j can communicate in N if there exists a chain of arcs in A of the form $\{i_1, i_2\}$, $\{i_2, i_3\}$, $\{i_3, i_4\}$, $\ldots$, $\{i_{k-1}, i_k\}$ with $i = i_1$ and $j = i_k$. We say that i and j are connected in N with respect to A if either $i = j$ or i and j can communicate in N. Graphs of this nature will be termed communication graphs. It is obvious that the number of distinct communication graphs on N is equal to $2^{n(n-1)/2}$.

Example 1 *Let $N = \{1,2,3\}$. Then the eight possible communication graphs are*



Recall that a collection $\{C_1, C_2, \ldots, C_k\}$ of subsets of N is said to be a partition for N if

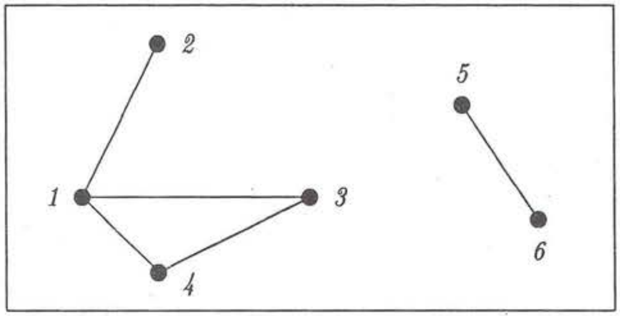
$$\bigcup_{i=1}^{n} C_i = N \quad \text{and} \quad C_i \cap C_j = \emptyset \quad \text{for all} \quad i,j \in N,\ i \neq j.$$

A variety of problems related to communication networks have been investigated by Cohen [2]. In particular, the notion of structural centrality has been considered by MacKenzie [6], and for a game theoretic approach the reader can consult the work of Grofman and Owen [3].

Any subset of N is called a coalition and denoted by S. Such a set is said to be connected if each pair in S is connected in N. S is called a component of N if it is connected and if for all connected sets T with $T \supset S$ we have $T = S$.

The components of N form a partition of N and will be denoted by N/A. One can speak of communication within the coalition S provided only chains as before are used with $i_1,\ i_2, \ldots, i_k \in S$. We will write S/A to indicate the components in S with respect to the communication within S. In other words, S/A can be interpreted as the collection of smaller coalitions into which S would break up if players could only communicate along the arcs in A. For more on coalition formation, see Shenoy [11].

Example 2 *Let $N = \{1,2,3,4,5,6\}$ and suppose that the arc set is $A = \{\{1,2\},\{1,3\},\{1,4\},\{3,4\},\{5,6\}\}$. Then the communication graph is represented by*



Now consider the coalitions $L = \{2,3,4\}$ *and* $M = \{5,6\}$. *Then* $L/A = \{\{2\},\{3,4\}\}$ *and* $M/A = \{\{5,6\}\}$. *Furthermore,* $N/A = \{\{1,2,3,4\},\{5,6\}\}$. *Note that* L *is connected in* N *but not within* L.

Remark 1 *For a communication graph, the partition* N/A *is the natural way to describe the coalition structure. In fact, although two players may not have a direct communication between themselves, they can still communicate through an agreeable mutual third party.*

2. Communication games

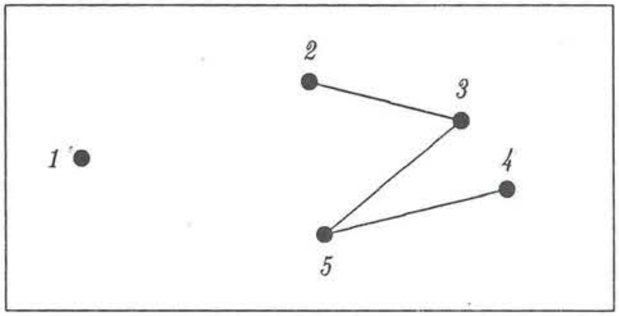
Let $< N, v >$ be a cooperative game in characteristic function form. Then v maps each coalition $S \in 2^N$ into the real number $v(S)$ to be interpreted as the reward the numbers of S would have to divide among themselves if they were to communicate with each other directly or indirectly.

Definition 2.1 *The communication game corresponding to the situation described above is* $< N, v_A >$ *where for all* $S \in 2^N$, *the characteristic function is*

$$v_A(S) = \sum_{T \in S/A} v(T).$$

Note that $v_A(\{i\}) = v(i)$ for any arc set A for all $i \in N$. Also, $v_A(S) = v(S)$ if v is additive, $v_A(S) \leq v(S)$ if v is superadditive, and $v_A(S) = v(S)$ when S is connected. This implies that v_A can be viewed as the characteristic function of the game where the players are restricted to communicate only along the arcs in A.

