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Calculation of the integral squared error for large dynamic systems with many commensurate delays [1]

by

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In the paper, an analytic solution is presented for the problem of calculation of $\int \varepsilon^2 dt$ for large dynamic systems with many time delays. The system of equations used for determination of unknown polynomials $C_i(s)$, $i = 0, 1, \ldots, n$ is underdetermined. With the use of certain symmetry properties this difficulty was overcome and the unknowns C_i have been uniquely determined by analytic formulae, which are noticeable by their simplicity and the lowest possible order. The method proposed here gives analytic formulae and solution algorithms much simpler than those proposed by Penrose [2]. An example is given which makes it possible to compare both methods. The results obtained may be used to compute optimal regulator settings in parametric optimization.

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Introduction

The calculation of the integral squared dynamic error for systems without time delay was generally worked out, and the analytic formulae were obtained in many papers (Feldbaum 1957), (James, Nichols, Philips 1974), (Newton, Gould, Kaizer 1957). The analogous problem for the systems with one delay was solved in (Górecki H., Popek L. 1982), (Grabowski P. 1983), (Walton K., Górecki H. 1984).

The present work is a generalization for the systems with many commensurate delays. At the end of the paper some practical examples are given.

The analytic formula for integral squared error of the system enables establishing of the optimal values of the parameters of controllers.

1. Statement of the problem

Let us consider the integral

$$J = \int_0^\infty \varepsilon^2(t)dt \tag{1}$$

where $\varepsilon(t)$ is the transient error of stable system. By virtue of Parseval's theorem the integral (1) can be calculated from the formula

$$J = \frac{1}{2\Pi j} \int_{-j\infty}^{j\infty} E(s)E(-s)ds \tag{2}$$

where $E(s)$ denotes the Laplace transform of $\varepsilon(t)$. Let us assume that

$$E(s) = \frac{B(s)}{A(s)} \tag{3}$$

where

$$\left.\begin{array}{rcl} A(s) & = & A_0(s) + A_1(s)e^{-s\tau} + \ldots + A_n(s)e^{-ns\tau} \\ A_0(s) & \not\equiv & 0 \\ A_n(s) & \not\equiv & 0 \\ A_i(s) & \not\equiv & 0 \quad 1 \le i \le n-1 \quad \text{for at least one "i"} \\ B(s) & = & B_0(s) + B_1(s)e^{-s\tau} + \ldots + B_n(s)e^{-ns\tau} \end{array}\right\} \tag{4}$$

$A_0(s), A_1(s), \ldots, A_n(s), B_0(s), B_1(s), \ldots, B_n(s)$ are polynomials of the operator "s" and $\tau > 0$ is a real and positive number representing time delay. We have made the basic assumption about asymptotic stability of the system. In consequence, all the poles of $E(s)$ or equivalently all zeroes of $A(s)$ lie in the left half of the complex s-plane. This fact is the corner-stone of the presented method.

The substitution of (3) into (2) gives

$$J = \frac{1}{2\Pi j}\int_{-j\infty}^{j\infty} \frac{B(s)B(-s)}{A(s)A(-s)}ds \tag{5}$$

Using partial fractions the integrand of (5) may be separated into two parts, namely

$$\begin{aligned} J &= \frac{1}{2\Pi j}\int_{-j\infty}^{j\infty} \frac{B(s)B(-s)}{A(s)A(-s)}ds = \frac{1}{2\Pi j}\int_{-j\infty}^{j\infty} \frac{C(s)}{\mu(s)A(s)}ds + \\ &+ \frac{1}{2\Pi j}\int_{-j\infty}^{j\infty} \frac{C(-s)}{\mu(s)A(-s)}ds \end{aligned} \tag{6}$$

Denoting

$$\left.\begin{aligned} e^{-s\tau} &= z \\ B(-s) &= \bar{B} \\ A(-s) &= \bar{A} \\ C(-s) &= \bar{C} \end{aligned}\right\} \qquad (7)^2$$

we express our problem as follows.

We want to split the product into the sum

$$\begin{aligned} E\cdot\bar{E} &= \frac{B_0 + B_1 z + \ldots + B_n z^n}{A_0 + A_1 z + \ldots + A_n z^n} \cdot \frac{\bar{B}_0 + \bar{B}_1 z^{-1} + \ldots + \bar{B}_n z^{-n}}{\bar{A}_0 + \bar{A}_1 z^{-1} + \ldots + \bar{A}_n z^{-n}} \\ &= \frac{C_0 + C_1 z + \ldots + C_n z^n}{\mu[A_0 + A_1 z + \ldots + A_n z^n]} + \frac{\bar{C}_0 + \bar{C}_1 z^{-1} + \ldots + \bar{C}_n z^{-n}}{\mu[\bar{A}_0 + \bar{A}_1 z^{-1} + \ldots + \bar{A}_n z^{-n}]} \end{aligned} \tag{8}$$

Comparing the coefficient of the same powers of "z" in both sides of equation (8) we obtain $(2n+1)$ equations for the $(2n+2)$ unknowns $C_0, C_1, \ldots, C_n, \bar{C}_0, \bar{C}_1,$

[2] We omit the dependence on "s" for simplicity.

$\ldots, \bar{C}_n$, so the solution of this system is not unique, the system is underdetermined.

$$A \cdot C =$$

$$= \begin{bmatrix} A_0 & 0 & \ldots & 0 & 0 & \bar{A}_n & 0 & \ldots & 0 & 0 \\ A_1 & A_0 & \ldots & 0 & 0 & \bar{A}_{n-1} & \bar{A}_n & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n-1} & A_{n-2} & \ldots & A_0 & 0 & \bar{A}_1 & \bar{A}_1 & \ldots & \bar{A}_n & 0 \\ A_n & A_{n-1} & \ldots & A_1 & A_0 & \bar{A}_0 & \bar{A}_1 & \ldots & \bar{A}_{n-1} & \bar{A}_n \\ 0 & A_n & \ldots & A_2 & A_1 & 0 & \bar{A}_0 & \ldots & \bar{A}_{n-2} & \bar{A}_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & A_n & A_{n-1} & 0 & 0 & \ldots & \bar{A}_0 & \bar{A}_1 \\ 0 & 0 & \ldots & 0 & A_n & 0 & 0 & \ldots & 0 & \bar{A}_0 \end{bmatrix} \cdot \begin{bmatrix} \bar{C}_n \\ \bar{C}_{n-1} \\ \vdots \\ \bar{C}_1 \\ \bar{C}_0 \\ C_0 \\ C_1 \\ \vdots \\ C_{n-1} \\ C_n \end{bmatrix} =$$

$$= \mu \cdot \begin{bmatrix} B_0 & 0 & \ldots & 0 & 0 \\ B_1 & B_0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{n-1} & B_{n-2} & \ldots & B_0 & 0 \\ B_n & B_{n-1} & \ldots & B_1 & B_0 \\ 0 & B_n & \ldots & B_2 & B_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & B_n & B_{n-1} \\ 0 & 0 & \ldots & 0 & B_n \end{bmatrix} \cdot \begin{bmatrix} \bar{B}_n \\ \bar{B}_{n-1} \\ \vdots \\ \bar{B}_1 \\ \bar{B}_0 \end{bmatrix} \tag{9}$$

We will choose the solution which has the property of symmetry. Observe that in order that the solutions for unknowns $C_0, \ldots, C_n$ be polynomials and not rational functions of the operator "s" we have introduced the unknown polynomial $\mu(s)$.

Denoting in the equation (9) by

$$\begin{bmatrix} \bar{D}_n \\ \bar{D}_{n-1} \\ \vdots \\ \bar{D}_1 \\ \bar{D}_0 \\ D_0 \\ D_1 \\ \vdots \\ D_{n-1} \\ D_n \end{bmatrix} = \begin{bmatrix} B_0 & 0 & \dots 0 & 0 \\ B_1 & B_0 & \dots 0 & 0 \\ \vdots & \vdots & \ddots \vdots & \vdots \\ B_{n-1} & B_{n-2} & \dots B_0 & 0 \\ B_n & B_{n-1} & \dots B_1 & B_0 \\ 0 & B_n & \dots B_2 & B_1 \\ \vdots & \vdots & \ddots \vdots & \vdots \\ 0 & 0 & \dots B_n & B_{n-1} \\ 0 & 0 & \dots 0 & B_n \end{bmatrix} \cdot \begin{bmatrix} \bar{B}_n \\ \bar{B}_{n-1} \\ \vdots \\ \bar{B}_1 \\ \bar{B}_0 \end{bmatrix} \tag{10}$$

we can write the equation (9) as follows

$$AC = \mu D \tag{11}$$

From the linear equation (11) we deduce that the solution for C has the following general form

$$C = MD \tag{12}$$

where the matrix M can be determined after the substitution of (12) into (11)

$$(AM - \mu E)D = 0 \tag{13}$$

As the vector D is not a zero-vector, we conclude that

$$AM = \mu E \tag{14}$$

where E denotes a unit matrix.

For establishing the structure of the unknown matrix M we will use a special matrix $\exists$

$$\exists = \begin{bmatrix} 0 & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & 0 \end{bmatrix} \tag{15}$$

which has the following properties:

1^0 $\exists\exists = E$ (16)

This corresponds to the square root of the unity matrix.

2^0 Premultiplication of the matrix X by matrix $\exists$ corresponds to the change of the order of rows of the matrix X.

3^0 Postmultiplication of the matrix X by matrix $\exists$ corresponds to the change of the order of columns of the matrix X.

