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A silent duel under arbitrarness of movements

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The paper considers a silent duel in which the players have one bullet each, the accuracy functions are arbitrary as well as the payoff function and the players can move as they like. Strategies for both players are established under such conditions.

1. Introduction

Let us consider a game which will be called $G(1,1)$. It consists in a duel fought by Players I and II, who can move as they want. The maximal speed of Player I is v_1, the maximal speed of Player II is v_2 and it is assumed that $v_1 > v_2 \geq 0$. The players have one bullet each and this fact is known to both of them. It is also known that the duel is silent: neither player hears the shot of his opponent.

At the beginning of the duel the players are at a distance of 1 from each other. Let $P_1(s)$ (resp. $P_1(s)$) be the probability of achieving success (destroying the opponent) by Player I (resp. II) when the distance between the players is $1-s$. The functions $P_1(s)$, $P_2(s)$ will be called accuracy functions. It is assumed that they are increasing and continuous in $[0,1]$, have continuous second derivatives in $(0,1)$ and that $P_i(s)=0$ for $s\leq 0$, $P_i(s)=1$, $i=1,2$.

Player I gains $k>0$ if the success is achieved only by him, gains $-l<0$ if it is only Player II who achieves success, gains w if both Players achieve success and gains 0 if none of them does, $-l\leq w\leq k$. The duel is a zero–sum game.

As it will be seen from the sequel, we can suppose without any loss of generality that $v_1=1$ and that Player II is motionless. It is assumed also that at the beginning of the duel Player I is in the point 0 and Player II is in the point 1.

About definitions and results in the theory of the game of timing see [3, 4, 5, 7, 9, 10, 15, 17].

2. Auxiliary duel

In order to solve $G(1,1)$, as defined in the previous section, determination of optimal strategies in an auxiliary game $G_0(1,1)$ will be useful. Consider one–bullet silent duel with accuracy functions $P_1(s)$, $P_2(s)$ in which Player I approaches Player II with constant velocity $v=1$ all the time, even after firing of his bullet. Player I gains $k>0$ if it is only him who achieves success etc., similarly as in the duel defined in previous section.

Denote by $K_0(s,t)$ the expected gain of Player I if he shoots at the moment $s\in[0,1]$ and Player II shoots at the moment $t\in[0,1]$. It is assumed that

$$K_0(s,t)=\begin{cases} kP_1(s) & \text{if } \ s<t,\\ kP_1(s)-lP_2(s)-(k-l-w)P_1(s)P_2(s) & \text{if } \ s=t,\\ -lP_2(t)+k(1-P_2(t))P_1(s) & \text{if } \ s>t.\end{cases}$$

It is easy to see that $K_0(s,t)$ is the expected payoff in the duel in which Player II is not allowed to fire after the shot of Player I.

Denote by ξ_0^a the strategy of Player I in the game $G_0(1,1)$ in which he fires his shot at a random moment s distributed according to the density $pf_1(s)$ in the interval $[a,1]$, $0<a<1$, and discrete probability $1-p$, $0<p<1$, in the point 1. This distribution is chosen in such a way that if $t\in[a,1)$ then

$$\begin{aligned} K_0(\xi_0^a;t) &= \qquad\qquad (1)\\ &= p\left[k\int_a^t P_1(s)f_1(s)ds+\int_t^1(-lP_2(t)+k(1-P_2(t))P_1(s))f_1(s)ds\right]\\ &\quad +(1-p)(k-(k+l)P_2(t))=\text{const.}\end{aligned}$$

In the above formula function $K_0(\xi_0^a;t)$ is the expected gain of Player I if Player I applies strategy ξ_0^a and Player II fires his shot at t.