Example 1 *Let* $< N, v >$ *be the 5-person game with* $v(N) = 5$, $v(2,3,4,5) = 2$, $v(1,2,3) = 2$ *and* $v(S) = 0$ *otherwise. If* $A = \{\{2,3\},\{3,5\},\{4,5\}\}$ *then the communication graph is*



and for corresponding communication game $< N, v_A >$

$$\begin{aligned} v_A(N) &= v(2,3,4,5) + v(1) = 2 \\ v_A(\{2,3,4,5\}) &= v(2,3,4,5) = 2 \\ v_A(S) &= 0 \quad \textit{for all other coalitions.} \end{aligned}$$

Example 2 *For the communication situation of Example 2 in Section 1, suppose that* $v(S) = |S| - 1$ *for all* $S \in 2^N \setminus \{\emptyset\}$. *Then the corresponding communication game is defined by*

$$\begin{aligned} v_A(L) &= v_A(\{2,3,4\}) = v(2) + v(3,4) = 1 \\ v_A(M) &= v_A(\{5,6\}) = v(5,6) = 1 \\ v_A(\{3,4,5,6\}) &= v_A(3,4) + v(5,6) = 2 \end{aligned}$$

and so on. In particular

$$v_A(N) = v(1,2,3,4) + v(5,6) = 3 + 1 = 4$$

i.e. less then $v(N) = 5$.

Let G^n denote the class of all n-person games.

Definition 2.2 *For each* $T \in 2^N \setminus \{\emptyset\}$ *the* *T-unaminity game* $< N, u_T >$ *is defined by*

$$u_T(S) = \begin{cases} 1 & \textit{if } T \subset S \\ 0 & \textit{otherwise} \end{cases}$$

(In particular, for the N–unanimity game, $u_N(S) = 0$ if $S \neq N$ and $u_N(N) = 1$). Let us point out that this game has only one sensible solution: only (and all) the players in T contribute something to a coalition, while the others do not. Furthermore, the players in T are undistinguishable. Hence, each of the members in T should receive the amount $\frac{1}{|T|}$ and the other players nothing, which is how much they are capable of obtaining by acting on their own.

To show that G^n is a $(2^n - 1)$–dimensional linear space, is enough to prove that the set $\{u_T | T \in 2^N \setminus \{\emptyset\}\}$ is a basis for G^n. In fact, this set of games is a linearly independent set. Also, each game $v \in G^n$ can be written as

$$v = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T u_T \quad \text{where} \quad c_T = \sum_{S: S \subset T} (-1)^{|T|-|S|} v(S).$$

We recall that core of a T–unaminity game $< N, u_T >$ is given by the set $C(u_T) = \text{conv}\{e^i | i \in T\}$, i.e. the convex hull generated by the vectors e^i.

Finally, players i and j are said to be symmetric in a game $< N, v >$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \in N \setminus \{i, j\}$. Now, it is not difficult to verify that in a T–unaminity game, players $i, j \in T$ are symmetric and so are players $i, j \in N \setminus T$.

Consider now a communication situation with N players and arc set A fixed. Let $L_A : G^n \to G^n$ be the map defined by $L_A(v) = v_A$. Then it is easy to see that L_A is linear. In fact, for $v, w \in G^n$ and $\lambda, \mu \in \Re$, $L_A(\lambda v + \mu w) = (\lambda v + \mu w)_A$ and since by definition,

$$\begin{aligned} (\lambda v + \mu w)_A(S) &= \sum_{T \in S/A} (\lambda v + \mu w)(T) \\ &= \lambda \sum_{T \in S/A} v(T) + \mu \sum_{T \in S/A} w(T) \\ &= \lambda v_A(S) + \mu w_A(S) \end{aligned}$$

for all $S \in 2^N$, it follows that $L_A(\lambda v + \mu w) = \lambda v_A + \mu w_A = \lambda L_A(v) + \mu L_A(w)$.

Suppose now that v is superadditive. Then by defonition,

$$v(S_1 \cup S_2) \geq v(S_1) + v(S_2) \quad \text{for} \quad S_1, S_2 \in 2^N, S_1 \cap S_2 = \emptyset$$

and

$$\begin{aligned} &L_A(v)(S_1) + L_A(v)(S_2) = v_A(S_1) + v_A(S_2) \\ &\quad = \sum_{T \in S_1/A} v(T) + \sum_{T \in S_2/A} v(T) \leq \sum_{T \in S_1 \cup S_2/A} v(T) = v(S_1 \cup S_2) \end{aligned}$$

shows that also v_A is superadditive. In particular, if v is additive, that is $v(S_1 \cup S_2) = v(S_1) + v(S_2)$ for all $S_1, S_2 \in 2^N$ with $S_1 \cap S_2 = \emptyset$, then

$$L_A(v)(S) = \sum_{T \in S/A} v(T) = \sum_{T \in S/A} \sum_{i \in T} v(i) = \sum_{i \in S} v(i) = v(S)$$

and therefore $L_A(v) = v$.

Finally, for the T–unanimity communication game $< N, (u_T)_A >$ one can prove that

$$(u_T)_A(S) = \begin{cases} 1 & \text{if there is a connected set } K \text{ such that } T \subset K \subset S \\ 0 & \text{otherwise} \end{cases}$$

3. The Shapley value

Let $\Phi(v) = (\Phi_1(v), \Phi_2(v), \ldots, \Phi_n(v))$ be an imputation of the game in characteristic function form $< N, v >$. To turn this vector in $\Re^n$ into a solution for the game, Shapley [10] has imposed four axioms which read as follows.

