Control
and Cybernetics
VOL. 20 (1991) No. 2

An approach to linear model reduction

by

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An approach to reduce a high-order linear system to a reduced order model is described. A reduced order model is assumed to have the same response to an input as that of the high-order system and the configuration of the reduced model is then determined by employing a modified model reference technique. The use of the convolution summation formula for mathematical manipulation of the problem is found to be effective in finding an error at the input side. The input error and the Taylor's series expansion of the sampled impulse response of the reduced model around the parameters of an assumed model are used to determine the parameters of the reduced model. The determined reduced model is found to satisfy the impulse response requirement of the original system with a reasonable degree.

Introduction

The need for model reduction has been recognized as one of the most significant problems in dynamic system theory [1,2]. Considerable efforts have been made during the last three decades to solve the problem for obtaining an approximate low-order model from a high-order system. The techniques for model reduction so far developed can be divided into two groups. In one group, the parameters of the reduced model (RM) are obtained by keeping the dominant poles of its transfer function the same as those of the transfer function of the high-order system [3–10] and in the second, irrespective of the poles' positions of the transfer function of the original system, the parameters of the RM are obtained using some optimization criteria depending on the error which is defined either on the impulse responses of the high-order system and the RM or on their output responses to an input other than the impulse [11–16]. It is seen that in the existing techniques, the knowledge of the configuration of the system to be reduced is necessary. The error introduced in reducing the system increases as the difference in the order of the high-order system and the RM increases. Such an error which is referred to the output side makes the RM unacceptable in many cases like in regulators and in projective control.

An approach to the linear model reduction is described which avoids the knowledge of the configuration of the system to be reduced and provides a possibility of matching the outputs of the high-order system and the RM. In this approach, a modified model reference technique (MRT) is used in which the high-order system is replaced by a reduced order model whose response to an input is assumed to be the same as that of the original system. An assumed model (model) having the same order as that of the RM and its parameters different from those of the RM is incorporated. For ensuring a match of the outputs of the RM and the model in the MRT, their inputs are to be different which gives rise to an error at the input side. This error is called an input error (IE) and can readily be expressed as a function of the difference between the impulse responses of the RM and the model by the use of a real convolution summation formula. The parameters of the RM can be obtained by minimising a defined cost function comprised either of the error in the impulse responses or of the IE. The procedure for minimising these errors is, however, involved with a complex nonlinear computational process. An alternative technique which involves a simpler computational procedure is adopted in this paper which deals with the use of the IE for estimating the parameters of the RM through the determination of the departures of the RM parameters from those of the model. These parameter departures can be determined by expanding the impulse response of the RM around the parameters of the model and solving a multivariable polynomial equation thus obtained. The proposed approach is exemplified by determining the parameters of an RM which is found to satisfy the impulse response requirement of the considered high-order system to a reasonable degree.

DEFINITION. Let the output of a high–order system be $\mathbf{Y}_1(t)$ with its input $\mathbf{X}_1(t)$. A low–order system is said to be a reduced form of the given one if the low–order one has either an output $\mathbf{Y}_1(t)$ with an input close to $\mathbf{X}_1(t)$ or an output close to $\mathbf{Y}_1(t)$ with an input $\mathbf{X}_1(t)$.

The above definition does not specify the degree of the closeness between the inputs of the high–order system and the RM in the first case and the closeness between the outputs in the second case. It is possible to find different forms of the RM for a given high order system depending upon the criterion imposed on the closeness.

1. Basic considerations of the approach

Let the input and output of a causal, linear, time–invariant multi–input multi–output (MIMO) high–order system along with a zero order hold element (Z.O.H.) be $\mathbf{X}_1(n)$ and $\mathbf{Y}_1(n)$ respectively. According to the above definition, this high–order system is replaced by a RM as shown in Fig.1. The model in the figure is a known parameter MIMO system having the same order as that of the RM. $\mathbf{X}_2(n)$ and $\mathbf{Y}_2(n)$ are the input and output of the model along with a Z.O.H. respectively. The IE is the difference between the inputs of the high–order system and the model and is comprised of the two components arising out of (i) the departure in the parameters of the RM and the model and (ii) the departure in the orders of the the high–order system and the RM.

