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The Pontryagin Minimum Principle for a strongly nonlinear two point boundary value problem

by

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A necessary optimality condition in form of the Pontryagin Minimum Principle is derived for an optimal control problem in which the behaviour is monitored by a strongly nonlinear second order ordinary differential equation with homogeneous Dirichlet boundary conditions. The proof is based on the standard needle-like variation of the optimal control and on a generalized Green representation formula for the solutions of linear two point boundary value problems.
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1. Introduction

We are going to consider the following optimal control problem for a strongly nonlinear second order ordinary differential equation with homogeneous Dirichlet boundary conditions :

Find

$$\inf J(u,y), \quad J(u,y) = \int_0^1 g(x,u(x),y(x),y'(x))dx, \tag{1}$$

subject to

$$u \in U_{ad} = \{u \in L_m^\infty(0,1) : u(x) \in Q \ a.e. \ x \in (0,1)\}, \tag{2}$$

$$\left.\begin{array}{l} -(d/dx)a(x,y(x),y'(x)) + b(x,u(x),y(x),y'(x)) = 0, \ x \in (0,1), \\ y(0) = y(1) = 0. \end{array}\right\} \tag{3}$$

Here $L_m^\infty(0,1)$ stands for the m-fold Cartesian product of $L^\infty(0,1)$ and $Q \subset R^m$, $m \geq 1$, is an arbitrarily given set containing at least two elements. The given real-valued functions $a = a(x,s,t)$, $b = b(x,r,s,t)$ and $g = g(x,r,s,t)$ are defined for $x \in (0,1)$, $r \in Q$ and $s,t \in R$ and satisfy certain assumptions to be specified below.

The main characteristic feature of the above control problem is that the state of the system is given by the solution of a boundary value problem for a nonlinear second order differential equation. A survey on the relevant literature has been given in [10]. Here we want to mention only [2,3,12,13,14,17], in which both analytical and numerical aspects were investigated for certain control problems with linear second order ordinary differential equations whose coefficients (all or some of them) may act as control functions. Book [11] shall summarize the findings of the authors in the domain.

The present paper is devoted to deriving the Pontryagin Minimum Principle for the control problem (1-3). It is a free continuation of [8,9], in which we have proved a Pontryagin-like Minimum Principle for a linear respective quasilinear second order ordinary differential equation, whose all coefficients are depending nonlinearly on the control parameters. Because of the example given in [8,15] we could not expect the Pontryagin Minimum Principle to be valid if the leading coefficient of the differential equation (that means "coefficient" a in (3)) depends on the control. (This statement is rather surprising in view of paper [16].) However, if the leading coefficient is not depending on the control parameters, then optimality conditions derived in [8,9] yield the corresponding Pontryagin Minimum Principles. Thus, we extend our previous results and show that a Pontryagin Minimum Principle is also valid in the case when the state equation is given by the strongly nonlinear two point boundary value problem (3). We overcome the special difficulties connected with the strong nonlinearity of (3) by means of [7]. In that paper we have given an explicit formula for the solution of a linear two point boundary value problem with measurable coefficients (and a right hand side belonging to $H^{-1}(0,1)$). Note also that in general (3) not can be rewritten as a system of two first order ordinary differential equations in normal form. It is know that control problems for implicit differential equations possess certain peculiarities (cf., e.g.,[5]).

Like in [8,9], in order to derive the wanted optimality condition we use a needle-like variation of the optimal control. In section 2 we introduce some notations and formulate the needed assumptions. In section 3 we study the solution of (3) related to the varied control and prepare the proof of the final result, which is given in the last section 4.

2. Notations and assumptions

Most of the notations used in this paper are standard. So we shall use $|\cdot|$ for the Euclidean norm in R^n, $n \geq 1$, $\|\cdot\|_C$ for the norm in $C[0,1]$ and $\|\cdot\|_p$ for the norm in $L^p(0,1)$, $1 \leq p \leq \infty$. $H_0^1(0,1)$ stands for the usual Sobolev space, whose elements vanish at the ends of the interval $(0,1)$, and in which the norm is given by $\|y\|_0 = \|y'\|_2$. We recall that $H_0^1(0,1)$ is continuously embedded into $C[0,1]$ and that there are two elementary inequalities

$$|y(x)| \leq \|y\|_0 \qquad \forall x \in [0,1], \qquad \|y\|_2 \leq \|y\|_0 \tag{4}$$

satisfied by each $y \in H_0^1(0,1)$.

Now we give the assumptions A1–A3; the assumptions A4 and A5 will be given below. In all what follows lower indices s and t mean the partial derivatives of the corresponding function with respect to these variables. $f \in$CAR denotes a function $f : (0,1) \times R \times R \longrightarrow R$ satisfying the Carathéodory conditions.