Axiom 1 (Symmetry) *If $\pi: N \to N$ is a permutation of the player set and if the characteristic function w is defined on N by $w(S) = v(\pi S)$. Then for all $i \in N$, $\Phi_i(w) = \Phi_{\pi(i)}(v)$.*

This condition states that the value assigned to a player is independent of the labelling. In particular, it implies that in a symmetric game, all players are assigned equal value.

Axiom 2 (Efficiency) $\sum_{i=1}^{n} \Phi_i(v) = v(N)$.

This condition, known as Pareto optimality or group rationality, ensures that the value $\Phi(v)$ is indeed an imputation for the game.

Axiom 3 (Dummy Property) *If $v(S - \{i\}) = v(S)$ for all S, then $\Phi_i(v) = 0$.*

In other words, if a player adds nothing to any coalition, then his value is zero, and such a player is called a dummy.

Axiom 4 (Additivity) *If v and v' are two characteristic functions defined on the same set of players N as $w = v + v'$ is the characteristic function on N, then $\Phi(w) = \Phi(v) + \Phi(v')$.*

It is easy to see that the function defined by $(v+v')(S) = v(S) + v'(S)$ for all $S \in 2^N$ is also a characteristic function defined on 2^N. Furthermore, if a player participates simultaneously in two games, then one can think of him as taking part in a single game with characteristic function $v + v'$ and expecting to gain the sum of the rewards for the two separate games. On the other hand, it is hard to imagine that a player taking part in the game $v + v'$ will behave as if he were playing only one of them.

The Shapley value is the unique vector $\Phi(v)$ which satisfies the axioms above, and the components of it can be computed according to the following formula:

$$\Phi_i(v) = \sum_{S:S \ni i} \frac{(n-|S|)!(|S|-1)!}{n!}[v(S) - v(S - \{i\})].$$

An alternative expression is given by

$$\Phi_i(v) = \frac{1}{n} \sum_{S:S \ni i} \frac{v(S) - v(S - \{i\})}{\binom{n-1}{|S|-1}}.$$

Here, as in the previous formula, the quantity $v(S) - v(S - \{i\})$ represents the contribution made to the coalition $S - \{i\}$ by the new joining member i. One plausible interpretation goes as follows. Suppose that the grand coalition is formed gradually (i.e. by one player at the time) and randomly (i.e. according to a device for which each player has the same probability $\frac{1}{n}$ of being selected). Then the Shapley value can be viewed as the expected value of the marginal contribution made by a player to the coalition already formed before him. On the other hand, note that for each S, the number $(n - |S|)!(|S| - 1)!$ gives the number of permutations of N in which elements of $S - \{i\}$ come first (in some order), then i, and finally the rest of the elements (in some order) join in.

Examples 1a. *The Shapley value for an additive game is equal to*

$$\Phi(v) = (v(1), v(2), \ldots, v(n)).$$

1b. *For a 2-person game, the Shapley value is given by* $\Phi(v) = (\Phi_1(v), \Phi_2(v))$ *where*

$$\Phi_1(v) = v(1) + \tfrac{v(N)-v(1)-v(2)}{2} \text{ and } \Phi_2(v) = v(2) + \tfrac{v(N)-v(1)-v(2)}{2}.$$

1c. *Let* $< N, v >$ *be the 3-person game with* $v(\emptyset) = 0,\ v(1) = 4,\ v(2) = 6,\ v(3) = 3,\ v(1,2) = 12,\ v(1,3) = 14,\ v(2,3) = 16,$ *and* $v(1,2,3) = 20.$

The the Shapley solution for the first player is

$$\begin{aligned}\Phi_1(v) &= \frac{1}{3}\left[\frac{v(1)-v(\emptyset)}{\binom{2}{0}} + \frac{v(1,2)-v(2)}{\binom{2}{1}} + \frac{v(1,3)-v(3)}{\binom{2}{1}} + \frac{v(1,2,3)-v(2,3)}{\binom{2}{2}}\right] \\ &= \frac{1}{3}\left[4 + \frac{12-6}{2} + \frac{14-3}{2} + (20-16)\right] = \frac{11}{2}\end{aligned}$$

and analogous calculation lead to $\Phi_2(v) = \frac{15}{2}$, $\Phi_3(v) = 7$ *for the second and the third player respectively.*

Lemma 3.1 *Let* $< N, v >$ *be an n-personal game with the property that* $v(S) = 0$ *if* $1 \notin S$ *or* $2 \notin S$*. Then* $\Phi_1(v) = \Phi_2(v)$.

PROOF:

$$\begin{aligned}\Phi_1(v) &= \sum_{S:S\not\ni 1} \frac{|S|! - (n-|S|-1)!}{n}[(S\cup\{1\}) - v(S)] \\ &= \sum_{\substack{S:S\not\ni 1\\ S\not\ni 2}} \gamma(S,n)[(S\cup\{1\}) - v(S)] + \sum_{\substack{S:S\not\ni 1\\ S\ni 2}} \gamma(S,n)[(S\cup\{1\}) - v(S)] \\ &= 0 + \sum_{\substack{S:S\not\ni 1\\ S\ni 2}} \gamma(S,n)[(S\cup\{1\})] \\ &= \sum_{\substack{S:S\not\ni 1\\ S\ni 2}} \gamma(S,n)[(S\cup\{2\})] \\ &= \sum_{\substack{S:S\not\ni 2\\ S\ni 1}} \gamma(S,n)[(S\cup\{2\}) - v(S)] + \sum_{\substack{S:S\not\ni 2\\ S\ni 1}} \gamma(S,n)[(S\cup\{2\}) - v(S)] \\ &= \Phi_2(v).\end{aligned}$$

■

Lemma 3.2 *For any game* $< N, v >$, $\Phi(-v) = \Phi(v)$.