4^0 Premultiplication and postmultiplication of the X by matrix $\exists$ gives as the final result the angle Π of turn of the matrix X, which we denote

$$\exists X \exists = X_{\Pi} \tag{17}$$

THEOREM 1 *Due to the special structure of the matrix A in equation (9) the following relation holds*

$$\bar{A}_{\Pi} = A \tag{18}$$

This means that the matrix A is equal to the matrix $\bar{A}_{\Pi}$ which is turned by the angle Π, and in which the argument "s" is substituted by $(-s)$.

The proof may done by inspection of the matrix eq. (9).

THEOREM 2 *The same property (18) holds for the unknown matrix M.*

$$\bar{M}_{\Pi} = M \tag{18'}$$

PROOF: We start with the relation (14)

$$AM = \mu E$$

We apply the operation (17) to the both sides of the equation (14)

$$\exists AM \exists = \exists[\mu E]\exists = [E\mu]_{\Pi} = \mu E \tag{19}$$

because μ is the polynomial, and the unit matrix turned by the angle Π is again the unit matrix.

The left-handed side of equation (19) can be written as follows

$$\exists AM \exists = \exists AEM \exists \tag{20}$$

Using the identity (16) we can replace the matrix E by the product $\exists\exists$ and we obtain that

$$\exists AM \exists = \exists AEM \exists = \exists A \exists\exists M \exists \tag{21}$$

Now we apply the identity (17) to the relation (21) and using (19) we obtain finally that

$$A_\Pi M_\Pi = \mu E \tag{22}$$

Replacing the argument "s" in the relation (22) by $(-s)$ yields the relation

$$A_\Pi M_\Pi = \bar{\mu}\bar{E} = \mu E \tag{23}$$

if necessary condition

$$\bar{\mu} = \mu \tag{23'}$$

holds (see App.1.)
Comparing (14) with (23) we have that

$$A_\Pi M_\Pi = AM \tag{24}$$

Using the relation (18) in (24) we obtain

$$A\bar{M}_\Pi = AM \tag{25}$$

Hence we obtain the final result

$$\bar{M}_\Pi = M \tag{26}$$

Returning to the equation (9) we observe that the matrix A has the dimensions $(2n+1) \times (2n+2)$, and its multiplication by M is possible if the dimensions of M are $(2n+2) \times (2n+1)$, and so we conclude that the matrix E has the dimensions $(2n+1) \times (2n+1)$.

Assuming that the matrix N has the form

$$M = \begin{bmatrix} M_\alpha & M_\varepsilon & M_\gamma \\ M_\beta & M_\eta & M_\delta \end{bmatrix} \tag{27}$$

and taking into account the equality (26) we obtain

$$\bar{M}_\pi = \begin{bmatrix} \bar{M}_{\delta\pi} & \bar{M}_{\eta\pi} & \bar{M}_{\beta\pi} \\ \bar{M}_{\gamma\pi} & \bar{M}_{\varepsilon\pi} & \bar{M}_{\alpha\pi} \end{bmatrix} = \begin{bmatrix} M_\alpha & M_\varepsilon & M_\gamma \\ M_\beta & M_\eta & M_\delta \end{bmatrix} \tag{28}$$

Comparing submatrices we have

$$\bar{M}_{\delta\pi} = M_\alpha \tag{29}$$

$$\bar{M}_{\gamma\pi} = M_\beta \tag{30}$$

$$\bar{M}_{\eta\pi} = M_{\varepsilon} \tag{31}$$

$$\bar{M}_{\varepsilon\pi} = M_{\eta} \tag{32}$$

$$\bar{M}_{\beta\pi} = M_{\gamma} \tag{33}$$

$$\bar{M}_{\alpha\pi} = M_{\delta} \tag{34}$$

Using the relations (33) and (34) in (27) we obtain the final form of the matrix

$$M = \begin{bmatrix} M_\alpha & M_\varepsilon & \bar{M}_{\beta\pi} \\ M_\beta & M_\eta & \bar{M}_{\alpha\pi} \end{bmatrix} \tag{35}$$

and the equation (14) can be written in the form

$$\begin{bmatrix}
A_0 & 0 & \dots & 0 & 0 & \bar{A}_n & 0 & \dots & 0 & 0 \\
A_1 & A_0 & \dots & 0 & 0 & \bar{A}_{n-1} & \bar{A}_n & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
A_{n-1} & A_{n-2} & \dots & A_0 & 0 & \bar{A}_1 & \bar{A}_1 & \dots & \bar{A}_n & 0 \\
A_n & A_{n-1} & \dots & A_1 & A_0 & \bar{A}_0 & \bar{A}_1 & \dots & \bar{A}_{n-1} & \bar{A}_n \\
0 & A_n & \dots & A_2 & A_1 & 0 & \bar{A}_0 & \dots & \bar{A}_{n-2} & \bar{A}_{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & A_n & A_{n-1} & 0 & 0 & \dots & \bar{A}_0 & \bar{A}_1 \\
0 & 0 & \dots & 0 & A_n & 0 & 0 & \dots & 0 & \bar{A}_0
\end{bmatrix} \cdot$$

$$\cdot \left[\begin{array}{cccc|c|cccc}
\alpha_{01} & \alpha_{02} & \dots & \alpha_{0n} & \varepsilon_0 & \bar{\beta}_{nn} & \dots & \bar{\beta}_{n2} & \bar{\beta}_{n1} \\
\alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} & \varepsilon_1 & \bar{\beta}_{n-1,n} & \dots & \bar{\beta}_{n-1,2} & \bar{\beta}_{n-1,1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\alpha_{n-1,1} & \alpha_{n-1,2} & \dots & \alpha_{n-1,n} & \varepsilon_{n-1} & \bar{\beta}_{1n} & \dots & \bar{\beta}_{1,2} & \bar{\beta}_{1,1} \\
\alpha_{n,1} & \alpha_{n,2} & \dots & \alpha_{n,n} & \varepsilon_n & \bar{\beta}_{0n} & \dots & \bar{\beta}_{02} & \bar{\beta}_{01} \\
\beta_{01} & \beta_{02} & \dots & \beta_{0n} & \eta_0 & \bar{\alpha}_{nn} & \dots & \bar{\alpha}_{n2} & \bar{\alpha}_{n1} \\
\beta_{11} & \beta_{12} & \dots & \beta_{1n} & \eta_1 & \bar{\alpha}_{n-1,n} & \dots & \bar{\alpha}_{n-1,2} & \bar{\alpha}_{n-1,1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\beta_{n-1,1} & \beta_{n-1,2} & \dots & \beta_{n-1,n} & \eta_{n-1} & \bar{\alpha}_{1n} & \dots & \bar{\alpha}_{1,2} & \bar{\alpha}_{1,1} \\
\beta_{n,1} & \beta_{n,2} & \dots & \beta_{n,n} & \eta_n & \bar{\alpha}_{0n} & \dots & \bar{\alpha}_{02} & \bar{\alpha}_{01}
\end{array}\right] =$$

$$= \mu \left[\begin{array}{cccc|c|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{36}$$

The equation (12) can be now written in the form

$$\begin{bmatrix} \bar{C}_n \\ \bar{C}_{n-1} \\ \vdots \\ \bar{C}_1 \\ \bar{C}_0 \\ C_0 \\ C_1 \\ \vdots \\ C_{n-1} \\ C_n \end{bmatrix} = \tag{37}$$

$$= \left[\begin{array}{cccc|c|cccc} \alpha_{01} & \alpha_{02} & \dots & \alpha_{0n} & \varepsilon_0 & \bar{\beta}_{nn} & \dots & \bar{\beta}_{n2} & \bar{\beta}_{n1} \\ \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} & \varepsilon_1 & \bar{\beta}_{n-1,n} & \dots & \bar{\beta}_{n-1,2} & \bar{\beta}_{n-1,1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha_{n-1,1} & \alpha_{n-1,2} & \dots & \alpha_{n-1,n} & \varepsilon_{n-1} & \bar{\beta}_{1n} & \dots & \bar{\beta}_{1,2} & \bar{\beta}_{1,1} \\ \alpha_{n,1} & \alpha_{n,2} & \dots & \alpha_{n,n} & \varepsilon_n & \bar{\beta}_{0n} & \dots & \bar{\beta}_{02} & \bar{\beta}_{01} \\ \hline \beta_{01} & \beta_{02} & \dots & \beta_{0n} & \eta_0 & \bar{\alpha}_{nn} & \dots & \bar{\alpha}_{n2} & \bar{\alpha}_{n1} \\ \beta_{11} & \beta_{12} & \dots & \beta_{1n} & \eta_1 & \bar{\alpha}_{n-1,n} & \dots & \bar{\alpha}_{n-1,2} & \bar{\alpha}_{n-1,1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \beta_{n-1,1} & \beta_{n-1,2} & \dots & \beta_{n-1,n} & \eta_{n-1} & \bar{\alpha}_{1n} & \dots & \bar{\alpha}_{1,2} & \bar{\alpha}_{1,1} \\ \beta_{n,1} & \beta_{n,2} & \dots & \beta_{n,n} & \eta_n & \bar{\alpha}_{0n} & \dots & \bar{\alpha}_{02} & \bar{\alpha}_{01} \end{array}\right] \begin{bmatrix} \bar{D}_n \\ \bar{D}_{n-1} \\ \vdots \\ \bar{D}_1 \\ \bar{D}_0 \\ D_0 \\ D_1 \\ \vdots \\ D_{n-1} \\ D_n \end{bmatrix}$$

From the set of equations (37) we need only the subset for determining $C_0, C_1, \dots, C_n$, because the unknowns $\bar{C}_0, \bar{C}_1, \dots, \bar{C}_n$ can be determined from $C_0, C_1, \dots, C_n$ by replacing the operator "s" by $(-s)$.