A1: $a, a_s, a_t \in$CAR

$$\left.\begin{array}{l} b(\cdot,u(\cdot),\cdot,\cdot),\ b_s(\cdot,u(\cdot),\cdot,\cdot),\ b_t(\cdot,u(\cdot),\cdot,\cdot), \\ g(\cdot,u(\cdot),\cdot,\cdot),\ g_s(\cdot,u(\cdot),\cdot,\cdot),\ g_t(\cdot,u(\cdot),\cdot,\cdot), \end{array}\right\} \in \text{CAR} \qquad \forall u \in U_{ad}$$

A2 (i): For any $\lambda > 0$ there is a positive constant $\mu_{1\lambda}$ such that

$$| a(x,s,t) | \leq \mu_{1\lambda}(1+ | t |)$$

for a.e. $x \in (0,1)$, $\forall s \in R$ with $| s | \leq \lambda$, $\forall t \in R$.

(ii): There is a positive constant α and for any $\lambda > 0$ positive constant $\mu_{2\lambda}$ such that

$\alpha \leq a_t(x,s,t)$ for a.e. $x \in (0,1)$, $\forall s,t \in R$,

$| a_s(x,s,t) |$, $a_t(x,s,t) \leq \mu_{2\lambda}$

for a.e. $x \in (0,1)$, $\forall s,t \in R$ with $| s | + | t | \leq \lambda$.

(iii): There are two positive constants μ and δ such that

$| a_s(x,s,t) - a_s(x,\sigma,\tau) |, | a_t(x,s,t) - a_t(x,\sigma,\tau) | \leq \mu(| s-\sigma | + | t-\tau |)$

for a.e. $x \in (0,1)$, $\forall s,t,\sigma,\tau \in R$ with $| s-\sigma |, | t-\tau | < \delta$.

A3 (i): For any $\lambda > 0$ there are the function $h_{1\lambda} \in L^1(0,1)$ and a positive constant $\mu_{3\lambda}$ such that

$$| b(x,r,s,t) |, | g(x,r,s,t) | \leq h_{1\lambda}(x) + \mu_{3\lambda} | t |^2$$

for a.e. $x \in (0,1)$, $\forall \{r,s\} \in Q \times R$ with $| r | + | s | \leq \lambda$, $\forall t \in R$.

(ii): For any $\lambda > 0$ there are two functions $h_{2\lambda} \in L^1(0,1)$ and $h_{3\lambda} \in L^2(0,1)$ such that

$$| b_s(x,r,s,t) |, | g_s(x,r,s,t) | \leq h_{2\lambda}(x),$$
$$| b_t(x,r,s,t) |, | g_t(x,r,s,t) | \leq h_{3\lambda}(x),$$

for a.e. $x \in (0,1)$, $\forall \{r,s,t\} \in Q \times R \times R$ with $| r | + | s | + | t | \leq \lambda$.

(iii): For any $\lambda > 0$ there are two positive constants $\mu_{4\lambda}$ and δ_λ such that

$$\left.\begin{array}{l} | b_s(x,r,s,t) - b_s(x,r,\sigma,\tau) |, \\ | b_t(x,r,s,t) - b_t(x,r,\sigma,\tau) |, \\ | g_s(x,r,s,t) - g_s(x,r,\sigma,\tau) |, \\ | g_t(x,r,s,t) - g_t(x,r,\sigma,\tau) | \end{array}\right\} \leq \mu_{4\lambda}(| s-\sigma | + | t-\tau |)$$

for a.e. $x \in (0,1)$, $\forall r \in Q$ with $| r | \leq \lambda$ and $\forall s,t,\sigma,\tau \in R$ with $| s-\sigma |$, $| t-\tau | < \delta_\lambda$.

The first conclusion of these assumptions is that

$a(\cdot, y(\cdot), y'(\cdot)) \in L^2(0,1) \quad \forall y \in H_0^1(0,1)$,

$b(\cdot, u(\cdot), y(\cdot), y'(\cdot)), g(\cdot, u(\cdot), y(\cdot), y'(\cdot)) \in L^1(0,1)\ \forall u \in U_{ad}, \forall y \in H_0^1(0,1)$.