PROOF:

$$\begin{aligned}\Phi_i(v) &= \sum_{S:S\not\ni i} \frac{|S|!(n-|S|-1)!}{n}[-v(S\cup\{i\}) + v(S)] \\ &= -\sum_{S:S\not\ni i} \frac{|S|!(n-|S|-1)!}{n}[v(S\cup\{i\}) - v(S)]\end{aligned}$$

$\Phi_i(v)$.

■

We return now to communication situation and the corresponding communication games. To see how the outcomes of a communication game depend on the cooperation structure, consider the next

Example 2 *Let* $< N, v >$ *be the 3-person game where* $V(1) = 1$, $V(2) = V(3) = 0$, $V(1,2) = 2$, $V(1,3) = 4$, $V(2,3) = 8$, $V(1,2,3) = 13$. *The eight possible communication graphs are of course those listed in Example 1 of the Section 1. Note that* $v_A(\{1\}) = 1$ *and* $v_A(\{2\}) = v_A(\{3\}) = 0$ *for all* A. *The other values of the characteristic functions for the corresponding communication games are reported in the table below for* $|S| \geq 2$ *together with the Shapley value* $\Phi(v_A)$. *Observe that for the originall game the Shapley value was* $\Phi(v) = (\frac{6}{2}, \frac{9}{2}, \frac{11}{2})$.

A	$v_A(\{1,2\})$	$v_A(\{1,3\})$	$v_A(\{2,3\})$	$v_A(\{1,2,3\})$	$\Phi(v_A)$
$\emptyset$	1	1	0	1	$(1,0,0)$
$\{\{1,2\}\}$	2	1	0	2	$(\frac{3}{2}, \frac{1}{2}, 0)$
$\{\{1,3\}\}$	1	4	0	4	$(\frac{5}{2}, 0, \frac{3}{2})$
$\{\{2,3\}\}$	1	1	8	9	$(1,4,4)$
$\{\{1,2\},\{1,3\}\}$	2	4	0	13	$(\frac{17}{3}, \frac{19}{6}, \frac{25}{6})$
$\{\{1,2\},\{2,3\}\}$	2	1	8	13	$(\frac{5}{2}, \frac{11}{2}, 5)$
$\{\{1,3\},\{2,3\}\}$	1	4	8	13	$(\frac{17}{6}, \frac{13}{3}, \frac{35}{6})$
$\{\{1,2\},\{1,3\},\{2,3\}\}$	2	4	8	13	$(3, \frac{9}{2}, \frac{11}{2})$

The bargaining solutions proposed by Harsanyi [4] for n-person games are a generalization of the Shapley value for games without side payments.

4. Coalition Structures

In cooperative games with n players the grand coalition does not necessarily form. When only smaller coalitions may form, it seems reasonable to consider the problem of how the outcome of a game should depend on which players communicate with each other. When for some reason the player set splits up into disjoint groups $C_1, C_2, \ldots, C_k$ which represent a partition of N, we speak of a coalition structure $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$. Games with coalition structure have been studied by Aumann and Drèze [1] as well as by Wallmeier [13].

Example 1 *Let* $< N, V >$ *be the 3-person (additive) game with* $v(S) = 0$ *if* $|S| = 1$, $v(S) = 100$ *if* $|S| = 2$, $v(N) = 120$. *Experience shows that in playing* *such a game, often a 2–person coalition may form, each player taking half of 100. This corresponds to one of the 3 coalition structures* $\mathcal{C}_1 = \{\{1\},\{2,3\}\}$, $\mathcal{C}_2 = \{\{2\},\{1,3\}\}$ *and* $\mathcal{C}_3 = \{\{3\},\{1,2\}\}$.

Example 2 *Consider the T–unanimity game* u_T. *In playing this sort of game often T will form and the members of this coalition will then divide the gain of 1 in some acceptable way. This corresponds to the coalition structure* $\mathcal{C} = \{T, \{i\}_{i\in N\setminus T}\}$.

Remarks *When the grand coalition forms, the corresponding coalition structure is the trivial one where* $\mathcal{C} = \{N\}$. *Note also that for a game with n players, the number of distinct coalition structures is* $2^{n-1} - 1$.

For a game $< N, v >$ with coalition structure $\mathcal{C}$, the pre–imputations are payoff configurations (w.r.t. $\mathcal{C}$) $x \in \Re^n$ such that $\sum_{i\in C} x_i = v(C)$ for all $C \in \mathcal{C}$. The set $I^*(v,\mathcal{C})$ of all such pre–imputations is $(n - |\mathcal{C}|)$–dimensional, and the elements if $I^*(v,\mathcal{C})$ are said to be $\mathcal{C}$–efficient.