The unknowns $C_0, C_1, \ldots, C_n$ can be calculated form the lower subset of the set (37), namely

$$\begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-1} \\ C_n \end{bmatrix} = \tag{38}$$

$$= \begin{bmatrix} \beta_{01} & \beta_{02} & \ldots & \beta_{0n} & \eta_0 & \bar{\alpha}_{nn} & \ldots & \bar{\alpha}_{n2} & \bar{\alpha}_{n1} \\ \beta_{11} & \beta_{12} & \ldots & \beta_{1n} & \eta_1 & \bar{\alpha}_{n-1,n} & \ldots & \bar{\alpha}_{n-1,2} & \bar{\alpha}_{n-1,1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \beta_{n-1,1} & \beta_{n-1,2} & \ldots & \beta_{n-1,n} & \eta_{n-1} & \bar{\alpha}_{1n} & \ldots & \bar{\alpha}_{1,2} & \bar{\alpha}_{1,1} \\ \beta_{n,1} & \beta_{n,2} & \ldots & \beta_{n,n} & \eta_n & \bar{\alpha}_{0n} & \ldots & \bar{\alpha}_{02} & \bar{\alpha}_{01} \end{bmatrix} \begin{bmatrix} \bar{D}_n \\ \bar{D}_{n-1} \\ \vdots \\ \bar{D}_1 \\ \bar{D}_0 \\ D_0 \\ D_1 \\ \vdots \\ D_{n-1} \\ D_n \end{bmatrix}$$

In order to determine the elements of the matrix M we consider the equation (36). At the first stage we determine the elements of the matrices M_α and M_β (see (35)).

We can write two matrix equations as follows :

$$\begin{bmatrix} A_0 & 0 & \ldots & 0 & 0 \\ A_1 & A_0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n-1} & A_{n-2} & \ldots & A_0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{01} & \alpha_{02} & \ldots & \alpha_{0n} \\ \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n-1,1} & \alpha_{n-1,2} & \ldots & \alpha_{n-1,n} \\ \alpha_{n,1} & \alpha_{n,2} & \ldots & \alpha_{n,n} \end{bmatrix} +$$

$$+ \begin{bmatrix} \bar{A}_n & 0 & \ldots & 0 & 0 \\ \bar{A}_{n-1} & \bar{A}_n & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{A}_1 & \bar{A}_2 & \ldots & \bar{A}_n & 0 \end{bmatrix} \begin{bmatrix} \beta_{01} & \beta_{02} & \ldots & \beta_{0n} \\ \beta_{11} & \beta_{12} & \ldots & \beta_{1n} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1,1} & \beta_{n-1,2} & \ldots & \beta_{n-1,n} \\ \beta_{n,1} & \beta_{n,2} & \ldots & \beta_{n,n} \end{bmatrix} =$$

$$= \mu \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{39}$$

$$\begin{bmatrix} A_n & A_{n-1} \dots A_1 & A_0 \\ 0 & A_n \dots A_2 & A_1 \\ \vdots & \vdots \ddots \vdots & \vdots \\ 0 & 0 \dots A_n & A_{n-1} \\ 0 & 0 \dots 0 & A_n \end{bmatrix} \begin{bmatrix} \alpha_{01} & \alpha_{02} & \dots & \alpha_{0n} \\ \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n-1,1} & \alpha_{n-1,2} & \dots & \alpha_{n-1,n} \\ \alpha_{n,1} & \alpha_{n,2} & \dots & \alpha_{n,n} \end{bmatrix} +$$

$$+ \begin{bmatrix} \bar{A}_0 & \bar{A}_1 \dots \bar{A}_{n-1} & \bar{A}_n \\ 0 & \bar{A}_0 \dots \bar{A}_{n-2} & \bar{A}_{n-1} \\ \vdots & \vdots \ddots \vdots & \vdots \\ 0 & 0 \dots \bar{A}_0 & \bar{A}_1 \\ 0 & 0 \dots 0 & \bar{A}_0 \end{bmatrix} \begin{bmatrix} \beta_{01} & \beta_{02} & \dots & \beta_{0n} \\ \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1,1} & \beta_{n-1,2} & \dots & \beta_{n-1,n} \\ \beta_{n,1} & \beta_{n,2} & \dots & \beta_{n,n} \end{bmatrix} =$$

$$= \mu \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{40}$$

Now, to reduce the number of calculations we assume that

$$\begin{bmatrix} \alpha_{01} & \alpha_{02} & \dots & \alpha_{0n} \\ \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n-1,1} & \alpha_{n-1,2} & \dots & \alpha_{n-1,n} \\ \alpha_{n,1} & \alpha_{n,2} & \dots & \alpha_{n,n} \end{bmatrix} = \begin{bmatrix} \bar{A}_0 & \bar{A}_1 \dots \bar{A}_{n-1} & \bar{A}_n \\ 0 & \bar{A}_0 \dots \bar{A}_{n-2} & \bar{A}_{n-1} \\ \vdots & \vdots \ddots \vdots & \vdots \\ 0 & 0 \dots \bar{A}_0 & \bar{A}_1 \\ 0 & 0 \dots 0 & \bar{A}_0 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \varphi_{01} & \varphi_{02} & \dots & \varphi_{0n} \\ \varphi_{11} & \varphi_{12} & \dots & \varphi_{1n} \\ \vdots & \vdots & & \vdots \\ \varphi_{n-1,1} & \varphi_{n-1,2} & \dots & \varphi_{n-1,n} \\ \varphi_{n,1} & \varphi_{n,2} & \dots & \varphi_{n,n} \end{bmatrix} \tag{41}$$

where the matrix M_φ is to be determined. The substitution of (41) into (40) gives :

$$\begin{bmatrix} A_n & A_{n-1} & \dots & A_1 & A_0 \\ 0 & A_n & \dots & A_2 & A_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_n & A_{n-1} \\ 0 & 0 & \dots & 0 & A_n \end{bmatrix} \cdot \begin{bmatrix} \bar{A}_0 & \bar{A}_1 & \dots & \bar{A}_{n-1} & \bar{A}_n \\ 0 & \bar{A}_0 & \dots & \bar{A}_{n-2} & \bar{A}_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \bar{A}_0 & \bar{A}_1 \\ 0 & 0 & \dots & 0 & \bar{A}_0 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \varphi_{01} & \varphi_{02} & \dots & \varphi_{0n} \\ \varphi_{11} & \varphi_{12} & \dots & \varphi_{1n} \\ \vdots & \vdots & & \vdots \\ \varphi_{n-1,1} & \varphi_{n-1,2} & \dots & \varphi_{n-1,n} \\ \varphi_{n,1} & \varphi_{n,2} & \dots & \varphi_{n,n} \end{bmatrix} + \begin{bmatrix} \bar{A}_0 & \bar{A}_1 & \dots & \bar{A}_{n-1} & \bar{A}_n \\ 0 & \bar{A}_0 & \dots & \bar{A}_{n-2} & \bar{A}_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \bar{A}_0 & \bar{A}_1 \\ 0 & 0 & \dots & 0 & \bar{A}_0 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \beta_{01} & \beta_{02} & \dots & \beta_{0n} \\ \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1,1} & \beta_{n-1,2} & \dots & \beta_{n-1,n} \\ \beta_{n,1} & \beta_{n,2} & \dots & \beta_{n,n} \end{bmatrix} = \mu \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} \tag{42}$$

Taking into account that the first two matrices in (42) are permutable and different from a zero matrix (see App. 2) we can obtain that

$$\begin{bmatrix} \beta_{01} & \beta_{02} & \dots & \beta_{0n} \\ \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1,1} & \beta_{n-1,2} & \dots & \beta_{n-1,n} \\ \beta_{n,1} & \beta_{n,2} & \dots & \beta_{n,n} \end{bmatrix} =$$

$$= - \begin{bmatrix} A_n & A_{n-1} & \dots & A_1 & A_0 \\ 0 & A_n & \dots & A_2 & A_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_n & A_{n-1} \\ 0 & 0 & \dots & 0 & A_n \end{bmatrix} \begin{bmatrix} \varphi_{01} & \varphi_{02} & \dots & \varphi_{0n} \\ \varphi_{11} & \varphi_{12} & \dots & \varphi_{1n} \\ \vdots & \vdots & & \vdots \\ \varphi_{n-1,1} & \varphi_{n-1,2} & \dots & \varphi_{n-1,n} \\ \varphi_{n,1} & \varphi_{n,2} & \dots & \varphi_{n,n} \end{bmatrix} \tag{43}$$

In such a way two unknown matrices M_α and M_β are replaced by one unknown matrix M_φ.

Returning to the equation (39) it is evident that due to the fact that the last columns in the matrices A and $\bar{A}$ have only zero elements, the elements of the last rows of the matrices M_α and M_β have no influence on the final result.

Direct observation of (41) and (43) gives that

$$\left.\begin{aligned} \alpha_{nj} &= \bar{A}_0 \varphi_{nj} \\ \beta_{nj} &= -A_n \varphi_{nj} \end{aligned}\right\} \quad \text{for } j = 1 \text{ to } n \tag{44}$$

In every column of the matrix M we have one degree of freedom because the number of equations is $(2n+1)$, and the number of the elements in the column is $(2n+2)$ eq. (36). For that reason in every column of the matrix M we can arbitrarily choose the value of one element.

The simplest is to assume that $\varphi_{nj} = 0$ for $j = 1, \ldots, n$ because then there is $\alpha_{nj} = \beta_{nj} = 0$ for $j = 1, \ldots, n$.