We obtain

$$\frac{K_0(\xi_0^a;t)}{\partial t} = p[(l+kP_1(t))P_2(t)f_1(t)- \\ - P_2'(t)\int_t^1 (l+kP_1(s))f_1(s)ds\Big] - (1-p)(k+l)P_2'(t) = 0, \tag{2}$$

$$\frac{\partial^2 K_0(\xi_0^a;t)}{\partial t} = p[(lP_2'(t)+kP_1'(t)P_2(t)+kP_1(t)P_2'(t))f_1(t) \\ +(l+kP_1(t))P_2(t)f_1'(t) - P_2''(t)\int_t^1 (l+kP_1(s))f_1(s)ds + \\ +(l+kP_1(t))P_2'(t)f_1(t)] - (1-p)(k+l)P_2''(t) = 0. \tag{3}$$

Eliminating the integral from equations (2) and (3) we obtain

$$(l+kP_1(t))P_2(t)f_1'(t) + [2(l+kP_1(t))P_2'(t)+kP_1'(t)P_2(t)]f_1'(t)- \\ -\frac{P_2''(t)}{P_2'(t)}(l+kP_1(t))P_2(t)f_1(t) = 0$$

from where we get

$$f_1(t) = C\frac{P_2'(t)}{P_2^2(t)(P_1(t)+\frac{l}{k})} \tag{4}$$

for the constant C satisfying equation

$$C\int_a^1 \frac{P_2'(t)dt}{P_2^2(t)(P_1(t)+\frac{l}{k})} = 1. \tag{5}$$

Moreover, from (1) and (4) we obtain

$$\begin{aligned} K_0(\xi_0^a;t) &= pCk\left[\int_a^1 \frac{P_1(s)P_2'(s)ds}{P_2^2(s)(P_1(s)+\frac{l}{k})} - P_2(t)\int_t^1 \frac{P_2'(s)}{P_2^2(s)}ds\right] \\ &\quad +(1-p)(k-(k+l)P_2(t)) \\ &= pCk\left[\int_a^1 \frac{P_1(s)P_2'(s)ds}{P_2^2(s)(P_1(s)+\frac{l}{k})} + P_2(t) - 1\right] \\ &\quad +(1-p)(k-(k+l)P_2(t)) \\ &= pCk\left[\int_a^1 \frac{P_1(s)P_2'(s)ds}{P_2^2(s)(P_1(s)+\frac{l}{k})} - 1\right] + (1-p)k \quad = \quad \text{const.} \end{aligned} \tag{6}$$

if

$$pCk = (1-p)(k+l). \tag{7}$$

Let η_0^a be the strategy of Player II in $G_0(1,1)$, in which Player II chooses at the random the moment t of the shot, according to the density $f_2(t)$ in $[a,1]$, to obtain

$$\begin{aligned} K_0(s;\eta_0^a) &= \\ &= \int_a^s (-lP_2(t)+k(1-P_2(t))P_1(s))f_2(t)dt + \int_s^1 kP_1(s)f_2(t)dt = \text{const.} \end{aligned}$$

if $s \in [a,1)$. The function $K_0(s;\eta_0^a)$ is the expected gain of Player I if Player II applies the strategy η_0^a and Player I fires the shot at s.

Proceeding in the same way as before we obtain

$$f_2(s) = D\frac{P_1'(t)}{P_2(t)(P_1(t)+\frac{l}{k})^2}, \tag{8}$$

$$D\int_a^1 \frac{P_1'(s)}{P_2(s)(P_1(s)+\frac{l}{k})^2} = 1, \tag{9}$$

$$D = P_1(a) + \frac{l}{k}, \tag{10}$$

$$K_0(s,\eta_0^a) = D\frac{kP_1(a)}{P_1(a)+\frac{l}{k}} = kP_1(a) \tag{11}$$

if $a \le s < 1$.

Assuming that $K_0(\xi_0^a;t) = K_0(s;\eta_0^a) = \text{const}$ for $s,t \in [a,1]$ we obtain additional equation from (6) and (11)

$$pCk\left[\int_a^1 \frac{P_1(s)P_2'(s)ds}{P_2^2(s)(P_1(s)+\frac{l}{k})} - 1\right] + (1-p)k = kP_1(a). \tag{12}$$

From equations (5), (7), (9), (10) and (12) we determine the unknown parameters C, D, a, p. Let us notice that we have five equations but only four unknown quantities.