The imputation set and the core, with respect to the coalition structure $\mathcal{C}$ are defined by

$$I(v,\mathcal{C}) := \{x \in I^*(v,\mathcal{C}) \mid x_i \geq v(i) \quad \text{for all} \quad i \in N\}$$

and

$$C(v,\mathcal{C}) := \{x \in I^*(v,\mathcal{C}) \mid x(S) \geq v(S) \quad \text{for all} \quad S \in 2^N\}.$$

Both of them are convex and compact sets, and they can be empty.

Moreover, for a game $< N, v >$ with coalition structure $\mathcal{C}$, $x \in I(v,\mathcal{C})$ dominates $y \in I(v,\mathcal{C})$ via coalition S if

$$x_i > y_i \quad \text{for all} \quad i \in S \quad \text{and} \quad x(S) \leq v(S).$$

and therefore the D–core w.r.t. $\mathcal{C}$ is given by

$$D \subset (v,\mathcal{C}) := \{x \in i(v,\mathcal{C}) | x \text{ is undominated by the elements of } I(v,\mathcal{C})\}.$$

Also, a stable set M w.r.t $\mathcal{C}$ in $< N, v >$ is a subset of $I(v,\mathcal{C})$ such that

$$M \cap \text{dom}\,(M) = \emptyset \quad \text{and} \quad M \cup \text{dom}\,(M) = I(v,\mathcal{C}).$$

Examples *For the game of Example 1, it is easy to see that* $(50,50,0) \in I(v,\mathcal{C}_3)$ *and that* $C(v,\mathcal{C}_k) = \emptyset$ *for* $k = 1,2,3$. *Note that also* $C(v) = \emptyset$. *For the coalition structure of Example 2, we have*

$$C(v,\mathcal{C}) = I(v,\mathcal{C}) = \{x \in \Re^n_+ \mid x(N \setminus T) = 0, \sum_{i \in T} x_i v(T)\}.$$

Let $N = \{1,2,3\}$ *and suppose that the coalition structure is given by the partition* $\mathcal{C} = \{\{1,2\},\{3\}\}$. *If the characteristic function is* $v_a: G^3 \to \Re$ *with* $v_a(S) = 0$ *if* $|S| = 1$, $v_a(1,2) = 3$, $v_a(1,3) = v_a(2,3) = a$, $v_a(N) = 4$ *where* $a \in [0,4]$, *then*

$$\begin{aligned}
I^*(v_a,\mathcal{C}) &= \{x \in \Re^3 \mid x_1 + x_2 = 3, x_3 = 0\} \\
I(v_a,\mathcal{C}) &= \{x \in \Re^3 \mid x_i \geq 0, x_1 + x_2 = 3, x_3 = 0\} \\
&\quad \mathrm{conv}\{(0,3,0),(3,0,0)\} \\
C(v_a,\mathcal{C}) &= \{x \in \Re^3 \mid x_i \geq 0, x_3 = 0, x_1 + x_2 = 3, x_1 + x_3 \geq a, \\
&\quad x_2 + x_3 \geq a, x_1 + x_2 + x_3 \geq 4,\} \\
&= \{x \in \Re^3 \mid x_i \geq 0, x_3 = 0, x_1 + x_2 = 3, x_1 \geq a, x_2 \geq a, \\
&\quad x_1 + x_2 \geq 4,\} \\
&= \emptyset.
\end{aligned}$$

Remark *Coalition structure need not be restricted to supperadditive games. Take for instance* $< N, v >$ *to be the non-superadditive 4-person game with* $v(1,2) = v(3,4) = 3$, $v(1,3) = 1$, $v(N) = 5$ *and* $v(S) = 0$ *for the other coalitions. If the coalitional structure is* $\mathcal{C} = \{\{1,2\},\{3,4\}\}$ *then*

$$\begin{aligned}
I^*(v,\mathcal{C}) &= \{x \in \Re^4 \mid x_1 + x_2 = 3, x_3 + x_4 = 3\} \\
I(v,\mathcal{C}) &= \{x \in \Re^4_+ \mid x_1 + x_2 = 3, x_3 + x_4 = 3\} \\
&\quad \mathrm{conv}\{(3,0,3,0),(0,3,3,0),(3,0,0,3),(0,3,0,3)\} \\
C(v,\mathcal{C}) &= \mathrm{conv}\{(3,0,3,0),(0,3,3,0),(3,0,0,3),(1,2,0,3),(0,3,1,2)\}.
\end{aligned}$$

Recall that the reasonable set $R(v)$ for a game $< N, v >$ is defined by

$$R(v) := \{x \in \Re^n \mid v(i) \leq x_i \leq \max_{S:S \ni i}(v(S) - v(S \setminus \{i\}))\}.$$

Obviously, $C(v) \subset R(v)$ and therefore the reasonable set can be thought of as a " core catcher ". For games with coalition structures we have a similar result, namely

Lemma 4.1 *The core $C(v,\mathcal{C})$ is a subset of the reasonable set $R(v)$.*

PROOF: If $x \in C(v,\mathcal{C})$ then $x_i \geq v(i)$ for all $i \in N$. Suppose that $S \in \mathcal{C}$ with $i \in S$. Then

$$x_i + x(S-i) = v(S)$$

and hence

$$\begin{aligned} x_i &= v(S) - x(S-i) \\ &\leq v(S) - x(S-i) \\ &\leq \max_{S:S\ni i}(v(S) - v(S\setminus\{i\})) \end{aligned}$$

i.e. $C(v,\mathcal{C}) \setminus R(v)$. ■

A one–point solution for games with coaliton structures, which generalizes the Shapley value, is described in the next theorem due to Aummann and Drèze [1].