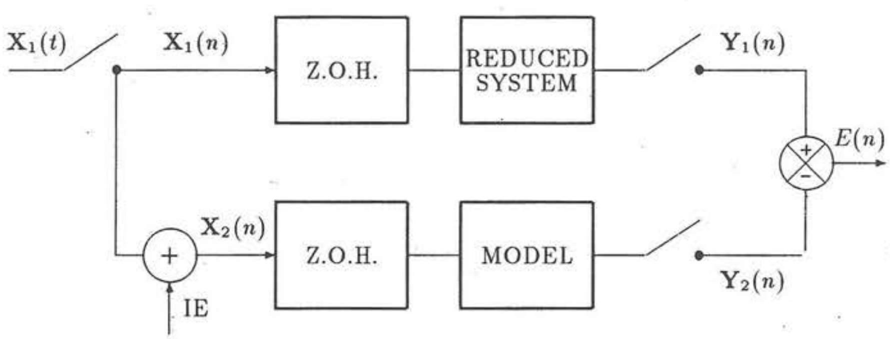

Figure 1. The idea of the proposed approach.

The response of the jth output of the MIMO system is a total effect of the response to all its inputs. The real convolution summation describing the relationship between the jth output and inputs of the MIMO (p inputs, q outputs) RM applicable at the nth instant of sampling time is expressed as

$$y_{1_j}(n) = \sum_{k=0}^{n} \sum_{i=1}^{p} x_{1_i}(n-k) h_{1_{ij}}(k) \quad n = 0, 1, \ldots, N; \;\; j = 1, 2, \ldots, q \quad (1)$$

where x_{1_i} is the sampled signal of the ith input and $h_{1_{ij}}(.,.)$ is the response at the jth output of the RM along with a Z.O.H. to an impulse at the ith input (sampled impulse response) and is a function of the parameters of the RM.

A similar expression is also written for the model along with a Z.O.H. as

$$y_{2_j}(n) = \sum_{k=0}^{n} \sum_{i=1}^{p} x_{2_i}(n-k) h_{2_{ij}}(k) \quad n = 0, 1, \ldots, N; \; j = 1, 2, \ldots, q \quad (2)$$

where $x_{2_i}(.,.)$ is the sampled signal of the ith input of the model and $h_{2_{ij}}(.,.)$ is the sampled response at the jth output to an impulse at the ith input of the model and is a function of the model parameters.

For ensuring a match between the outputs of the RM and the model ($\mathbf{Y}_1(n) - \mathbf{Y}_2(n) = E(n) = 0$), the model should have an input $x_2(.,.)$ which is called a requested input to the model. In such a case, $y_{2_j}(n) = y_{1_j}(n) = y(n)$ for $j = 1, 2, \ldots, q$. Oné then obtains

$$\sum_{k=0}^{n} \sum_{i=1}^{p} [x_{1_i}(n-k) h_{1_{ij}}(k) - x_{2_i}(n-k) h_{2_{ij}}(k)] = 0, \quad n = 0, 1, \ldots, N \quad (3)$$

Since the commutative property holds in the real convolution summation, subtracting and adding one term $x_{1_i}(.,.)h_{2_{ij}}(.,.)$ to (3) and rearranging, the following expression is obtained

$$\sum_{k=0}^{n} \sum_{i=1}^{p} x_{1_i}(n-k)[h_{1_{ij}}(k) - h_{2_{ij}}(k)] = \\ \sum_{k=0}^{n} \sum_{i=1}^{p} [x_{2_i}(n-k) - x_{1_i}(n-k)] h_{2_{ij}}(k) \\ j = 1, 2, \ldots, q; \quad n = 0, 1, \ldots, N \quad (4)$$

This is written in the matrix form as

$$\sum_{k=0}^{n} [H_{1_j}^T(k) - H_{2_j}^T(k)] X_1(n-k) = \sum_{k=0}^{n} H_{2_j}^T(k) [X_2(n-k) - X_1(n-k)]$$

$$j = 1, 2, \ldots, q; \quad n = 0, \ldots, N \tag{5}$$

where

$$\begin{aligned} X_1(n-k) &= [x_{1_1}(n-k)x_{1_2}(n-k)\ldots x_{1_p}(n-k)]^T \in R^{p\times 1} \\ X_2(n-k) &= [x_{2_1}(n-k)x_{2_2}(n-k)\ldots x_{2_p}(n-k)]^T \in R^{p\times 1} \\ H_{1_j}(k) &= [h_{1_{1j}}(k.,.)h_{1_{2j}}(k.,.)\ldots h_{1_{pj}}(k.,.)]^T \in R^{p\times 1} \\ H_{2_j}(k) &= [h_{2_{1j}}(k.,.)h_{2_{2j}}(k.,.)\ldots h_{2_{pj}}(k.,.)]^T \in R^{p\times 1} \end{aligned}$$