By eliminating from the above five equations parameters C and D we obtain the system of equations

$$(k+l)(1-p)\int_a^1 \frac{P_2'(t)dt}{P_2^2(t)(P_1(t)+\frac{l}{k})} = kp, \tag{13}$$

$$(kP_1(a)+l)\int_a^1 \frac{P_2'(t)dt}{P_2(t)(P_1(t)+\frac{l}{k})^2} = k, \tag{14}$$

$$(1-p)\left[(k+l)\int_a^1 \frac{P_1(t)P_2'(t)dt}{P_2^2(t)(P_1(t)+\frac{l}{k})^2} - l\right] = kP_1(a) \tag{15}$$

with unknown quantities p and a.

From (13) and (14) we obtain

$$(k+l)(1-p)\int_a^1 \frac{P_2'(t)}{P_2^2(t)}dt = kP_1(a)+l$$

or, computing the integral,

$$(k+l)(1-p) = \frac{P_2(a)(kP_1(a)+l)}{1-P_2(a)}. \tag{16}$$

On the other hand, integration by parts leads to equations

$$\begin{aligned}
\int_a^1 \frac{P_2'(t)dt}{P_2^2(t)(P_1(t)+\frac{l}{k})} &= -\frac{k}{k+l} + \frac{1}{P_2(a)(P_1(a)+\frac{l}{k})} \\
&\quad - \int_a^1 \frac{P_1'(t)dt}{P_2(t)(P_1(t)+\frac{l}{k})^2}, \\
\int_a^1 \frac{P_1(t)P_2'(t)dt}{P_2^2(t)(P_1(t)+\frac{l}{k})} &= -\frac{k}{k+l} + \frac{P_1(a)}{P_2(a)(P_1(a)+\frac{l}{k})} \\
&\quad + \frac{l}{k}\int_a^1 \frac{P_1'(t)dt}{P_2(t)(P_1(t)+\frac{l}{k})^2}.
\end{aligned} \tag{17}$$

Now, we can obtain from (13) and (15)

$$(k+l)(1-p)\left[-\frac{k}{k+l} + \frac{1}{P_2(a)(P_1(a)+\frac{l}{k})} - \int_a^1 \frac{P_1'(t)dt}{P_2(t)(P_1(t)+\frac{l}{k})^2}\right] = kp, \tag{18}$$

$$(k+l)(1-p)\left[-1 + \frac{P_1(a)}{P_2(a)(P_1(a)+\frac{l}{k})} + \frac{l}{k}\int_a^1 \frac{P_1'(t)dt}{P_2(t)(P_1(t)+\frac{l}{k})^2}\right] = kP_1(a). \tag{19}$$

Assume that equations (14) and (16) have a solution. By introducing the values of p and $\int_a^1 \frac{P_1'(t)dt}{P_2(t)(P_1(t)+\frac{l}{k})^2}$ obtained from (14) and (16) into (18) and (19) we obtain identities. Then, the system of five equations, (5), (7), (9), (10), (12), has a solution C, D, p, a, $C > 0$, $D > 0$, $0 < p < 1$, $0 < a < 1$, if equations (14) and (16) have a solution p, a, $0 < p < 1$, $0 < a < 1$.

Let us consider function

$$\varphi(a) = (kP_1(a) + l) \int_a^1 \frac{P_1'(t)dt}{P_2(t)(P_1(t) + \frac{l}{k})^2}.$$

We have

$$\begin{aligned}\varphi'(a) &= kP_1'(a)\left[\int_a^1 \frac{P_1'(t)dt}{P_2(t)(P_1(t)+\frac{l}{k})^2} - \frac{1}{P_2(a)(P_1(a)+\frac{l}{k})}\right] \\ &\overset{(17)}{=} kP_1'(a)\left[-\int_a^1 \frac{P_2'(t)dt}{P_2^2(t)(P_1(t)+\frac{l}{k})} - \frac{k}{k+l}\right] < 0.\end{aligned}$$

It follows then that there exists at most one solution a, $0 < a < 1$, of the equation $\varphi(a) = 0$.