Theorem 4.2 *Let $N = \{1,2,\ldots,n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be a partition of the player set. Then there exists a unique function $f: G^n \to \Re^n$ with the following four properties:*

(i) Additivity: $f(v+w) = f(v) + f(w)$ for all $v, w \in G^n$

(ii) Dummy property: $f_i(v) = v(i)$ for all $v \in G^n$ and all dummy players i

(iii) $\mathcal{C}$–efficiency: $\sum_{i\in C} f_i(v) = v(C)$ for all $C \in \mathcal{C}$ and $v \in G^n$

(iv) $\mathcal{C}$–anonimity: $f(v^\sigma) = \sigma^*(f(v))$ for all $v \in G^n$ and all permutations $\sigma: N \to N$ with $\sigma(C) = C$ for all $C \in \mathcal{C}$.

The function $\Phi^{\mathcal{C}}$ is defined by $\Phi_i^{\mathcal{C}}(v) = \Phi_i(v|2^C)$ for $i \in C$, $C \in \mathcal{C}$, $v \in G^n$, where $v|2^C$, $C \in \mathcal{C}$ denotes the game with player set C and where the characteristic function is the restriction of v to 2^C.

PROOF: (a) First we show that $\Phi^{\mathcal{C}}$ has the required properties. For the additivity, note that if $i \in C \in \mathcal{C}$, then

$$\begin{aligned} \Phi_i^{\mathcal{C}}(v+w) &= \Phi_i(v+w|2^C) &= \Phi_i(v|2^C + w|2^C) \\ &= \Phi_i(v|2^C) + \Phi_i(w|2^C) &= \Phi_i^{\mathcal{C}}(v) + \Phi_i^{\mathcal{C}}(w) \end{aligned}$$

where the third equality follows from the additivity of the Shapley value. For the dummy property, it is obvious that if $i \in C \in \mathcal{C}$ is a dummy player in $< N, v >$

then the same holds in $< C, v|2^C >$ and therefore $\Phi_i^{\mathcal{C}}(v) = \Phi_i(v|2^C) = 0$ by the dummy property of Φ. The efficiency of Φ_i implies the $\mathcal{C}$–efficiency of $\Phi^{\mathcal{C}}$, i.e.

$$\sum_{i \in C} \Phi_i^{\mathcal{C}}(v) = \sum_{i \in C} \Phi_i(v|2^C) = (v|2^C)(C) = v(C).$$

From the anonimity property of the Shapley value, $\Phi(v^\sigma) = \sigma^*(\Phi(v))$ it follows that $\Phi^{\mathcal{C}}(v^\sigma) = \sigma^* \Phi^{\mathcal{C}}(v)$, i.e. the $\mathcal{C}$–anonimity of $\Phi^{\mathcal{C}}$.

(b) Now suppose that $f: G^n \to \Re^n$ satisfies the property (i)–(iv). Let $v \in G^n$. We have to show that if $i \in C \in \mathcal{C}$, then

$$f(v) = \Phi^{\mathcal{C}}(v) \quad \text{or} \quad f_i(v) = \Phi_i(v|2^C).$$

From Section 2 we know that a game $v \in G^n$ can be written as a linear combination $v = \sum_T c_T u_T$ of unanimity games where the summation is taken over all coalitions $\emptyset \neq T \subset \dot{C}$. Then by additivity, $f(v) = \sum_T f(c_T u_T)$. Now $\mathcal{C}$–anonimity implies that for $i, j \in C \cap T$

$$f_i(c_T u_T) = f_j(c_T u_T)$$

and the dummy property yields $f_i(c_T u_T) =)$ when $i \notin T$. From the $\mathcal{C}$–efficiency it follows that

$$\sum_{i \in C} f_i(c_T u_T) = c_T u_T(C) = \begin{cases} c_T & \text{if } T \subset C \\ 0 & \text{otherwise} \end{cases}$$

and thus

$$f_i(c_T u_T) = \begin{cases} |T|^{-1} & \text{if } T \subset C \\ 0 & \text{otherwise.} \end{cases}$$

from which we can write

$$f_i(v) = \sum_T f_i(c_T u_T) = \sum_{T:T \subset C} |T|^{-1} c_T e_i^T \quad \text{if} \quad i \in C.$$

But this implies that $f_i(v) = \Phi_i(v|2^C)$ because from $v = \sum_{T \subset N} c_T u^T$ it follows that

$$v|2^C = \sum_{T \subset N} c_T u^T |2^c = \sum_{T \subset C} c_T u^T.$$

■

5. Allocation rules

Let CS^n denote the set of n–person communication situations. An allocation rule is a map $F: CS^n \rightarrow \Re^n$ which assigns to a communication situation (v, A) a payoff vector $F(v, A)$. For instance, the map $F: CS^n \rightarrow \Re^n$ which assigns to a communication situation (v, A) the Shapley value $\Phi(v_A)$ of the corresponding communication game is an allocation rule. Before characterizing such a rule some definitions will be given.