By doing this we have equations (45), (46) and (47) in which we omit the last column in the matrices A_α and $\bar{A}_\beta$, and the last rows in M_α and M_β instead of equations (39), (41) and (43):

$$\begin{bmatrix} A_0 & 0 & \ldots & 0 \\ A_1 & A_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1} & A_{n-2} & \ldots & A_0 \end{bmatrix} \begin{bmatrix} \alpha_{01} & \alpha_{02} & \ldots & \alpha_{0n} \\ \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n-1,1} & \alpha_{n-1,2} & \ldots & \alpha_{n-1,n} \end{bmatrix} +$$

$$+ \begin{bmatrix} \bar{A}_n & 0 & \ldots & 0 \\ \bar{A}_{n-1} & \bar{A}_n & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_1 & \bar{A}_2 & \ldots & \bar{A}_n \end{bmatrix} \begin{bmatrix} \beta_{01} & \beta_{02} & \ldots & \beta_{0n} \\ \beta_{11} & \beta_{12} & \ldots & \beta_{1n} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1,1} & \beta_{n-1,2} & \ldots & \beta_{n-1,n} \end{bmatrix} =$$

$$= \mu \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \tag{45}$$

$$\begin{bmatrix} \alpha_{01} & \alpha_{02} & \dots & \alpha_{0n} \\ \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n-1,1} & \alpha_{n-1,2} & \dots & \alpha_{n-1,n} \end{bmatrix} =$$

$$= \begin{bmatrix} \bar{A}_0 & \bar{A}_1 & \dots & \bar{A}_{n-1} \\ 0 & \bar{A}_0 & \dots & \bar{A}_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{A}_0 \end{bmatrix} \begin{bmatrix} \varphi_{01} & \varphi_{02} & \dots & \varphi_{0n} \\ \varphi_{11} & \varphi_{12} & \dots & \varphi_{1n} \\ \vdots & \vdots & & \vdots \\ \varphi_{n-1,1} & \varphi_{n-1,2} & \dots & \varphi_{n-1,n} \end{bmatrix} \tag{46}$$

$$\begin{bmatrix} \beta_{01} & \beta_{02} & \dots & \beta_{0n} \\ \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1,1} & \beta_{n-1,2} & \dots & \beta_{n-1,n} \end{bmatrix} =$$

$$= - \begin{bmatrix} A_n & A_{n-1} & \dots & A_1 \\ 0 & A_n & \dots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{bmatrix} \begin{bmatrix} \varphi_{01} & \varphi_{02} & \dots & \varphi_{0n} \\ \varphi_{11} & \varphi_{12} & \dots & \varphi_{1n} \\ \vdots & \vdots & & \vdots \\ \varphi_{n-1,1} & \varphi_{n-1,2} & \dots & \varphi_{n-1,n} \end{bmatrix} \tag{47}$$

The substitution of (46) and (47) into (45) gives the set of equations for calculating the elements of the unknown matrix M_φ:

$$[M_1 \cdot M_2 - M_3 \cdot M_4] \cdot \begin{bmatrix} \varphi_{01} & \varphi_{02} & \dots & \varphi_{0n} \\ \varphi_{11} & \varphi_{12} & \dots & \varphi_{1n} \\ \vdots & \vdots & & \vdots \\ \varphi_{n-1,1} & \varphi_{n-1,2} & \dots & \varphi_{n-1,n} \end{bmatrix} = \mu \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{48}$$

$$\left.\begin{aligned}
M_1 &= \begin{bmatrix} A_0 & 0 & \dots & 0 \\ A_1 & A_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1} & A_{n-2} & \dots & A_0 \end{bmatrix} \\
M_2 &= \begin{bmatrix} \bar{A}_0 & \bar{A}_1 & \dots & \bar{A}_{n-1} \\ 0 & \bar{A}_0 & \dots & \bar{A}_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{A}_0 \end{bmatrix} \\
M_3 &= \begin{bmatrix} \bar{A}_n & 0 & \dots & 0 \\ \bar{A}_{n-1} & \bar{A}_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_1 & \bar{A}_2 & \dots & \bar{A}_n \end{bmatrix} \\
M_4 &= \begin{bmatrix} A_n & A_{n-1} & \dots & A_1 \\ 0 & A_n & \dots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{bmatrix}
\end{aligned}\right\} \tag{48'}$$

Take the matrix

$$P = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{bmatrix} =$$

$$= \begin{bmatrix} A_0 & 0 & \dots & 0 \\ A_1 & A_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1} & A_{n-2} & \dots & A_0 \end{bmatrix} \begin{bmatrix} \bar{A}_0 & \bar{A}_1 & \dots & \bar{A}_{n-1} \\ 0 & \bar{A}_0 & \dots & \bar{A}_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{A}_0 \end{bmatrix} +$$

$$- \begin{bmatrix} \bar{A}_n & 0 & \dots & 0 \\ \bar{A}_{n-1} & \bar{A}_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_1 & \bar{A}_2 & \dots & \bar{A}_n \end{bmatrix} \begin{bmatrix} A_n & A_{n-1} & \dots & A_1 \\ 0 & A_n & \dots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{bmatrix} \tag{49}$$

We can obtain the solution for the elements of the matrix M_{φ}:

$$\begin{bmatrix} \varphi_{01} & \varphi_{02} & \cdots & \varphi_{0n} \\ \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \vdots & \vdots & & \vdots \\ \varphi_{n-1,1} & \varphi_{n-1,2} & \cdots & \varphi_{n-1,n} \end{bmatrix} = \mu \frac{P_{adj}}{|P|} \tag{50}$$

Where the adjoint matrix P_{adj} is given by

$$P_{adj} = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{bmatrix} \tag{51}$$

$|P|$ – denotes the main determinant of the matrix P and for the sake of simplicity we assume that

$$\mu = |P| \tag{52}$$

Returning to the equations (41) and (43) we obtain

$$\begin{bmatrix} \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0n} \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n-1,1} & \alpha_{n-1,2} & \cdots & \alpha_{n-1,n} \\ \alpha_{n,1} & \alpha_{n,2} & \cdots & \alpha_{n,n} \end{bmatrix} =$$

$$= \begin{bmatrix} \bar{A}_0 & \bar{A}_1 & \cdots & \bar{A}_{n-1} & \bar{A}_n \\ 0 & \bar{A}_0 & \cdots & \bar{A}_{n-2} & \bar{A}_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{A}_0 & \bar{A}_1 \\ 0 & 0 & \cdots & 0 & \bar{A}_0 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{53}$$

$$\begin{bmatrix} \beta_{01} & \beta_{02} & \cdots & \beta_{0n} \\ \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1,1} & \beta_{n-1,2} & \cdots & \beta_{n-1,n} \\ \beta_{n,1} & \beta_{n,2} & \cdots & \beta_{n,n} \end{bmatrix} =$$

$$= - \begin{bmatrix} \bar{A}_n & \bar{A}_{n-1} & \dots & \bar{A}_1 & \bar{A}_0 \\ 0 & \bar{A}_n & \dots & \bar{A}_2 & \bar{A}_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \bar{A}_n & \bar{A}_{n-1} \\ 0 & 0 & \dots & 0 & \bar{A}_n \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & \dots & Q_{1n} \\ Q_{21} & Q_{22} & \dots & Q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \dots & Q_{nn} \\ 0 & 0 & \dots & 0 \end{bmatrix} \tag{54}$$

For the complete determination of the matrix M it is necessary to determine the elements $\varepsilon_i (i = 0, 1, \dots, n)$ and the elements $\eta_j (j = 0, 1, \dots, n)$. The starting point for our consideration is the equation (36). We can write that

$$\begin{bmatrix} A_0 & 0 & \dots & 0 & 0 \\ A_1 & A_0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n-1} & A_{n-2} & \dots & A_0 & 0 \\ A_n & A_{n-1} & \dots & A_1 & A_0 \end{bmatrix} \begin{bmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \vdots \\ \varepsilon_{n-1} \\ \varepsilon_n \end{bmatrix} +$$

$$+ \begin{bmatrix} \bar{A}_n & 0 & \dots & 0 & 0 \\ \bar{A}_{n-1} & \bar{A}_n & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{A}_1 & \bar{A}_2 & \dots & \bar{A}_n & 0 \\ \bar{A}_0 & \bar{A}_1 & \dots & \bar{A}_{n-1} & \bar{A}_n \end{bmatrix} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{n-1} \\ \eta_n \end{bmatrix} = \mu \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{55}$$

From this it is evident that in particular the following equation holds

$$A_0 \varepsilon_0 + \bar{A}_n \eta_0 = 0 \tag{56}$$

We have also that

$$\begin{bmatrix} 0 & A_n & \dots & A_2 & A_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_n & A_{n-1} \\ 0 & 0 & \dots & 0 & A_n \end{bmatrix} \begin{bmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \vdots \\ \varepsilon_{n-1} \\ \varepsilon_n \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & \bar{A}_0 & \dots & \bar{A}_{n-2} & \bar{A}_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \bar{A}_0 & \bar{A}_1 \\ 0 & 0 & \dots & 0 & \bar{A}_0 \end{bmatrix} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{n-1} \\ \eta_n \end{bmatrix} = \mu \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{57}$$

We can omit the first columns in the matrices of eq. (57) taking into account equation (56). Equation (57) takes the form

$$\begin{bmatrix} A_n \dots A_2 & A_1 \\ \vdots \ddots \vdots & \vdots \\ 0 \dots A_n & A_{n-1} \\ 0 \dots 0 & A_n \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_{n-1} \\ \varepsilon_n \end{bmatrix} +$$

$$+ \begin{bmatrix} \bar{A}_0 \dots \bar{A}_{n-2} & \bar{A}_{n-1} \\ \vdots \ddots \vdots & \vdots \\ 0 \dots \bar{A}_0 & \bar{A}_1 \\ 0 \dots 0 & \bar{A}_0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_{n-1} \\ \eta_n \end{bmatrix} = \mu \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{58}$$