Hence, the cost functional J is well defined over $U_{ad} \times H_0^1(0,1)$ and for fixed $u \in U_{ad}$ we may define a function $y \in H_0^1(0,1)$ to be (weak) solution to the boundary value problem (3) if

$$\int_0^1 (a(x,y,y')z' + b(x,u,y,y')z)dx = 0 \qquad \forall z \in H_0^1(0,1). \tag{5}$$

By means of the generalized Lemma of DuBois-Reymont (cf.,e.g.,[1]) it is easily checked that $y \in H_0^1(0,1)$ is a solution of (3) if and only if

$$a(x,y(x),y'(x)) = \int_0^x b(\xi,u(\xi),y(\xi),y'(\xi))d\xi = c(u) \quad \forall x \in [0,1] \tag{6}$$

with the constant

$$c(u) = a(x,y(x),y'(x))|_{x=0} \tag{7}$$

Throughout the whole paper a solution to any linear or nonlinear two point boundary value problem is to be understood in the above sense with the respective integral identity. Concerning (3) we formulate now the assumptions A4 and A5.

A4: For each $u \in U_{ad}$ the boundary value problem (3) has a unique solution $y(u) \in H_0^1(0,1)$ and there is a constant $C > 0$ with

$$\| y(u) \|_0 \leq C \qquad \forall u \in U_{ad}.$$

A5: For any $\lambda > 0$ there is a positive constant ν_λ such that

$$\int_0^1 [a_t(x,y,y')z'^2 + (a_s(x,y,y') + b_t(x,u,y,y'))zz'$$
$$+ b_s(x,u,y,y')z^2]dx \geq \nu_\lambda \| z \|_0^2$$

$\forall z \in V(0,1) = \{y \in H^1(0,1) : y(0) = 0 \text{ or } y(1) = 0\}$, $\forall \{u,y\} \in U_{ad} \times H_0^1(0,1)$ for which $y' \in L^\infty(0,1)$ and $\| |u| \|_\infty + \| y' \|_\infty < \lambda$.

We remark that, for example, in [4] the reader can find sufficient conditions for the unique solvability of (3) with fixed $u \in U_{ad}$. Assumption A5 ensures that certain linear boundary value problems, which will play an important role in the text, are uniquely solvable. We finish this section with some notations. So let $\{u_0, y_0\} \in L_m^\infty(0,1) \times H_0^1(0,1)$ be any fixed optimal solution to the control problem (1-3). The upper index "o" always indicates that the corresponding function is defined by means of this optimal solution. For example,

$a^0(x) = a(x,y_0(x),y_0'(x))$, $b^0(x) = b(x,u_0(x),y_0(x),y_0'(x))$,

$g^0(x) = g(x, u_0(x), \dot{y}_0(x), y_0'(x))$
but also
$a_s^0(x) = a_s(x, y_0(x), y_0'(x)), \ldots, g_t^0(x) = g(x, u_0(x), y_0(x), y_0'(x))$, $x \in (0,1)$.

Finally, for fixed $u \in Q$ by $\omega \subset (0,1)$ we denote the set, whose elements are Lebesgue points of a finite number of integrable functions occurring in what follows and for which the estimations in A2–A3 are satisfied. Obviously, ω has the measure one; in general, ω depends on u.

3. Preliminaries

As we have said in the introduction we are going to derive the Pontryagin Minimum Principle for the optimal control problem (1-3) using needle-like variations of the optimal control u_0 . To do so , let $u \in Q$ and $\xi \in \omega \subset (0,1)$ be any points and $\varepsilon \in (0, \varepsilon_0)$ a parameter. Then, for $\varepsilon_0 > 0$ sufficiently small we define

$$u_\varepsilon(x) = \begin{cases} u & \text{if } \ x \in E \\ u_0(x) & \text{if } \ x \in (0,1)\backslash E \end{cases} \qquad E = [\xi, \xi + \varepsilon). \tag{8}$$

Clearly, $u_\varepsilon \in U_{ad}$ and, hence, by assumption A4 there exists a unique solution $y_\varepsilon = y(u_\varepsilon) \in H_0^1(0,1)$ to the state equation (3) corresponding to u_ε. Thus, we have to investigate the behaviour of y_ε and $J(u_\varepsilon, y_\varepsilon)$ if ε tends to $+0$. At the moment, because of (4) and A4, we know that

$$||y_\varepsilon||_C \leq ||y_\varepsilon||_0 \leq C, \tag{9}$$

where here and in the whole following text C denotes a generic constant. Our first lemma gives a regularity statement on the state $y(u) \in H_0^1(0,1)$ for arbitrary $u \in U_{ad}$ and shows that $||y_\varepsilon'||_\infty$ is bounded by a constant not depending on the parameter ε.

LEMMA 1 *(i)* $y(u)' \in L^\infty(0,1) \qquad \forall u \in U_{ad}$
(ii) $\exists C > 0 : ||y_\varepsilon'||_\infty \leq C$.