Definition 5.1 *An allocation rule $F: CS^n \rightarrow \Re^n$ is called efficient if for all $(v, A) \in CS^n$ and all $C \in N/A$,*

$$\sum_{i \in C} F_i(v, A) = v(C).$$

This condition states that, if C is a connected component of N/A, then the members of C ought to allocate to themselves the total reward $v(C)$ available to them. Note that the allocation within a connected coalition C still depends on the actual graph.

Example 1 *An allocation rule might assign a higher payoff to player 1 in $A_1 = \{\{1,2\}, \{1,3\}, \{1,4\}\}$ then in $A_2 = \{\{1,2\}, \{2,3\}, \{3,4\}\}$ because his position is more crutial in communicating with other players in the former arc set. In each case, however, the above condition requires that*

$$\sum_{i=1}^{4} F_i(v, A_1) = \sum_{i=1}^{4} F_i(v, A_2) = v(1, 2, 3, 4).$$

Efficiency of F means that for $(v, A) \in CS^n$, the vector $F(v_A)$ is a $\mathcal{C}$–efficient payoff vector for v, where $\mathcal{C}$ is the partition N/A of N. The notion of $\mathcal{C}$–efficiency was discussed in Section 4.

Definition 5.2 *An allocation rule $F: CS^n \rightarrow \Re^n$ is called fair if for all $(v, A) \in CS^n$ and all $\{i, j\} \in A$, $F_i(v, A) - F_i(v, A \setminus \{i, j\}) = F_j(v, A) - F_j(v, A \setminus \{i, j\})$.*

Note that this condition asserts that two players should gain equally from their bilateral agreement. Also, fairness of F implies that if in a communication situation (v, A) with $\{i, j\} \in A$ the direct communication between i and j is broken (and this results in the communication situation $(v, A \setminus \{i, j\})$), then both players i and j lose the same amount of reward.

Fair allocation rules in the context of conference structures have been studied by Myerson [8].

Example 2 *Return to the table of Example 2 in Section 3, and suppose that the allocation rule is the Shapley value $\Phi(v_A)$. Then, if the direct communication between players 2 and 3 is broken, the amounts of lost reward are*

$$\Phi_2(v,\{\{1,2\},\{2,3\}\}) - \Phi_2(v,\{\{1,2\}\}) = \frac{11}{2} - \frac{1}{2} = 5$$

and

$$\Phi_3(v,\{\{1,2\},\{2,3\}\}) - \Phi_3(v,\{\{1,2\}\}) = 5 - 0 = 5$$

which are the same.

The following result is due to Myerson [7].

Theorem 5.3 *(i) There is a unique allocation rule which is efficient and fair.*

(ii) This rule assigns to $(v,A) \in CS^n$ the Shapley value $\Phi(v_A)$ of the communication game $< N, v_A >$ corresponding to v and A.

PROOF: (i) Suppose F^1 and F^2 are efficient and fair allocation rules, and let $v \in G^n$. We show by induction on $|A|$ that $F^1(v,A) = F^2(v,A)$ for all A. First of all, note that $F^1(v,\emptyset) = F^2(v,\emptyset) = (v(1), v(2), \ldots, v(n))$. Now take A with $|A| = k \geq 1$ and suppose that $F^1(v,B) = F^2(v,B)$ for all B with $|B| < k$. Then, for each pair $\{i,j\} \in A$, the fairness property implies that

$$F_i^1(v,A) - F_i^1(v,A\setminus\{i,j\}) = F_j^1(v,A) - F_j^1(v,A\setminus\{i,j\})$$

and

$$F_i^2(v,A) - F_i^2(v,A\setminus\{i,j\}) = F_j^2(v,A) - F_j^2(v,A\setminus\{i,j\}).$$

By the induction hypothesis we have

$$F_k^1(v,A\setminus\{i,j\}) = F_k^2(v,A\setminus\{i,j\}) \quad \text{for} \quad k \in \{i,j\}$$

and therefore $F_i^1(v,A) - F_i^2(v,A) = F_j^1(v,A) - F_j^2(v,A)$.
Since this holds true for all pairs i,j which can communicate directly i.e. for $\{i,j\} \in A$, the same formula applies for all i,j which are connected in N. Hence, for each component $C \in N/A$ there is an $\alpha_C \in \Re$ such that for all $i \in C$

$$F_i^1(v,A) - F_i^2(v,A) = \alpha_C.$$

Next, the efficiency of F^1 and F^2 together with the above relation imply that

$$|C|\alpha_C = \sum_{i \in C}(F_i^1(v,A) - F_i^2(v,A)) - v(C) - v(C) = 0$$

and therefore $\alpha_C = 0$. Thus, $F_i^1(v,A) = F_i^2(v,A)$ for all $i \in C$ i.e. $F^1(v,A) = F^2(v,A)$ and we have shown that there is at most one allocation rule which is both efficient and fair.