We prove that if there exists a solution a, $0 < a < 1$, of the equation (14) then there exists a solution p, $0 < p < 1$, of the equation (16). Since the integral at the left hand of equation (17) is positive then, for constant a being a solution of (14)

$$-\frac{k}{k+l} + \frac{1}{P_2(a)(P_1(a)+\frac{l}{k})} - \frac{1}{P_1(a)+\frac{l}{k}} > 0$$

or

$$\frac{P_2(a)(kP_1(a)+l)}{1-P_2(a)} < k + l$$

what ends the proof. Then the solution C, D, p, a of five equations exists, $C > 0$, $D > 0$, $0 < p < 1$, $0 < a < 1$, if there exists a solution a, $0 < a < 1$, of the equation (14).

Let $P_1(t) = t$, $P_2(t) = t^\alpha$, $\alpha > 0$. We obtain

$$\begin{aligned}(kP_1(a)+l)\int_a^1 \frac{P_1'(t)dt}{P_2(t)(P_1(t)+\frac{l}{k})^2} &= (ka+l)\int_a^1 \frac{dt}{t^\alpha(t+\frac{l}{k})^2} \\ &= (ka+l)\int_{\frac{k}{k+l}}^{\frac{k}{ka+l}} \left(\frac{x}{1-\frac{l}{k}x}\right)^\alpha dx \le (ka+l)\int_{\frac{k}{k+l}}^{\frac{k}{ka+l}} \frac{dx}{(1-\frac{l}{k}x)^\alpha} \\ &\overset{\alpha\to 0}{\longrightarrow} (1-a)\frac{k^2}{k+l} < k\end{aligned}$$

for each $0 < a < 1$. Then for these $P_1(s)$, $P_2(s)$ equation (14) has not a solution a, $0 < a < 1$, when α is small.

Lemma. *If there exists a solution a, $0 < a < 1$, of the equation (14) then for this a the strategy ξ_0^a is maximin and the strategy η_0^a is minimax in the game $G_0(1,1)$. The value of the game is $v_{11}^0 = kP_1(a)$.*

Proof: Let constant a be a solution of equation (14). We have proved that

$$K_0(\xi_0^a;t) = kP_1(a)$$

for $a \le t < 1$. Moreover

$$\begin{aligned} K_0(\xi_0^a;1) &= p\int_a^1 kP_1(s)f_1(s)ds \\ &\ge p\int_a^1 kP_1(s)f_1(s)ds + (1-p)(k-(k+l)P_2(1)) \\ &= \lim_{r\to 1-} K_0(\xi_0^a;t) \quad = \quad kP_1(a) \end{aligned}$$

since $K_0(\xi_0^a;t) = \text{const} = kP_1(a)$ for $a \le t < 1$.

If $t < a$

$$\begin{aligned} K_0(\xi_0^a;t) &= \\ &= p\int_a^1 (-lP_2(t) + k(1-P_2(t))P_1(s))f_1(s)ds + (1-p)(k-(k+l)P_2(t)) \\ &> p\int_a^1 (-lP_2(a) + k(1-P_2(a))P_1(s))f_1(s)ds + (1-p)(k-(k+l)P_2(a)) \\ &= K_0(\xi_0^a;a) \quad = \quad kP_1(a). \end{aligned}$$

Then

$$K_0(\xi_0^a;\eta) \ge kP_1(a)$$

for any strategy η of Player II.