PROOF: (i) For arbitrarily fixed $u \in U_{ad}$ let $y(u) \in H_0^1(0,1)$ be the corresponding solution of (3). In virtue of the Langrange formula and (6) we have

$$\begin{aligned} y(u)'(x) \int_0^1 a_t(x, y(u)(x), \theta y(u)'(x)) d\theta &= a(x, y(u)(x), y(u)'(x)) \\ - \quad a(x, y(u)(x), 0) &= \int_0^x b(\xi, u(\xi), y(u)(\xi), y(u)'(\xi)) d\xi + c(u) \\ - \quad a(x, y(u)(x), 0), & \quad \text{a.e.} \quad x \in (0,1). \end{aligned}$$

Choosing $\lambda > 0$ such that $|u(x)| + |y(u)(x)| \leq \lambda$ for a.e. $x \in (0,1)$ and applying the assumption A2 (i,ii) and A3 (i), we come to

$$\begin{aligned}
&|y(u)'(x)| \leq \alpha^{-1}\Big(\int_0^1 |b(x,u(x),y(u)(x),y(u)'(x))|dx + |c(u)| \\
&+ |a(x,y(u)(x),0|) \leq \alpha^{-1}(\|h_{1\lambda}\|_1 + \mu_{3\lambda}\|y(u)\|_0^2 + |c(u)| + \mu_{1\lambda}), \\
&\text{a.e.} \quad x \in (0,1),
\end{aligned}$$

and statement (i) is proved.

(ii) Because of (8) and (9) we can take a constant $\lambda \geq 0$ (not depending on $\varepsilon \in (0,\varepsilon_0)$) such that $|u_\varepsilon(x)| + |y_\varepsilon(x)| \leq \lambda$, a.e. $x \in (0,1)$, and (10) considered for $u = u_\varepsilon$ yields

$$|y_\varepsilon'(x)| \leq \alpha^{-1}(\|h_{1\lambda}\|_1 + \mu_{3\lambda}|y_\varepsilon|_0^2 + |c(u_\varepsilon)| + \mu_{1\lambda}), \quad \text{a.e. } x \in (0,1).$$

Furthermore, again using (6), A2 (i) and A3 (i) we obtain

$$\begin{aligned}
|c(u_\varepsilon)| &\leq \int_0^1 |a(x,y_\varepsilon(x),y_\varepsilon'(x))|dx + \int_0^1 |b(x,u_\varepsilon(x),y_\varepsilon(x),y_\varepsilon'(x))|dx \\
&\leq \mu_{1\lambda}(1 + \|y_\varepsilon\|_0) + \|h_{1\lambda}\|_1 + \mu_{3\lambda}\|y_\varepsilon\|_0^2.
\end{aligned}$$

Because of the both last estimations and (9) the claimed second statement (ii) is also proved. ∎

Next we introduce some auxiliary functions by setting

$$\left.\begin{aligned}
a_{1\varepsilon}(x) &= \int_0^1 a_t(x,y_0+\theta(y_\varepsilon-y_0),y_0'+\theta(y_\varepsilon'-y_0'))d\theta, \\
a_{2\varepsilon}(x) &= \int_0^1 a_s(x,y_0+\theta(y_\varepsilon-y_0),y_0'+\theta(y_\varepsilon'-y_0'))d\theta, \\
b_{1\varepsilon}(x) &= \int_0^1 b_t(x,u_\varepsilon,y_0+\theta(y_\varepsilon-y_0),y_0'+\theta(y_\varepsilon'-y_0'))d\theta, \\
b_{2\varepsilon}(x) &= \int_0^1 b_s(x,u_\varepsilon,y_0+\theta(y_\varepsilon-y_0),y_0'+\theta(y_\varepsilon'-y_0'))d\theta, \\
g_{1\varepsilon}(x) &= \int_0^1 g_t(x,u_\varepsilon,y_0+\theta(y_\varepsilon-y_0),y_0'+\theta(y_\varepsilon'-y_0'))d\theta, \\
g_{2\varepsilon}(x) &= \int_0^1 g_s(x,u_\varepsilon,y_0+\theta(y_\varepsilon-y_0),y_0'+\theta(y_\varepsilon'-y_0'))d\theta,
\end{aligned}\right\} \tag{11}$$

and

$$\left.\begin{array}{rcl}\Delta b(x) &=& b(x,u_\varepsilon(x),y_0(x),y_0'(x)) - b^0(x),\\ \Delta g(x) &=& g(x,u_\varepsilon(x),y_0(x),y_0'(x)) - g^0(x),\end{array}\right\} \tag{12}$$

where $x \in (0,1)$. Note that $\Delta b(x) = b(x,u;y_0(x),y_0'(x) - b^0(x)$ if $x \in E$ and $\Delta b(x) = 0$ if $x \in (0,1)\backslash E$ and analogously for Δg. By Lemma 1, assumptions A2 (ii) and A3 (i,ii) we may easily conclude that these functions have the following properties: .