We postulate now, on the basis of eq. (58), that

$$\begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_{n-1} \\ \varepsilon_n \end{bmatrix} = \begin{bmatrix} \bar{A}_0 \dots \bar{A}_{n-2} & \bar{A}_{n-1} \\ \vdots \ddots \vdots & \vdots \\ 0 \dots \bar{A}_0 & \bar{A}_1 \\ 0 \dots 0 & \bar{A}_0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_{n-1} \\ \psi_n \end{bmatrix} \tag{59}$$

and due to the fact that the matrices in (58) are non-zero and permutable

$$\begin{bmatrix} \eta_1 \\ \vdots \\ \eta_{n-1} \\ \eta_n \end{bmatrix} = - \begin{bmatrix} A_n \dots A_2 & A_1 \\ \vdots \ddots \vdots & \vdots \\ 0 \dots A_n & A_{n-1} \\ 0 \dots 0 & A_n \end{bmatrix} \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_{n-1} \\ \psi_n \end{bmatrix} \tag{60}$$

Taking into account equation (56) we can also write

$$\left.\begin{aligned} \varepsilon_0 &= \bar{A}_n \psi_0 \\ \eta_0 &= -A_0 \psi_0 \end{aligned}\right\} \tag{61}$$

where the elements $\psi_0, \psi_1, \dots, \psi_n$ are to be determined. We can rewrite the equation (55) in the form

$$\begin{bmatrix} A_0 & \dots 0 & 0 \\ \vdots & \ddots \vdots & \vdots \\ A_{n-2} & \dots A_0 & 0 \\ A_{n-1} & \dots A_1 & A_0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_{n-1} \\ \varepsilon_n \end{bmatrix} + \begin{bmatrix} \bar{A}_n & \dots 0 & 0 \\ \vdots & \ddots \vdots & \vdots \\ \bar{A}_2 & \dots \bar{A}_n & 0 \\ \bar{A}_1 & \dots \bar{A}_{n-1} & \bar{A}_n \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_{n-1} \\ \eta_n \end{bmatrix} =$$

$$= \mu \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} - \varepsilon_0 \begin{bmatrix} A_1 \\ \vdots \\ A_{n-1} \\ A_n \end{bmatrix} - \eta_0 \begin{bmatrix} \bar{A}_{n-1} \\ \vdots \\ \bar{A}_1 \\ \bar{A}_0 \end{bmatrix} \tag{62}$$

The substitution of the equation (59), (60) and (61) into the equation (62) gives

$$[M_1 \cdot M_2 - M_3 \cdot M_4] \cdot \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_{n-1} \\ \psi_n \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + \left[A_0 \begin{bmatrix} \bar{A}_{n-1} \\ \vdots \\ \bar{A}_1 \\ \bar{A}_0 \end{bmatrix} - \bar{A}_n \begin{bmatrix} A_1 \\ \vdots \\ A_{n-1} \\ A_n \end{bmatrix} \right] \cdot \psi_0 \tag{63}$$

$$M_1 = \begin{bmatrix} A_0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ A_{n-2} & \dots & A_0 & 0 \\ A_{n-1} & \dots & A_1 & A_0 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \bar{A}_0 & \dots & \bar{A}_{n-2} & \bar{A}_{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \bar{A}_0 & \bar{A}_1 \\ 0 & \dots & 0 & \bar{A}_0 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \bar{A}_n & 0 & \dots & 0 \\ \bar{A}_{n-1} & \bar{A}_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_1 & \bar{A}_2 & \dots & \bar{A}_n \end{bmatrix}$$

$$M_4 = \begin{bmatrix} A_n & A_{n-1} & \dots & A_1 \\ 0 & A_n & \dots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{bmatrix}$$

Using the notation (49) we rewrite (63) in the form

$$\begin{bmatrix} P_{11} & \dots & P_{12} & P_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ P_{n-1,1} & \dots & P_{n-1,n-1} & P_{n-1,n} \\ P_{n1} & \dots & P_{n2} & P_{nn} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_{n-1} \\ \psi_n \end{bmatrix} = \mu \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} P_{1n} \\ \vdots \\ P_{n-1,n} \\ P_{nn} \end{bmatrix} \psi_0 \tag{64}$$

It is evident that we can state the following

THEOREM 3

$$A_0 \begin{bmatrix} \bar{A}_{n-1} \\ \vdots \\ \bar{A}_1 \\ \bar{A}_0 \end{bmatrix} - \bar{A}_n \begin{bmatrix} A_1 \\ \vdots \\ A_{n-1} \\ A_n \end{bmatrix} = \begin{bmatrix} P_{1n} \\ \vdots \\ P_{n-1,n} \\ P_{nn} \end{bmatrix} \tag{65}$$

PROOF: We have the following general statement

$$P_{r,n} = \sum_{i=1}^{r} [A_{r-i}\bar{A}_{n-i} - A_i\bar{A}_{n-r+i}] \tag{66}$$

This yields

$$P_{r,n} = \sum_{i=1}^{r} A_{r-i}\bar{A}_{n-i} - \sum_{j=r-1}^{0} A_{r-j}\bar{A}_{n-j} \tag{67}$$

putting $i = r - j$ in (67) we obtain

$$\sum_{j=0}^{r-1} A_{r-j}\bar{A}_{n-j} = A_r\bar{A}_n + \sum_{j=1}^{r-1} A_{r-j}\bar{A}_{n-j}$$

Finally

$$P_{r,n} = \sum_{i=1}^{r-1} A_{r-i}\bar{A}_{n-i} + A_0\bar{A}_{n-r} - A_r\bar{A}_n - \sum_{j=1}^{r-1} A_{r-j}\bar{A}_{n-j} =$$

$$= A_0\bar{A}_{n-r} - A_r\bar{A}_n \tag{68}$$

Using the notation (51) we obtain from (64) after using Cramer's rule that

$$\begin{bmatrix} \psi_1 \\ \vdots \\ \psi_{n-1} \\ \psi_n \end{bmatrix} = \begin{bmatrix} Q_{1n} \\ \vdots \\ Q_{n-1,n} \\ Q_{nn} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \psi_0 \end{bmatrix} = \begin{bmatrix} Q_{1n} \\ \vdots \\ Q_{n-1,n} \\ Q_{nn} + \psi_0 \end{bmatrix} \tag{69}$$

From the equation (59) we see that:

$$\varepsilon_n = \bar{A}_0 \psi_n \tag{70}$$

Similarly, from the equation (26) and (31), (32) it is evident that

$$\left.\begin{array}{lrcl} & \varepsilon_n & = & \bar{\eta}_0 \\ \text{or} & & & \\ & \bar{\varepsilon}_n & = & \eta_0 \end{array}\right\} \tag{71}$$

but

$$\eta_0 = -A_0 \psi_0 \qquad \text{(see eq. (61))}.$$

Hence

$$\bar{\eta}_0 = -\bar{A}_0 \bar{\psi}_0 \tag{72}$$

Comparing (72) and (71) we obtain

$$\varepsilon_n = -\bar{A}_0 \bar{\psi}_0 \tag{73}$$

which together with (70) gives

$$\bar{A}_0 \psi_n = -\bar{A}_0 \bar{\psi}_0 \tag{74}$$

From (74) we have that

$$\psi_n + \bar{\psi}_0 = 0 \tag{75}$$

From (69) we have also that

$$\psi_n = Q_{nn} + \psi_0 \tag{76}$$

Combining (75) with (76) gives

$$\psi_0 + \bar{\psi}_0 + Q_{nn} = 0 \tag{77}$$

We have also from (71) that $\eta_0 = \bar{\varepsilon}_n$ which with (61) gives

$$\bar{\varepsilon}_n = -A_0 \psi_0 \tag{78}$$

We have also from (70)

$$\bar{\varepsilon}_n = A_0 \bar{\psi}_n \tag{79}$$

Comparing (78) and (79) we receive

$$-A_0 \psi_0 = A_0 \bar{\psi}_n$$

from which we have

$$\psi_0 + \bar{\psi}_n = 0 \tag{80}$$

Taking into account (76) and

$$\bar{\psi}_n = \bar{Q}_{nn} + \bar{\psi}_0 \tag{81}$$

we find that

$$\psi_0 + \bar{\psi}_0 + \bar{Q}nn = 0 \tag{82}$$

Comparing (82) with (77) we see that

$$Q_{nn} = \bar{Q}_{nn} \tag{83}$$

This means that Q_{nn} is an even function of "s".