On the other hand

$$K_0(s;\eta_0^a) \ge kP_1(a)$$

for $a \le s \le 1$ and if $s < a$

$$K_0(s;\eta_0^a) = kP_1(s) < kP_1(a).$$

Then

$$K_0(\xi;\eta_0^a) \le kP_1(a)$$

for any strategy ξ of Player I. The lemma is proved. ■

3. Main result

Let us return to the duel $G(1,1)$ defined at the beginning of the paper. Assume that there exists a solution a, $0 < a < 1$ of equation (14). Let the constant a_k be defined for a given natural n as follows:

$$a_0 = a, \quad p\int_{a_{i-1}}^{a_i} f_1(s)ds = \frac{1}{n}, \quad i = 1,\ldots,n_0, \quad a_{n_0+1} = 1,$$

n_0 is defined from the inequalities

$$p > p\int_a^{a_{n_0}} f_1(s)ds \geq p - \frac{1}{n}.$$

Define the strategy ξ^ε of Player I in the game $G(1,1)$ as follows:
If there exists a solution a of equation (14) (case 1) Player I moves back and forth with maximal speed in the following manner: at first between 0 and a_1, then between 0 and a_2, ..., finally between 0 and a_{n_0+1}. At the i-th step, $i = 1,\ldots,n_0+1$, he can shoot at random only if he is between the point a_{i-1} and a_i and goes forward and he shoots with the probability density $pf_1(s)$. If he has fired his shot at the i-th step he reaches the point a_i, evades to 0 and never approaches Player II. If Player has not fired between the points 0 and 1 and survives, he fires when he is at 1 as soon as possible.

If no solution a, $0 < a < 1$, of (14) exists (case 2), Player I, following ξ^ε, does not approach Player II.

The strategy η_0 of Player II is defined in case 1 as follows: if Player I reaches the point t first time and his velocity is $v_1(\tau)$ fire at random at time τ with density $v_1(\tau)f_2(t(\tau))$. Otherwise do not fire.

It is assumed that the function $v_1(\tau)$ is piecewise continuous.

In case 2, when equation (14) has no solution a, $0 < a < 1$, strategy η^0 is defined similarly but firing has probability density $v_1(\tau)f_2^0(t(\tau))$ where function $f_2^0(t)$ is defined in (8), for $a = 0$, where $D = D_0$ satisfies equation

$$D_0\int_0^1 \frac{P_1'(s)ds}{P_2(s)(P_1(s)+\frac{l}{k})^2} = 1. \tag{20}$$

Theorem. *The strategy ξ^ε is ε-maximin and strategy η_0 is minimax in the game $G(1,1)$. The value of the game is $v_{11} = kP_1(a)$ if there is a solution a, $0 < a < 1$, of the equation (14) and $v_{11} = 0$ in the other case.*

PROOF: Assume Player I applies strategy ξ^ε and that equation (14) has a solution a, $0 < a < 1$. We say that Player II fires the shot in (i, a') if he shoots when Player I is at the point a' and if this happens during the first Player's approach to a_i or during evasion from a_{i-1}. Denote also by (i, a') the thus defined strategy of Player II. We obtain

$$\begin{aligned}
K(\xi^\varepsilon; i, a') &> \\
&> p\left[\int_a^{a_{i-1}} kP_1(s)f_1(s)ds + \int_{a_i}^1 (-lP_2(a') + k(1-P_2(a'))P_1(s))f_1(s)ds\right] \\
&\quad +(1-p)(k-(k+l)P_2(a')) - \frac{l}{n} \\
&\geq p\left[\int_a^{a_{i-1}} kP_1(s)f_1(s)ds + \int_{a_i}^1 (-lP_2(a_i) + k(1-P_2(a_i))P_1(s))f_1(s)ds\right] \\
&\quad +(1-p)(k-(k+l)P_2(a_i)) - \frac{l}{n} \\
&\geq p\left[\int_a^{a_i} kP_1(s)f_1(s)ds + \int_{a_i}^1 (-lP_2(a_i) + k(1-P_2(a_i)P_1(s))f_1(s)ds\right] \\
&\quad +(1-p)(k-(k+l)P_2(a_i)) - \frac{k+l}{n} \\
&= kP_1(a) - \varepsilon
\end{aligned}$$

where $\varepsilon = \frac{k+l}{n}$.