(ii) To prove the existence of such an allocation rule, let $v \in G^n$ and take an undirect graph A on N, denoted by (N,A). Note that $v_A = \sum_{C \in N/A} \lambda^C$ where $\lambda^C \in G^n$ is defined as

$$\lambda^C(S) := \sum_{T \in S \cap C/A} v(T) \quad \text{for all} \quad S \in 2^N$$

and therefore

$$v_A(S) = \sum_{T \in S/A} v(T) = \sum_{C \in N/A} (\sum_{T \in S \cap C/A} v(T)) = \sum_{C \in N/A} \lambda^C(S)$$

where use was made of the fact that elements of S/A lie completely in a component of N with respect to the arc set A.
Now the additivity property of the Shapley value implies that

$$\Phi(v_A) = \sum_{C \in N/A} \Phi(\lambda^C).$$

Let $\hat{C} \in N/A$. For the proof of the efficiency property we must show that

$$\sum_{i \in \hat{C}} \Phi_i(v_A) = v(\hat{C}).$$

First of all, $\Phi_i(\lambda^C) = 0$ for all $i \in \hat{C}$ and $C \neq \hat{C}$ because i is a dummy player in λ^C if $i \in \hat{C} \neq C$. Furthermore, $\sum_{i \in \hat{C}} \Phi_i(v_A) = \lambda^{\hat{C}}(N)$ in view of the efficiency and dummy player property of the Shapley value. Hence,

$$\sum_{i \in \hat{C}} \Phi_i(v_A) = \sum_{C \in N/A} \sum_{T \in \hat{C}/A} \Phi_i(\lambda^C) = \lambda^{\hat{C}}(N)$$

which combined with

$$\lambda^{\hat{C}}(N) = \sum_{T \in N \cap \hat{C}/A} v(T) = \sum_{T \in \hat{C}/A} v(T) = v_A(\hat{C}) = v(\hat{C})$$

yields the desired result.

Finally to prove the fairness of the rule, let $v \in G^n$, (N, A) with $\{i,j\} \in A$ and introduce the game $w: 2^N \to \Re$ with

$$w \dot{:} v_A - v_{A/\{i,j\}}.$$

Since $w(S) = 0$ for all S with $\{i,j\} \not\subset S$, Lemma 3.1 implies that $\Phi_i(w) - \Phi_j(w)$. Using the additivity property of Φ and Lemma 3.2 we then obtain

$$\Phi_i(v_A) = \Phi_i(v_{A/\{i,j\}}) = \Phi_j(v_A) - \Phi_j(v_{A/\{i,j\}})$$

and the fairness of this allocation rule is demonstrated. ■

Remark *When the communication graph A is complete, then $< N, v_A >= < N, v >$ and $\Phi(v_a) = \Phi(v)$.*

Example 3 *Let $N = \{1,2,3\}$ and suppose that the characteristic function is $v(1) = v(2) = v(3) = 0$, $v(1,3) = v(2,3) = 6$, $v(1,2) = v(1,2,3) = 12$. Then the Shapley value is given by $\Phi(v) = (5,5,2)$, while the nucleolus as well as the bargaining set select the allocation $(6,6,0)$. The efficient and fair allocation rule is as follows:*

$$F(\emptyset) = (0,0,0), \quad F(\{1,2\}) = (6,6,0),$$

$$F(\{1,3\}) = (3,0,3), \quad F(\{2,3\}) = (0,3,3),$$

$$F(\{1,2\},\{1,3\}) = (7,4,1), \quad F(\{1,2\},\{2,3\}) = (4,7,1),$$

$$F(\{1,3\},\{2,3\}) = (3,3,6), \quad F(\{1,2\},\{1,3\},\{2,3\}) = (5,5,2).$$

As pointed out by Myerson [7], the core of the original game is $C(v) = (5,5,2)$, which appears to be a rather unstable allocation since players 1 and 2 could earn 12 units of reward for themselves which exceeds $5 + 5 = 10$ units as suggested also by the Shapley value. However, when we consider the associated communication graphs, the payoff vector $(5,5,2)$ is part of a fair and efficient allocation rule. Consequently, if any one of the players were to break either or both of his communication links, then his fair allocation would decrease. For instance, if both players 1 and 2 were to simultaneously break their communication links with player 3, then both would benefit, but each would gain even more by continuing to communicate with 3 while the other alone broke his communication with 3.

Definition 5.4 *An allocation rule* $F: CS^n \rightarrow \Re^n$ *is said to be* *coalition formation friendly* *in* (v, A) *if for all* $\{i, j\} \in A$,

$$F_i(v, A) \geq F_i(v, A \setminus \{i, j\}).$$

Tijs [12] has shown that the allocation rule described in Theorem 5.3 is coalition formation friendly provided the game $< N, v >$ *is superadditive.*

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Gry z narzucona strukturą porozumiewania

W pracy rozważono klasę gier n–osobowych, z wypłatami pobocznymi, w których współpraca między graczami jest ograniczona w sensie porozumiewania się przez narzucenie pewnych struktur grafowych. Pokazano, że istnieje jedyna reguła przydziału, zarówno sprawna jak i sprawiedliwa, w zasadzie dana przez wartość Shapleya ograniczonej gry. Jeśli ponadto gra jest superaddytywna, to reguła przydziału jest stabilna.

Игры с навязанной структурой согласования

В работе рассмотрен класс игр с n–игроками, с дополнительными выигрышами, в которых сотрудничество между игроками ограничено, в смысле возможности согласования, в результате навязания некоторых графовых структур. Показано, что существует единственное правило распределения, как эффективное так и справедливое, в принципе заданное значением Шепленя ограниченной игры. В случае, если кроме этого игра является сверхаддитивной, то правило распределения является устойчивым.