$$\left.\begin{array}{lll} a_{1\varepsilon}, a_{2\varepsilon} \in L^\infty(0,1) & \text{with} & \alpha \le a_{1\varepsilon}(x) \le C,\ \|a_{2\varepsilon}(x)| \le C,\\ & & \text{a.e.}\quad x \in (0,1),\\ b_{1\varepsilon}, g_{1\varepsilon} \in L^2(0,1) & \text{with} & \|b_{1\varepsilon}\|_2, \|g_{1\varepsilon}\|_2 \le C,\\ b_{2\varepsilon}, g_{2\varepsilon} \in L^1(0,1) & \text{with} & \|b_{2\varepsilon}\|_1, \|g_{2\varepsilon}\|_1 \le C,\\ \Delta b, \Delta g \in L^1(0,1) & \text{with} & \|\Delta b\|_1, \|\Delta g\|_1 \in C_\varepsilon \end{array}\right\} \tag{13}$$

Here, to point out once more, C does not depend on $\varepsilon \in (0,\varepsilon_0)$. In the proof of both inequalities of the last line of (13) we have to use additionally the fact that by definition of ω $\xi \in \omega$ is a Lebesgue-point of $b^0, b(\cdot,u,y_0(\cdot),y_0'(\cdot)) \in L^1(0,1)$ and $g^0, g(\cdot,u,y_0(\cdot),y_0'(\cdot)) \in L^1(0,1)$, respectively.

With the functions $a_{1\varepsilon},\ldots,b_{2\varepsilon}$ defined in (11) and Δb defined in (12) we consider the following linear second order boundary value problem :

$$\left.\begin{array}{l} -(d/dx)(a_{1\varepsilon}(x)\rho'(x) + a_{2\varepsilon}(x)\rho(x)) + b_{1\varepsilon}(x)\rho'(x) + b_{2\varepsilon}(x)\rho(x)\\ = -\Delta b(x), x \in (0,1), \qquad \rho(0) = \rho(1) = 0, \end{array}\right\} \tag{14}$$

for which $\rho \in H_0^1(0,1)$ is said to be a solution if

$$\left.\begin{array}{l} \displaystyle\int_0^1 [(a_{1\varepsilon}(x)\rho'(x) + a_{2\varepsilon}(x)\rho(x))z'(x) + (b_{1\varepsilon}(x)\rho'(x) +\\ \displaystyle b_{2\varepsilon}(x)\rho(x))z(x)]dx = -\int_0^1 \Delta b(x)z(x)dx \quad \forall z \in H_0^1(0,1). \end{array}\right\} \tag{15}$$

Taking into account Lemma 1 we find a constant $\lambda > 0$ such that $\|\,|u_\varepsilon|\,\|_\infty + \|(1-\theta)y_0' + \theta y_\varepsilon'\|_\infty \le \lambda\ \forall\theta \in [0,1]$. Thus, in A5 we can substitute $u = u_\varepsilon$ and $y = y_0 + \theta(y_\varepsilon - y_0) = (1-\theta)y_0 + \theta y_\varepsilon, \theta \in [0,1]$. If we intergrate the resulting inequality over $\theta \in [0,1]$, then we see that this boundary value problem is with respect to $\varepsilon \in (0,\varepsilon_0)$ uniformly coercive on $H_0^1(0,1)$. In other words, by the generalized Lax-Milgram-Theorem the boundary value problem (14) has a unique solution $\rho_\varepsilon \in H_0^1(0,1)$. To study ρ_ε as $\varepsilon \longrightarrow +0$ we could try to use the coercitivity of (14). Doing this and considering (4), we would find

$$\nu_\lambda \|\rho_\varepsilon\|_0^2 \le \int_0^1 |\Delta b||\rho_\varepsilon|dx \le \|\Delta b\|_1 \|\rho_\varepsilon\|_0,$$

which by (13) would give

$$\|\rho_\varepsilon\|_0 = 0(\varepsilon^{1-\delta}) \qquad \text{for each } \delta \in (0,1)$$

However, since this estimate is not sufficient for deriving the desired optimality condition we are forced to apply deeper results concerning linear boundary value problems with measurable coefficients. The properties of the coefficients stated above in (13) and certain properties of (14), which essentially are consequences of assumption A5, allow us to apply [7]. There we have proved the existence of a generalized Green function $G_\varepsilon = G_\varepsilon(x,\xi), x,\xi \in (0,1)$, having properties

$$G_\varepsilon, G_{\varepsilon x} \in L^\infty((0,1)\times(0,1)) \text{ with } |G_\varepsilon(x,\xi)|,\ |G_{\varepsilon x}(x,\xi)| \le C, \quad \text{a.e. } x,\xi \in (0,1),$$

and using which the solution $\rho_\varepsilon \in H_0^1(0,1)$ to (14) and its derivative ρ_ε' can be written as

$$\rho_\varepsilon(x) = -\int_0^1 G_\varepsilon(x,\xi)\Delta b(\xi)d\xi,\ \rho_\varepsilon'(x) = -\int_0^1 G_{\varepsilon x}(x,\xi)\Delta b(\xi)d\xi, \text{ a.e. } x \in (0,1).$$

By (13) these formulas yield the crucial estimates

$$|\rho_\varepsilon(x)|, |\rho_\varepsilon'(x)| \le C_\varepsilon, \quad \text{a.e.} \quad x \in (0,1), \tag{16}$$

whose importance will become evident in the next lemma.