If Player II fires only when Player I reaches the point 1 the best for him is to fire as soon as possible. For such a strategy η we obtain from the definition of the function $K_0(s,t)$

$$\begin{aligned}
K(\xi^\varepsilon; \eta) &= p\int_a^1 kP_1(s)f_1(s)ds + (1-p)w \\
&\geq p\int_a^1 kP_1(s)f_1(s)ds + (1-p)(k-(k+l)P_2(1)) \\
&= \lim_{t\to 1^-} K_0(\xi_0^a; t) \quad = \quad kP_1(a).
\end{aligned}$$

Wherefrom it follows that

$$K(\xi^\varepsilon; \eta) \geq kP_1(a) - \varepsilon$$

for any strategy η of Player II.

Suppose also that Player I had fired the shot in the point a' and later he evaded. Assume that he reached this point for the first time. For such a strategy (denote it also by a') we have: if $a \leq a' < 1$, then

$$\begin{aligned} K(a';\eta^0) &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (21)\\ &= \int_a^{a'}(-lP_2(t)+k(1-P_2(t))P_1(a'))f_2(t)dt+\int_{a'}^1 kP_1(a')f_2(t)dt\\ &= kP_1(a') \end{aligned}$$

and if $0 \le a' < a$, then

$$K(a';\eta^0) = kP_1(a') \le kP_1(a)$$

Suppose that the farthest point reached by Player I is a' but he fired the shot later, in $a'' \le a'$. For such a strategy ξ we have, if $a \le a' < 1$,

$$\begin{aligned} &K(\xi;\eta^0)\\ &= \int_a^{a'}(-lP_2(t)+k(1-P_2(t))P_1(a''))f_2(t)dt+\int_{a'}^1 kP_1(a'')f_2(t)dt\\ &\le kP_1(a) \end{aligned}$$

by (21), and also if $0 \le a' < a$

$$K(\xi,\eta^0) = kP_1(a'') \le kP_1(a).$$

Since approaching of Player II after the shot (of Player I) is for Player I not better than evasion when Player II applies η^0 then

$$K(\xi,\eta^0) \le kP_1(a).$$

for any strategy of Player I.

Suppose now that the equation (14) has not a solution a, $0 < a < 1$. In this case Player I assures for himself the value 0 simply by evasion.

As we remember, in this case Player II applies the distribution $f_2^0(t)$ defined similarly as that given by (8), distribution for $a = 0$ and $D = D_0$ satisfying the equation (20) i.e. $D_0 \ge P_1(a) + \frac{l}{k} = \frac{l}{k}$ (compare with (9), (10) and (14)).

Suppose that Player I shoots at point a' and evades. Assume that he reaches this point for the first time. We obtain for $0 \le a' < 1$

$$\begin{aligned} K(a';\eta^0) &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (22)\\ &= \int_0^{a'}(-lP_2(t)+k(1-P_2(t))P_1(a'))f_2^0(t)dt+\int_{a'}^1 kP_1(a')f_2^0(t)dt\\ &= kP_1(a')\int_{a'}^1 f_2^0(t)dt-(kP_1(a)+l)\int_0^{a'} P_2(t)f_2^0(t)dt\\ &= \frac{k^2}{l}\left(\frac{l}{k}-D_0\right) \quad \le \quad 0. \end{aligned}$$

Suppose that the farthest point reached by Player I is a' but he fired a shot later in $a'' \leq a'$. We obtain for such a strategy ξ

$$\begin{aligned} & K(\xi;\eta^0) \\ = & \int_0^{a'} (-lP_2(t) + k(1-P_2(t))P_1(a''))f_2^0(t)dt + \int_{a'}^1 kP_1(a'')f_2^0(t)dt \\ \leq & \int_0^{a'} (-lP_2(t) + k(1-P_2(t))P_1(a'))f_2^0(t)dt + \int_{a'}^1 kP_1(a')f_2^0(t)dt \\ \leq & \ 0 \end{aligned}$$

by (22). Since here also approaching of Player II after the shot of Player I is for Player I not better than evasion, when Player II applies η^0, then