Let us denote

$$\left.\begin{array}{rcl} p & = & p(s^2) \\ \text{and} \qquad & & \\ q & = & s \cdot q(s^2) \end{array}\right\} \tag{84}$$

Then we can write

$$\left.\begin{array}{rcl} \psi_0 & = & p+q \\ \bar{\psi}_0 & = & p-q \end{array}\right\} \tag{85}$$

Adding together (85) we have

$$\psi_0 + \bar{\psi}_0 = 2p \tag{86}$$

From (77) and (86) we have

$$p = -\frac{1}{2} Q_{nn} \tag{87}$$

and

$$\psi_0 = -\frac{1}{2} Q_{nn} + q \tag{88}$$

But we have one degree of freedom and q may be chosen arbitrarily. We put for simplicity

$$q = 0 \tag{89}$$

Finally

$$\psi_0 = -\frac{1}{2} Q_{nn} \tag{90}$$

On the basis of (60) and the second equation (61) we can write:

$$\begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{n-1} \\ \eta_n \end{bmatrix} = - \begin{bmatrix} A_0 & 0 & \dots & 0 & 0 \\ 0 & A_n & \dots & A_2 & A_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_n & A_{n-1} \\ 0 & 0 & \dots & 0 & A_n \end{bmatrix} \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{n-1} \\ \psi_n \end{bmatrix} \tag{91}$$

Using now (90),(76) and (69) we finally find:

$$\begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{n-1} \\ \eta_n \end{bmatrix} = - \begin{bmatrix} A_0 & 0 & \dots & 0 & 0 \\ 0 & A_n & \dots & A_2 & A_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_n & A_{n-1} \\ 0 & 0 & \dots & 0 & A_n \end{bmatrix} \begin{bmatrix} -\frac{1}{2} Q_{nn} \\ Q_{1n} \\ \vdots \\ Q_{n-1,n} \\ \frac{1}{2} Q_{nn} \end{bmatrix} \tag{92}$$

Returning to (38) we can write:

$$\begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-1} \\ C_n \end{bmatrix} = \begin{bmatrix} \beta_{01} & \beta_{02} & \dots & \beta_{0n} \\ \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1,1} & \beta_{n-1,2} & \dots & \beta_{n-1,n} \\ \beta_{n,1} & \beta_{n,2} & \dots & \beta_{n,n} \end{bmatrix} \begin{bmatrix} \bar{D}_n \\ \bar{D}_{n-1} \\ \vdots \\ \bar{D}_1 \end{bmatrix} + \begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{n-1} \\ \eta_n \end{bmatrix} [D_0] +$$

$$+ \begin{bmatrix} \bar{\alpha}_{nn} & \dots & \bar{\alpha}_{n2} & \bar{\alpha}_{n1} \\ \bar{\alpha}_{n-1,n} & \dots & \bar{\alpha}_{n-1,2} & \bar{\alpha}_{n-1,1} \\ \vdots & & \vdots & \vdots \\ \bar{\alpha}_{1n} & \dots & \bar{\alpha}_{1,2} & \bar{\alpha}_{1,1} \\ \bar{\alpha}_{0n} & \dots & \bar{\alpha}_{02} & \bar{\alpha}_{01} \end{bmatrix} \begin{bmatrix} D_1 \\ \vdots \\ D_{n-1} \\ D_n \end{bmatrix} \tag{93}$$

where:

$$\begin{bmatrix} \beta_{01} & \beta_{02} & \dots & \beta_{0n} \\ \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1,1} & \beta_{n-1,2} & \dots & \beta_{n-1,n} \\ \beta_{n,1} & \beta_{n,2} & \dots & \beta_{n,n} \end{bmatrix} \begin{bmatrix} \bar{D}_n \\ \bar{D}_{n-1} \\ \vdots \\ \bar{D}_1 \end{bmatrix} =$$

$$= - \begin{bmatrix} A_n & A_{n-1} & \dots & A_1 & A_0 \\ 0 & A_n & \dots & A_2 & A_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_n & A_{n-1} \\ 0 & 0 & \dots & 0 & A_n \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & \dots & Q_{1n} \\ Q_{21} & Q_{22} & \dots & Q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \dots & Q_{nn} \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \bar{D}_n \\ \bar{D}_{n-1} \\ \vdots \\ \bar{D}_1 \end{bmatrix} \tag{94}$$

$$\begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{n-1} \\ \eta_n \end{bmatrix} [D_0] = - \begin{bmatrix} A_0 & 0 & \dots & 0 & 0 \\ 0 & A_n & \dots & A_2 & A_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_n & A_{n-1} \\ 0 & 0 & \dots & 0 & A_n \end{bmatrix} \begin{bmatrix} -\frac{1}{2}Q_{nn} \\ Q_{1n} \\ \vdots \\ Q_{n-1,n} \\ \frac{1}{2}Q_{nn} \end{bmatrix} [D_0] \tag{95}$$

$$\begin{bmatrix} \bar{\alpha}_{nn} & \dots & \bar{\alpha}_{n2} & \bar{\alpha}_{n1} \\ \bar{\alpha}_{n-1,n} & \dots & \bar{\alpha}_{n-1,2} & \bar{\alpha}_{n-1,1} \\ \vdots & & \vdots & \vdots \\ \bar{\alpha}_{1n} & \dots & \bar{\alpha}_{1,2} & \bar{\alpha}_{1,1} \\ \bar{\alpha}_{0n} & \dots & \bar{\alpha}_{02} & \bar{\alpha}_{01} \end{bmatrix} \begin{bmatrix} D_1 \\ \vdots \\ D_{n-1} \\ D_n \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & A_0 \\ 0 & 0 & \dots & 0 & A_0 & A_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & A_0 & \dots & A_{n-3} & A_{n-2} & A_{n-1} \\ A_0 & A_1 & \dots & A_{n-2} & A_{n-1} & A_n \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \dots & \bar{Q}_{1n} \\ \bar{Q}_{21} & \bar{Q}_{22} & \dots & \bar{Q}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{Q}_{n1} & \bar{Q}_{n2} & \dots & \bar{Q}_{nn} \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} D_n \\ D_{n-1} \\ \vdots \\ D_1 \end{bmatrix} \tag{96}$$

Denoting by

$$\begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_{n-1} \\ U_n \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & \dots & Q_{1n} \\ Q_{21} & Q_{22} & \dots & Q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \dots & Q_{nn} \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \bar{D}_n \\ \bar{D}_{n-1} \\ \vdots \\ \bar{D}_1 \end{bmatrix} \tag{97}$$

$$\begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_{n-1} \\ V_n \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}Q_{nn} \\ Q_{1n} \\ \vdots \\ Q_{n-1,n} \\ \frac{1}{2}Q_{nn} \end{bmatrix} [D_0] \tag{98}$$

We can write (93) in the form

$$\begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-1} \\ C_n \end{bmatrix} = - \begin{bmatrix} A_n & A_{n-1} & \dots A_1 & A_0 \\ 0 & A_n & \dots A_2 & A_1 \\ \vdots & \vdots & \ddots \vdots & \vdots \\ 0 & 0 & \dots A_n & A_{n-1} \\ 0 & 0 & \dots 0 & A_n \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_{n-1} \\ U_n \end{bmatrix} +$$

$$- \begin{bmatrix} A_0 & 0 & \dots 0 & 0 \\ 0 & A_n & \dots A_2 & A_1 \\ \vdots & \vdots & \ddots \vdots & \vdots \\ 0 & 0 & \dots A_n & A_{n-1} \\ 0 & 0 & \dots 0 & A_n \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_{n-1} \\ V_n \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & 0 & \dots & 0 & A_0 \\ 0 & 0 & \dots & A_0 & A_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & A_0 & \dots & A_{n-2} & A_{n-1} \\ A_0 & A_1 & \dots & A_{n-1} & A_n \end{bmatrix} \begin{bmatrix} \bar{U}_0 \\ \bar{U}_1 \\ \vdots \\ \bar{U}_{n-1} \\ \bar{U}_n \end{bmatrix} \tag{99}$$

or finally

$$C_0 = \begin{bmatrix} A_n \\ A_{n-1} \\ \vdots \\ A_1 \\ A_0 \end{bmatrix}^T \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_{n-1} \\ V_0 \end{bmatrix} \tag{100}$$

$$C_n = \begin{bmatrix} A_n \\ A_{n-1} \\ \vdots \\ A_1 \\ A_0 \end{bmatrix}^T \begin{bmatrix} -V_n \\ \bar{U}_{n-1} \\ \vdots \\ \bar{U}_1 \\ \bar{U}_0 \end{bmatrix} \tag{101}$$

$$
\begin{bmatrix} C_1 \\ \vdots \\ C_{n-1} \end{bmatrix} = - \begin{bmatrix} A_n & \dots & A_2 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_n \end{bmatrix} \begin{bmatrix} U_1 + V_1 \\ \vdots \\ U_{n-1} + V_{n-1} \end{bmatrix} +
$$

$$
+ \begin{bmatrix} A_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ A_{n-2} & \dots & A_0 \end{bmatrix} \begin{bmatrix} \bar{U}_{n-1} \\ \vdots \\ \bar{U}_1 \end{bmatrix} - \begin{bmatrix} A_1 \\ \vdots \\ A_{n-1} \end{bmatrix} V_n \tag{102}
$$

After calculation of the unknown polynomials $C_0, \dots, C_n$ we can calculate the integral

$$
J_2 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} E(s)E(-s)ds \tag{103}
$$

$$
= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{B_0(s) + \dots + B_n(s)e^{-sn\tau}}{A_0(s) + \dots + A_n(s)e^{-sn\tau}} \cdot \frac{B_0(-s) + \dots + B_n(-s)e^{sn\tau}}{A_0(-s) + \dots + A_n(-s)e^{sn\tau}} ds
$$

$$
\begin{aligned}
J_2 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{C_0(s) + C_1(s)e^{-s\tau} + \dots + C_n(s)e^{-sn\tau}}{|P(s)|[A_0(s) + A_1(s)e^{-s\tau} + \dots + A_n(s)e^{-sn\tau}]} ds \\
+ \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{C_0(-s) + C_1(-s)e^{s\tau} + \dots + C_n(-s)e^{sn\tau}}{|P(-s)|[A_0(-s) + A_1(-s)e^{s\tau} + \dots + A_n(-s)e^{sn\tau}]} ds
\end{aligned} \tag{104}
$$

Observe that for physical realizability and for the existence of the integral the degrees of polynomials $B_0(s), \dots, B_n(s)$ must be less than of $A_0(s), \dots, A_n(s)$.

Suppose that the degrees of $A_0(s), \dots, A_n(s)$ are "m", than these $B_0(s), \dots,$ $B_n(s)$ are at most $(m-1)$ and in the consequence the degrees of $C_0(s), \dots, C_n(s)$ are at most $(3m-2) - 2m = m-2$.