LEMMA 2 *It is true that*

$$\rho_\varepsilon(x) = y_\varepsilon(x) - y_0(x), \quad \text{a.e.} \quad x \in (0,1), \tag{17}$$

where $\rho_\varepsilon, y_\varepsilon = y(u_\varepsilon)$ *and* $y_0 = y(u_0)$ *are defined above.*

PROOF: Indeed, by the respective definitions of y_ε and y_0 we have

$$\begin{aligned} 0 &= \int_0^1 (a(x,y_\varepsilon,y_\varepsilon') - a^0(x))z'dx \\ &+ \int_0^1 (b(x,u_\varepsilon,y_\varepsilon,y_\varepsilon') - b(x,u_\varepsilon,u_0,y_0'))zdx \\ &+ \int_0^1 (b(x,u_\varepsilon,y_0,y_0') - b^0(x))zdx \qquad \forall z \in H_0^1(0,1). \end{aligned}$$

Applying Langrange formula to the first two intergrands and using the functions $a_{1\varepsilon},\ldots,b_{2\varepsilon}$ and Δb defined in (11) and (12), respectively, we see that this identity has the form

$$
\begin{aligned}
0 &= \int_0^1 (a_{1\varepsilon}(y'_\varepsilon - y'_0) + a_{2\varepsilon}(y_\varepsilon - y_0)]z'dx \\
&+ \int_0^1 (b_{1\varepsilon}(y'_\varepsilon - y'_0) + b_{2\varepsilon}(y_\varepsilon - y_0)]zdx \\
&+ \int_0^1 \Delta b z dx \qquad \forall z \in H_0^1(0,1).
\end{aligned}
$$

which shows that besides ρ_ε also $y_\varepsilon - y_0$ solves the boundary value problem (14) and , hence , relation (17) is true. ■

According to (16) and (17) we obtain the estimates

$$|y_\varepsilon(x) - y_0(x)|, |y'_\varepsilon(x) - y'_0(x)| \le C_\varepsilon, \quad \text{a.e. } x \in (0,1). \tag{18}$$

Another consequence of (16) and Lemma 2 is that now we can calculate the limits of the functions $a_{1\varepsilon}, \dots, g_{2\varepsilon}$ defined in (11) as $\varepsilon \longrightarrow +0$.

LEMMA 3 *If* $\varepsilon \longrightarrow +0$ *then*

$$a_{1\varepsilon} \longrightarrow a_t^0, \quad a_{2\varepsilon} \longrightarrow a_s^0 \quad \text{in} \quad L^\infty(0,1)$$

$$\left.\begin{array}{ll} b_{1\varepsilon} \longrightarrow b_t^0, & b_{2\varepsilon} \longrightarrow b_s^0, \\ g_{1\varepsilon} \longrightarrow g_t^0, & g_{2\varepsilon} \longrightarrow g_s^0 \end{array}\right\} \text{in} \quad L^1(0,1)$$

PROOF: As examples we prove the first and the last statement; the proofs of the other ones are analogous. Thereby we have to use the assumptions A2 (iii) and A3 (iii). So let $\delta > 0$ be taken from A2 (iii) and $\varepsilon \in (0, \varepsilon_0)$ so small that

$$|y_\varepsilon(x) - y_0(x)|, |y'_\varepsilon(x) - y'_0(x)| \le \delta, \quad \text{a.e. } x \in (0,1).$$