$$K(\xi;\eta^0) \leq 0$$

for any strategy ξ of Player I. This ends the proof of the theorem. ■

When $P_1(s) = P_2(s) \stackrel{\mathrm{def}}{=} P(s)$ all integrals in the paper can be computed explicitly. Moreover when $k = l$ we obtain from (14)

$$(1+P(a))\left[\log\frac{1+P(a)}{2P(a)} + \frac{1}{2}\right] = 2.$$

This equation has a solution a for which

$$P(a) \cong 0.177655$$

and we obtain from (16), (7) and (10)

$$p \cong 0.872793, \quad C \cong 0.291494, \quad D \cong 1.177655.$$

Duels under arbitrary moving, as far as author knows, were never considered before exept in the papers of the author (see [13, 14]).

About other results in the theory of the game of timing see [1, 2, 6, 8, 11, 12, 16].

References

[1] Cegielski A., Tactical problems involving uncertain actions, *Journal of Optimization Theory and Applications* **49,** (1986), 81–105.

[2] Cegielski A., Game of timing with uncertain number of shots, *Mathematica Japonica* **31** (1986), 503–532.

[3] FOX M., KIMELDORF G., Noisy duels, *SIAM Journal of Applied Mathematics* **17**, (1969), 353–361.

[4] KARLIN S., Mathematical methods and theory in games, programming, and economics II, Addison-Wesley, Reading, Mass., (1959).

[5] KIMELDORF G., Duels: an overview. Mathematics of conflict., Elsevier Science Publishers B.V. (North-Holland), (1983), 55–71.

[6] ORŁOWSKI K., RADZIK T., Non discrete duels with complete counteraction, *Matematische Operationsforschung und Statistik, Ser. Optimization,* **16** (1985), 257–263.

[7] ORŁOWSKI K., RADZIK T., Discrete silent duels with complete counteraction, *Matematische Operationsforschung und Statistik, Ser. Optimization,* **16** (1985), 419–430.

[8] POSITIELSKAYA L.N., Non-discrete noisy duels, *Technicheskaya Kibernetika,* **2,** (1984), 40–44 (in Russian).

[9] RESTREPO R., Tactical problems involving several actions, Contributions to the theory of games III, *Annals of Mathematical Studies* **39,** 1957, 313–335.

[10] STYSZYŃSKI A., An n–silent vs noisy duel with arbitrary accuracy functions, *Zastosowania Matematyki* **14** (1974), 205–225.

[11] TERAOKA Y., Noisy duels with uncertain existence of the shot, *International Journal of Game Theory* **5** (1976), 239–250.

[12] TERAOKA Y., A single bullet duel with uncertain information availabe to the duelists, *Bulletin of Mathematical Statistics* **18** (1979), 69–83.

[13] TRYBUŁA S., A noisy duel under arbitrary moving I, II, III, IV, V, VI, (to be published in *Zastosowania Matematyki*).

[14] TRYBUŁA S., A mixed duel under arbitrary moving, (to be published in *Zastosowania Matematyki*).

[15] VOROBYEV N.N., Foundations of the theory of games. Non-coalition games, Nauka, Moskva 1984 (in Russian).

[16] YANOVSKAYA YE. B., Duel–type games with continuous firing, *Engineering Cybernetics* **1** (1969), 15–18 (in Russian).

[17] ZHADAN V.G., Noisy duels with arbitrary accuracy functions, *Issledovanye Operaciy* **5** (1976), 156–177 (in Russian).

Cichy pojedynek przy dowolnym poruszaniu się

W pracy rozpatruje się cichy pojedynek, w którym pojedynkujący się (Gracze I i II) mają po jednym pocisku, funkcje celności i funkcja wypłaty są dowolne i gracze mogą poruszać się jak chcą. Wprowadzono strategie obu graczy dla tych warunków.

Тихий поединок при произвольном движении

В работе рассматривается тихий поединок, в котором состязающиеся (Игрок I и Игрок II) имеют по одному снаряду, а функции попадания и функция прибыли произвольны и игроки могут свободно перемещаться. Введены стратегии обоих игроков при втих условиях.