Consider now the integrals

$$
\int_{\supset} \frac{C_0(s) + C_1(s)e^{-s\tau} + \dots + C_n(s)e^{-sn\tau}}{|P(s)|[A_0(s) + A_1(s)e^{-s\tau} + \dots + A_n(s)e^{-sn\tau}]} ds
$$

$$
\int_{\subset} \frac{C_0(-s) + C_1(-s)e^{s\tau} + \dots + C_n(-s)e^{sn\tau}}{|P(-s)|[A_0(-s) + A_1(-s)e^{s\tau} + \dots + A_n(-s)e^{sn\tau}]} ds
$$

Putting $s = R^{j\phi}$ we see that these integrals are equal zero on the arcs when $R \longrightarrow \infty$, because the degrees of $C(s)$ are at most $(m-2)$, and that of $A(s)$ are equal "m".

For that reason we can write that

$$\int_{-j\infty}^{j\infty} \frac{C_0(s) + C_1(s)e^{-s\tau} + \cdots + C_n(s)e^{-sn\tau}}{A_0(s) + A_1(s)e^{-s\tau} + \cdots + A_n(s)e^{-sn\tau}} ds$$

$$= \int_{D} \frac{C_0(s) + C_1(s)e^{-s\tau} + \cdots + C_n(s)e^{-sn\tau}}{A_0(s) + A_1(s)e^{-s\tau} + \cdots + A_n(s)e^{-sn\tau}} ds \tag{105}$$

$$\int_{-j\infty}^{j\infty} \frac{C_0(-s) + C_1(-s)e^{s\tau} + \cdots + C_n(-s)e^{sn\tau}}{A_0(s) + A_1(s)e^{-s\tau} + \cdots + A_n(s)e^{-sn\tau}} ds$$

$$= \int_{G} \frac{C_0(-s) + C_1(-s)e^{s\tau} + \cdots + C_n(-s)e^{sn\tau}}{A_0(s) + A_1(s)e^{-s\tau} + \cdots + A_n(s)e^{-sn\tau}} ds \tag{106}$$

and now to apply Cauchy's Residuum Theorem.

Taking into account that denominator of the first integral has no roots in the right half-plane, and this of the second integral no roots in the left half-plane and using the relations (105), (106) we calculate the integral

$$J_2 = \frac{1}{2\pi j} \int_{D} \frac{C_0(s) + C_1(s)e^{-s\tau} + \cdots + C_n(s)e^{-sn\tau}}{|P(s)|[A_0(s) + A_1(s)e^{-s\tau} + \cdots + A_n(s)e^{-sn\tau}]} ds$$

$$+ \frac{1}{2\pi j} \int_{G} \frac{C_0(-s) + C_1(-s)e^{s\tau} + \cdots + C_n(-s)e^{sn\tau}}{|P(-s)|[A_0(-s) + A_1(-s)e^{s\tau} + \cdots + A_n(-s)e^{sn\tau}]} ds \tag{107}$$

This two integrals are equal by symmetry, it is sufficient to take the double value of one and finally we have:

$$J_2 = -2 \sum_{\substack{\text{Res} \\ s=s_i}} \frac{C_0(s) + C_1(s)e^{-s\tau} + \cdots + C_n(s)e^{-sn\tau}}{|P(s)|[A_0(s) + A_1(s)e^{-s\tau} + \cdots + A_n(s)e^{-sn\tau}]} ds \tag{108}$$

where s_i are these roots of equation $|P(s)| = 0$ which are lying in the right half-plane or on positive imaginary axis including the origin.

Appendix 1.

We prove now that condition (23') holds

$$\mu = \bar{\mu}$$

PROOF: It is evident from equality (52) that the following must be fulfilled:

$$|P| = |\bar{P}| \tag{109}$$

It is known [4, p.15] that

$$|P^T| = |P| \tag{110}$$

From the structure of matrix P (see (49)) it is evident that

$$P = M_1 M_2 - M_3 M_4 \tag{111}$$

It is also easy to observe (see (48')) that the following equalities hold :

$$\begin{aligned} \bar{M}_1^T &= M_2 \\ \bar{M}_2^T &= M_1 \\ \bar{M}_3^T &= M_4 \\ \bar{M}_4^T &= M_3 \end{aligned}$$

From (111) we have that

$$\bar{P}^T = \bar{M}_2^T \bar{M}_1^T - \bar{M}_4^T \bar{M}_3^T \tag{112}$$

Using the above equalities we obtain that

$$\bar{P}^T = P \tag{113}$$

Taking into account (110) we have

$$|\bar{P}^T| = |\bar{P}| \tag{114}$$

From the equation (114) we see that also

$$|\bar{P}^T| = |P| \tag{115}$$

Comparing (114) and (115) we see finally that

$$|\bar{P}| = |P| \tag{116}.$$

and due to the assumption (52)

$$\mu = |P|$$

we have

$$\mu = \bar{\mu}$$

which ends the proof.

Appendix 2.

LEMMA 1 *The two matrices in eq. (42) are permutable*

$$FG = GF \tag{117}$$

PROOF: We consider the matrices

$$F = \begin{bmatrix} A_n & A_{n-1} & \dots A_1 & A_0 \\ 0 & A_n & \dots A_2 & A_1 \\ \vdots & \vdots & \ddots \vdots & \vdots \\ 0 & 0 & \dots A_n & A_{n-1} \\ 0 & 0 & \dots 0 & A_n \end{bmatrix} \tag{118}$$

and

$$G = \begin{bmatrix} \bar{A}_0 & \bar{A}_1 & \dots & \bar{A}_{n-1} & \bar{A}_n \\ 0 & \bar{A}_0 & \dots & \bar{A}_{n-2} & \bar{A}_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \bar{A}_0 & \bar{A}_1 \\ 0 & 0 & \dots & 0 & \bar{A}_0 \end{bmatrix} \tag{119}$$

and the matrix $H_{(n+1)\times(n+1)}$ [1, p.24]

$$H = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{120}$$

$$H^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{120'}$$

and so on.
We can write

$$F = A_n E + A_{n-1} H + \dots + A_0 H^n \tag{121}$$

$$G = \bar{A}_0 E + \bar{A}_1 H + \cdots + \bar{A}_n H^n \tag{122}$$

Then it is easy to observe

$$FG = A_n \bar{A}_0 E + (A_{n-1}\bar{A}_0 + A_n \bar{A}_1)H + \cdots + A_0 \bar{A}_n H^{2n} \tag{123}$$

On the other hand

$$GF = \bar{A}_0 A_n E + (\bar{A}_0 A_{n-1} + \bar{A}_1 A_n)H + \cdots + \bar{A}_n A_0 H^{2n} \tag{124}$$

But we have for the products

$$\bar{A}_0 A_n = A_n \bar{A}_0 \tag{125}$$

and so on (as the products of polynomials) and we see that really

$$FG = GF$$

We compare now on a simple example the method of Penrose [2] for finding solutions of the underdetermined system with the proposed method.

EXAMPLE

PROPOSED METHOD

We consider now the system with one delay $n = 1$

$$E(s) = \frac{B_0(s) + B_1(s)e^{-s\tau}}{A_0(s) + A_1(s)e^{-s\tau}} \tag{126}$$

From relation (49) we have

$$P = [A_0\bar{A}_0 - A_1\bar{A}_1] \tag{127}$$

The adjoint matrix is

$$P_{adj} = Q_{11} = 1 \tag{128}$$

From (10),(97),(98) and (99) we have

$$\left.\begin{array}{rcl} \bar{D}_1 & = & B_0\bar{B}_1 \\ \bar{D}_0 & = & B_0\bar{B}_0 + B_1\bar{B}_1 \end{array}\right\} \tag{129}$$

$$\left.\begin{array}{rclrcl} U_0 & = & Q_{11}\bar{D}_1, & V_0 & = & -\frac{1}{2}Q_{11}D_0 \\ U_1 & = & 0, & V_1 & = & \frac{1}{2}Q_{11}D_0 \end{array}\right\} \tag{130}$$

$$\left.\begin{array}{rcl} C_0 & = & \frac{1}{2}A_0(B_0\bar{B}_0 + B_1\bar{B}_1) - A_1B_0\bar{B}_1 \\ C_1 & = & A_0\bar{B}_0B_1 - \frac{1}{2}A_1(B_0\bar{B}_0 + B_1\bar{B}_1) \end{array}\right\} \tag{131}$$

from (127) we have that

$$|P| = A_0\bar{A}_0 - A_1\bar{A}_1 \tag{132}$$

Finally from relation (108) we have

$$J_2 = -2 \sum_{\substack{\text{Res}\\ s=s_i}} \frac{C_0(s) + C_1(s)e^{-s\tau}}{|P(s)|[A_0(s) + A_1(s)e^{-s\tau}]} \tag{133}$$

where s_i are the roots of equation

$$|P(s)| = 0 \tag{134}$$

lying in the right half-plane.

$$J_2 = -2\sum_{\substack{\text{Res}\\ s=s_i}} \frac{[\frac{1}{2}A_0(B_1\bar{B}_1 + B_0\bar{B}_1) - A_1B_0\bar{B}_1] + [A_0\bar{B}_0B_1 - \frac{1}{2}A_1(B_0\bar{B}_0 + B_1\bar{B}_1)]e^{-s\tau}}{(A_0\bar{A}_0 - A_1\bar{A}_1)[A_0(s) + A_1(s)e^{-s\tau}]} \tag{135}$$

where s we calculate from relation

$$A_0(s)A_0(-s) - A_1(s)A_1(-s) = 0 \tag{136}$$

System with two delays $n = 2$

$$E(s) = \frac{B_0(s) + B_1(s)e^{-s\tau} + B_2(s)e^{-2s\tau}}{A_0(s) + A_1(s)e^{-s\tau} + A_2(s)e^{-2s\tau}} \tag{137}$$