By the definition of $a_{1\varepsilon}$, by (17), A2 (iii) and (16) we obtain

$$
\begin{aligned}
|a_{1\varepsilon}(x) - a_t^0(x)| &\le \int_0^1 |a_t(x, y_0(x) + \theta\rho_\varepsilon(x), y'_0(x) + \theta\rho'_\varepsilon(x)) - a_t^0(x)|d\theta \\
&\le \mu(|\rho_\varepsilon(x)| + |\rho'_\varepsilon(x)|) \le C_\varepsilon, \text{ a.e. } x \in (0,1),
\end{aligned}
$$

which proves already the first assertion. For the proof of the last one we take a $\lambda > 0$ so large that $|u_\varepsilon(x)| \le \lambda$, a.e. $x \in (0,1)$, and choose $\varepsilon \in (0, \varepsilon_0)$ so small that

$$|y_\varepsilon(x) - y_0(x)|, |y'_\varepsilon(x) - y'_0(x)| \le \delta_\lambda \quad \text{a.e. } x \in (0,1),$$

where now δ_λ is taken out from A3 (iii). Then by the same argument and because of the definition of u_ε we find the estimate

$$\int_0^1 |g_{2\varepsilon}(x) - g_s^0(x)|dx \le \int_0^1 |g_{2\varepsilon}(x) - g_s(x, u_\varepsilon(x), y_0(x), y'_0(x))|dx +$$

$$\begin{aligned}
&\int_0^1 |g_s(x, u_\varepsilon(x), y_0(x), y_0'(x) - g_s^0(x)|dx \\
&\leq \int_0^1 \int_0^1 |g_s(x, u_\varepsilon(x), y_0(x) + \theta\rho_\varepsilon(x), y_0'(x) + \theta\rho_\varepsilon'(x)) \\
&- g_s(x, u_\varepsilon(x), y_0(x), y_0'(x))|d\theta dx + \int_E |g_s(x, u, y_0(x), y_0'(x)) - g_s^0(x)|dx \\
&\leq \ \mu_{4\lambda} \int_0^1 (|\rho_\varepsilon(x)| + |\rho_\varepsilon'(x)|)dx + \int_\xi^{\xi+\varepsilon} |g_s(x, u, y_0(x), y_0'(x)) - g_s^0(x)|dx,
\end{aligned}$$

from which the last statement of the lemma follows. ■

4. Pontryagin Minimum Principle

After the preparations in the previous paragraph we are now in a position to prove very easily the sought Pontryagin Minimum Principle for the control problem (1-3). To formulate it in a customary way we first introduce the adjoint state by means of the linear boundary value problem

$$\left.\begin{aligned}
&-(d/dx)(a_t^0(x)z'(x) + b_t^0(x)z(x)) + a_s^0(x)z'(x) + b_s^0(x)z(x) \\
&= (d/dx)g_t^0(x) - g_s^0(x), \qquad x \in (0,1), \qquad z(0) = z(1) = 0,
\end{aligned}\right\} \tag{19}$$

for which $z \in H_0^1(0,1)$ is said to be a solution if

$$\left.\begin{aligned}
&\int_0^1 [(a_t^0(x)z'(x)+b_t^0(x)z(x))y'(x)+(a_s^0(x)z'(x)+b_s^0(x)z(x))y(x)]dx \\
&= -\int_0^1 (g_t^0(x)y'(x) + g_s^0(x)y(x))dx \qquad \forall y \in H_0^1(0,1).
\end{aligned}\right\} \tag{20}$$

Its unique solution $z_0 \in H_0^1(0,1)$ (cf. A5) is called the adjoint state. Our final result given in the theorem below

THEOREM. *Under the assumptions A1–A5 the necessary condition for* $\{u_0, y_0\} \in L_m^\infty(0,1) \times H_0^1(0,1)$ *to be an optimal solution of (1-3) is that*

$$\begin{aligned}
&g(x, u, y_0(x), y_0'(x)) + b(x, u, y_0(x), y_0'(x))z_0(x) \\
&\geq g(x, u_0(x), y_0(x), y_0'(x)) + b(x, u_0(x), y_0(x), y_0'(x))z_0(x) \\
&\forall u \in Q \quad \text{a.e.} \quad x \in (0,1),
\end{aligned}$$

where $z_0 \in H_0^1(0,1)$ *denotes the adjoint state defined by (19).*

PROOF: In order to prove the theorem we consider the difference

$$\begin{aligned}0 &\le J(u_\varepsilon,y_\varepsilon)-J(u_0,y_0)\\ &= \int_0^1 (g(x,u_\varepsilon(x),y_\varepsilon(x),y'_\varepsilon(x))-g(x,u_\varepsilon(x),y_0(x),y'_0(x)))dx\\ &+ \int_0^1 (g(x,u_\varepsilon(x),y_0(x),y'_0(x))-g^0(x))dx,\end{aligned}$$

which by means of the functions $g_{1\varepsilon}, g_{2\varepsilon}$ and Δg defined in (11) and (12), respectively, can be rewritten in

$$0 \le \int_E \Delta g(x)dx + \int_0^1 (g_{1\varepsilon}(x)\rho'_\varepsilon(x) + g_{2\varepsilon}(x)\rho_\varepsilon(x))dx. \tag{21}$$