The above expression is written in a compact form as

$$[\mathbf{H}_{1_j}^T(n) - \mathbf{H}_{2_j}^T(n)]\mathbf{X}_1(n) = \mathbf{H}_{2_j}^T(n)[\mathbf{X}_2(n) - \mathbf{X}_1(n)] \tag{6}$$

$$j = 1, 2, \ldots, q: \quad n = 0, 1, \ldots, N$$

where

$$\begin{aligned} \mathbf{X}_1(n) &= [X_1^T(n)X_1^T(n-1)\ldots X_1^T(0)]^T \in R^{(n+1)p\times 1} \\ \mathbf{X}_2(n) &= [X_2^T(n)X_2^T(n-1)\ldots X_2^T(0)]^T \in R^{(n+1)p\times 1} \\ \mathbf{H}_{1_j}(n) &= [H_{1_j}^T(0)H_{1_j}^T(1)\ldots H_{1_j}^T(n)]^T \in R^{(n+1)p\times 1} \\ \mathbf{H}_{2_j}(n) &= [H_{2_j}^T(0)H_{2_j}^T(1)\ldots H_{2_j}^T(n)]^T \in R^{(n+1)p\times 1} \end{aligned}$$

The set of q equations given in (6) is farther expressed in the matrix form as

$$[\mathsf{H}_1(n) - \mathsf{H}_2(n)]\mathbf{X}_1(n) = \mathsf{H}_2(n)[\mathbf{X}_2(n) - \mathbf{X}_1(n)] \quad n = 0, 1\ldots, N \tag{7}$$

where $[\mathbf{X}_2(n) - \mathbf{X}_1(n)]$ is termed as the input error (IE) and

$$\mathsf{H}_1(n) = \begin{bmatrix} \mathbf{H}_{1_1}^T(n) \\ \mathbf{H}_{1_2}^T(n) \\ \vdots \\ \mathbf{H}_{1_q}^T(n) \end{bmatrix} \in R^{q\times(n+1)p}, \quad \mathsf{H}_2(n) = \begin{bmatrix} \mathbf{H}_{2_1}^T(n) \\ \mathbf{H}_{2_2}^T(n) \\ \vdots \\ \mathbf{H}_{2_q}^T(n) \end{bmatrix} \in R^{q\times(n+1)p}$$

It is seen from (7) that the parameters of the RM can be obtained either by minimising the IE or by minimising the difference between the two impulse response matrices (the error in the impulse responses).

1.1. Technique of the minimization of the IE

Assume that the rank of the matrix $\mathsf{H}_2(n)$ is such that the matrix $[\mathsf{H}_2^T(n)\mathsf{H}_2(n)]$ obtained by multiplying both sides of (7) by $\mathsf{H}_2^T(n)$ is a positive definite matrix. Then $[\mathsf{H}_2^T(n)\mathsf{H}_2(n)]^{-1}$ exists and is also a positive definite matrix. From (7), the IE is then obtained as

$$[\mathbf{X}_2(n) - \mathbf{X}_1(n)] = [\mathsf{H}_2^T(n)\mathsf{H}_2(n)]^{-1}\mathsf{H}_2^T(n)[\mathsf{H}_1(n) - \mathsf{H}_2(n)]\mathbf{X}_1(n) \qquad (8)$$

$$n = 0, 1, \ldots, N$$

Since the number of samples N is usually greater than the number of the RM parameters r, the unique solution can not be found and is also not expected. The parameters of the RM can be found by employing the weighted least squares error method for which a cost function is defined as

$$J_x = \sum_{n=0}^{N}[\mathbf{X}_2(n) - \mathbf{X}_1(n)]^T\ R_1\ [\mathbf{X}_2(n) - \mathbf{X}_1(n)] \qquad (9)$$

where R_1 is a positive definite matrix of the appropriate dimension.