From relation (49) we have

$$P = \begin{bmatrix} A_0\bar{A}_0 - A_2\bar{A}_2, & A_0\bar{A}_1 - A_1\bar{A}_2 \\ \bar{A}_0A_1 - \bar{A}_1A_2, & A_0\bar{A}_0 - A_2\bar{A}_2 \end{bmatrix} \tag{138}$$

and the determinant

$$|P| = (A_0\bar{A}_0 - A_2\bar{A}_2)^2 - (\bar{A}_0A_1 - \bar{A}_1A_2)(A_0\bar{A}_1 - A_1\bar{A}_2) \tag{139}$$

The adjoint matrix is

$$P_{adj} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \tag{140}$$

where

$$\left.\begin{array}{lcl} Q_{11} & = & A_0\bar{A}_0 - A_2\bar{A}_2 \\ Q_{12} & = & A_1\bar{A}_2 - A_0\bar{A}_1 \\ Q_{21} & = & \bar{A}_1 A_2 - \bar{A}_0 A_1 \\ Q_{22} & = & A_0\bar{A}_0 - A_2\bar{A}_2 \end{array}\right\} \tag{141}$$

From (10) we have

$$\left.\begin{array}{lcl} \bar{D}_2 & = & B_0\bar{B}_2 \\ \bar{D}_1 & = & B_1\bar{B}_2 + B_0\bar{B}_1 \\ D_0 & = & B_2\bar{B}_2 + B_1\bar{B}_1 + B_0\bar{B}_0 \\ D_1 & = & \bar{B}_1 B_2 + \bar{B}_0 B_1 \\ D_2 & = & \bar{B}_0 B_2 \end{array}\right\} \tag{142}$$

The relations (97) give

$$\left.\begin{array}{lcl} U_0 & = & Q_{11}\bar{D}_2 + Q_{12}\bar{D}_1 \\ U_1 & = & Q_{21}\bar{D}_2 + Q_{22}\bar{D}_1 \\ U_2 & = & 0 \end{array}\right\} \tag{143}$$

and from (98)

$$\left.\begin{array}{lcl} V_0 & = & -\frac{1}{2}Q_{22}D_0 \\ V_1 & = & Q_{12}D_0 \\ V_2 & = & \frac{1}{2}Q_{22}D_0 \end{array}\right\} \tag{144}$$

Finally from (100),(101) and (102) we obtain that:

$$\left.\begin{array}{lcl} C_0 & = & -(A_2U_0 + A_1U_1 + A_0V_0) \\ C_2 & = & -A_2V_2 + A_1\bar{U}_1 = A_0\bar{U}_0 \\ C_1 & = & A_0\bar{U}_1 - A_1V_2 - A_2(U_1 + V_1) \end{array}\right\} \tag{145}$$

Penrose method [2, p.242]

The unknown vector C can be determined from the relation

$$C = (A^*)^+ \cdot D \tag{146}$$

where

$$(A^*)^+ = (AA^*)^{-1}A \tag{147}$$

In the case of the system with one delay

$$AA^* = \begin{bmatrix} A_0 & 0 & \bar{A}_1 & 0 \\ A_1 & A_0 & \bar{A}_0 & \bar{A}_1 \\ 0 & A_1 & 0 & \bar{A}_0 \end{bmatrix} \cdot \begin{bmatrix} \bar{A}_0 & \bar{A}_1 & 0 \\ 0 & \bar{A}_0 & \bar{A}_1 \\ A_1 & A_0 & 0 \\ 0 & A_1 & A_0 \end{bmatrix} =$$

$$
= \begin{bmatrix} A_0\bar{A}_0 + A_1\bar{A}_1 & 2A_0\bar{A}_1 & 0 \\ 2\bar{A}_0 A_1 & 2(A_0\bar{A}_0 + A_1\bar{A}_1) & 2A_0\bar{A}_1 \\ 0 & 2\bar{A}_0 A_1 & A_0\bar{A}_0 + A_0\bar{A}_0 \end{bmatrix} \tag{148}
$$

and using (143) we find finally

$$
(A^*)^+ = \frac{1}{(A_0\bar{A}_0 - A_1\bar{A}_1)^2(A_0\bar{A}_0 + A_1\bar{A}_1)} \cdot [K_1 \cdot K_2 \cdot K_3] \tag{149}
$$

where

$$
K_1 = \begin{bmatrix} \bar{A}_0[(A_0\bar{A}_0)^2 + (A_1\bar{A}_1)^2] - A_0\bar{A}_1^2(A_0\bar{A}_0 + A_1\bar{A}_1) \\ -A_0\bar{A}_0\bar{A}_1(A_0\bar{A}_0 + A_1\bar{A}_1) + 2A_0^2\bar{A}_1^3 \\ \bar{A}_1[(A_0\bar{A}_0)^2 + (A_1\bar{A}_1)^2] - A_0^2\bar{A}_1(A_0\bar{A}_0 + A_1\bar{A}_1) \\ -A_0A_1\bar{A}_1(A_0\bar{A}_0 + A_1\bar{A}_1) + 2A_0^3\bar{A}_1^2 \end{bmatrix}
$$

$$
K_2 = \begin{bmatrix} -A_0^2A_1(A_0\bar{A}_0 + A_1\bar{A}_1) + \frac{1}{2}\bar{A}_1(A_0\bar{A}_0 + A_1\bar{A}_1)^2 \\ \frac{1}{2}\bar{A}_0(A_0\bar{A}_0 + A_1\bar{A}_1)^2 - A_0\bar{A}_1^2(A_0\bar{A}_0 + A_1\bar{A}_1) \\ -\bar{A}_0A_1^2(A_0\bar{A}_0 + A_1\bar{A}_1) + \frac{1}{2}\bar{A}_0(A_0\bar{A}_0 + A_1\bar{A}_1)^2 \\ \frac{1}{2}A_1(A_0\bar{A}_0 + A_1\bar{A}_1)^2 - A_0^2\bar{A}_1(A_0\bar{A}_0 + A_1\bar{A}_1) \end{bmatrix}
$$

$$
K_3 = \begin{bmatrix} 2\bar{A}_0^3A_1^2 - \bar{A}_0A_1\bar{A}_1(A_0\bar{A}_0 + A_1\bar{A}_1) \\ -\bar{A}_0^2A_1(A_0\bar{A}_0 + A_1\bar{A}_1) + \bar{A}_1[(A_0\bar{A}_0)^2 + (A_1\bar{A}_1)^2] \\ 2\bar{A}_0^2A_1^3 - \bar{A}_0A_1\bar{A}_1(A_0\bar{A}_0 + A_1\bar{A}_1) \\ -\bar{A}_0A_1^2(A_0\bar{A}_0 + A_1\bar{A}_1) + A_0[(A_0\bar{A}_0)^2 + (A_1\bar{A}_1)^2] \end{bmatrix}
$$

Conclusion

From the relation (145) it is evident that both the nominator and denominator are much more complicated in comparison with the proposed method. In consequence the calculation of the integral J_2 is also very complicated especially for systems with multiple delays.

The method presented solves the problem of undetermined system of linear equations, taking additionally into account some symmetric features of the system. This in turn leads to the choice of the simplest solution from an infinite number. This simply means that the obtained formulae have the smallest number of terms.

The proposed method using some simmetrization procedure leads to the simplest analytic results for calculation of integral squared error of system with many delays. This method can be used also as a starting point for the calculation of such integral for the systems with distributed delays.

The obtained analytic results enable us to calculate the integral for very complicated systems, such as, for example, a distillation column, where the direct numerical calculation is impossible at all. The value of the integral may serve as the performance index of the system and as the basis for optimization with respect to parameters of controllers.

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Obliczanie całkowego błędu kwadratowego dla wielkich systemów dynamicznych z wieloma współmiernymi opóźnieniami

W pracy przedstawiono analityczne rozwiązanie zagadnienia obliczania $\int \epsilon^2 dt$ dla wielkich systemów dynamicznych z wieloma opóźnieniami. Układ równań użyty do wyznaczenia nieznanych wielomianów $C_i(s), i = 1, \ldots, n$ jest niedookreślony. Używając pewnych właściwości symetrii trudność tę pokonano i nieznane C_i wyznaczono jednoznacznie w postaci prostych wyrażeń analitycznych o najmniejszym możliwym rzędzie. Zaproponowana metoda prowadzi do prostszych wzorów i algorytmów obliczeniowych niż metoda przedstawiona przez Penrose'a [2]. Podano przykład pozwalający na porównanie obu metod. Otrzymane wyniki mogą być przydatne do obliczania optymalnych nastaw regulatora w optymalizacji parametrycznej.

Вычисление интегральной квадратной ошибки для больших динамических систем со многими соизмеримыми запаздываниями

В работе представлено аналитическое решение задачи вычисления $\int \epsilon^2 dt$ для больших динамических систем со многими запаздываниями. Система уравниений, используемая для определения неизвестных многочленов $C_i(s), i = 1, \ldots, n$ неполностью определена. Используя некоторые свойства симметрии эти затруднения преодолены и неизвестные C_i определены однозначно в виде простых аналитических выражений возможно наименьшего порядка. Предлагаемый метод приводит к более простым формулам и вычислительным алгоритмам, чем метод представленый Пенрозом [2]. Дается пример позволяющий сравнить оба метода. Полученные результаты могут быть полезны для вычисления оптимальной установки регулятора в параметрической оптимизации.