In the definition of ρ_ε , that means in (15) with $\rho=\rho_\varepsilon$, we take $z=z_0$ and in the definition of z_0, that means in (20) with $z=z_0$, we take $y=\rho_\varepsilon$. Then we using the both resulting relations we obtain inequality (21) in the form

$$0 \le \int_E (\Delta g(x) + \Delta b(x)z_0(x))dx + J(\varepsilon), \tag{22}$$

where

$$\begin{aligned}J(\varepsilon) &= \int_0^1 (g_{1\varepsilon}-g_t^0)\rho'_\varepsilon dx + \int_0^1 (g_{2\varepsilon}-g_s^0)\rho_\varepsilon dx\\ &+ \int_0^1 (a_{1\varepsilon}-a_t^0)z'_0\rho'_\varepsilon dx + \int_0^1 (a_{2\varepsilon}-a_s^0)z'_0\rho_\varepsilon dx\\ &+ \int_0^1 (b_{1\varepsilon}-b_t^0)z_0\rho'_\varepsilon dx + \int_0^1 (b_{2\varepsilon}-b_s^0)z_0\rho_\varepsilon dx.\end{aligned}$$

Now, because of (16) and (4) we have

$$\begin{aligned}|\varepsilon^{-1}J(\varepsilon)| &\le C[\|g_{1\varepsilon}-g_t^0\|_1 + \|g_{2\varepsilon}-g_s^0\|_1\\ &+ (\|a_{1\varepsilon}-a_t^0\|_\infty + \|a_{2\varepsilon}-a_s^0\|_\infty)\|z'_0\|_1\\ &+ (\|b_{1\varepsilon}-b_t^0\|_1 + \|b_{2\varepsilon}-b_s^0\|_1)\|z_0\|_0],\end{aligned}$$

which , by Lemma 3 , implies

$$J(\varepsilon)=0(\varepsilon) \qquad \text{as} \qquad \varepsilon \longrightarrow +0.$$

Therefore, if we divide inequality (22) by $\varepsilon \in (0,\varepsilon_0)$ and if afterwards ε tends to $+0$, then we obtain

$$\begin{aligned}0 &\le \Delta g(\xi) + \Delta b(\xi)z_0(\xi)\\ &= g(\xi,u,y_0(\xi),y'_0(\xi)) - g^0(\xi) + (b(\xi,u,y_0(\xi),y'_0(\xi)) - b^0(\xi))z_0(\xi).\end{aligned}$$

Since at the beginning of section 3 both $u \in Q$ and $\xi \in \omega \subset (0,1)$ were taken arbitrarily the theorem is proved. ■

Clearly, if the control set $Q \subset R^m$ is convex and if for a.e. $x \in (0,1)$ the functions $b(x,\cdot,\cdot,\cdot)$ and $g(x,\cdot,\cdot,\cdot)$ have their respective gradients b_r and g_r continuous on $Q \times R \times R$, then for the solution $\{u_0, y_0\} \in L^\infty_m(0,1) \times H^1_0(0,1)$ of the control problem (1-3) the above theorem yields the linearized (weak) Pontryagin Minimum Principle

$$(g^0_r(x) + b^0_r(x)z_0(x), u - u_0(x))_{R^m} \geq 0 \quad \forall u \in Q, \quad \text{a.e. } x \in (0,1),$$

where $(\cdot,\cdot)_{R^m}$ denotes the scalar product in R^m. In [6] optimality conditions of such a type has been proved for both unconstrained and constrained control problems with a quasilinear second order ordinary differential equation, whose leading coefficient may also depend on the control $u \in U_{ad}$. We remark that the functional analytic method used there cannot be applied to the control problem considered above. In a forthcoming paper we shall consider the optimal control problem (1-3) with additional integral constraints.

Book [11] contains illustrative examples demonstrating the use and advantages of the result here presented.

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Zasada minimum Pontriagina dla mocno nieliniowego zadania dwubrzegowego

W pracy wyprowadzono warunek konieczny optymalności w postaci zasady minimum Pontriagina dla zadania sterowania optymalnego z mocno nieliniowym równaniem różniczkowym cząstkowym drugiego rzędu i z jednorodnymi warunkami brzegowymi Dirichleta. W dowodzie użyto standardowego podejścia z igłową wariancją sterowania oraz uogólnionej funkcji Greena dla rozwiązania liniowego zadania dwubrzegowego.

Принцип минимума Понтрягина для сильно нелинейной двухграничной задачи

В работе представлено необходимое условие оптимальности в виде принципа минимума Понтрягина для задачи оптимального управления

с сильно нелинейным дифференциальным уравнением в частных производных второго порядка и однородными граничнымм условиями Дирихле. В доказательстве используется стандартный подход с игольчатой дисперсией управления и обобщенная функция Грина для решения линейной двухграничной задачи.