With reference to (8), J_x can be expressed as

$$J_x = \sum_{n=0}^{N} \mathbf{X}_1^T(n)[\mathsf{H}_1(n) - \mathsf{H}_2(n)]^T\ S\ [\mathsf{H}_1(n) - \mathsf{H}_2(n)]\mathbf{X}_1(n) \qquad (10)$$

$$J_x = \sum_{n=0}^{N} \|[\mathsf{H}_1(n) - \mathsf{H}_2(n)]\mathbf{X}_1(n)\|_S^2 \qquad (11)$$

where $S = \mathsf{H}_2(n)[\mathsf{H}_2^T(n)\mathsf{H}_2(n)]^{-1}\ R_1[\mathsf{H}_2^T(n)\mathsf{H}_2(n)]^{-1}\mathsf{H}_2^T(n)$. The subscript S of the norm sign indicates that the norm is weighted. The RM parameters can be obtained from the first-order necessary conditions for the defined cost function to be a minimum subject to a stability condition of the reduced system such that the *RM* parameters determine the position of the poles within the unit cycle [17, 18]. For such a stability, the reparametrization to be performed with the help of Cholesky factorization used in the Levinson–Durbin algorithm requires a nonlinear computational procedure.

$[\mathsf{H}_2^T(n)\mathsf{H}_2(n)]^{-1}\mathsf{H}_2^T(n)$ is the generalized inverse (Penrose–Moore inverse or pseudoinverse) which is denoted by $\mathsf{H}_2^+(n)$ [17, 20, 23]. It may be noted that $\mathsf{H}_2^+(n)$ can also be obtained by taking the inverse of the partitioned matrix $\mathsf{H}_2(n)$. The partition is performed after rearranging columns, if necessary, so that the left block of dimension q is nonsingular [17, 20]. In the case, where $\mathbf{H}_2(n)$ is not full row rank matrix, $[\mathbf{H}_2^T(n)\mathbf{H}_2(n)]^{-1}$ does not exist but the generalized inverse of $[\mathbf{H}_2^T(n)\mathbf{H}_2(n)]$ can be obtained. By means of permutation, the matrix $[\mathbf{H}_2^T(n)\mathbf{H}_2(n)]$ can be so arranged that its upper left block becomes a nonsingular submatrix having the same rank as that of $\mathbf{H}_2(n)$. The submatrix is invertible and the generalized inverse of $[\mathbf{H}_2^T(n)\mathbf{H}_2(n)]$ consists of the inverse of the said nonsingular submatrix in the upper left block with null matrices in other blocks. In this case, the matrix S in (10) and (11) is no longer positive definite and thus the minimization of J_x belongs to the ill-conditioned class [17]. The partition of the expression in the norm into the range and null space of the unknown $\mathbf{H}_1(n)$ is to be performed. The minimization is then carried out w.r.t. the unknown in the range space subject to the condition set in the null space [20]. Another technique of the minimization of the J_x is to use the "weighted generalized inverse solution" [20] in which a positive definite matrix having the same rank as that of $\mathbf{H}_2(n)$ is introduced to transform the "weighted generalised inverse solution" to the form of the "generalised inverse solution".

1.2. Technique of the minimization of the error in impulse responses

Multiplying both sides of (7) by a positive definite matrix R_2 and their respective transposes, one obtains

$$\sum_{n=0}^{N} \mathbf{X}_1^T(n)[\mathbf{H}_1(n) - \mathbf{H}_2(n)]^T R_2 [\mathbf{H}_1(n) - \mathbf{H}_2(n)]\mathbf{X}_1(n) =$$

$$= \sum_{n=0}^{N} [\mathbf{X}_2(n) - \mathbf{X}_1(n)]^T \mathbf{H}_2^T(n) R_2 \mathbf{H}_2(n) [\mathbf{X}_2(n) - \mathbf{X}_1(n)] \qquad (12)$$

It is seen that $[\mathbf{H}_1(n) - \mathbf{H}_2(n)]\mathbf{X}_1(n)$ on the L.H.S. of (12) is the difference between the nth sampled outputs $\mathbf{Y}_1(n)$ and $\mathbf{Y}_2(n)$.

If a pth order model is chosen such that it is expressed in the state variable equation as

$$Z_2(n+1) = \mathbf{A}_2 Z_2(n) + \mathbf{B}_2 \mathbf{X}_2(n) \qquad (13)$$

$$\mathbf{Y}_2(n) = \mathbf{C}_2 Z_2(n) \qquad (14)$$

where $\mathbf{X}_2(.,.) \in R^{p\times 1}, \mathbf{Y}_2(.,.) \in R^{q\times 1}$ are the input and output vectors. $\mathbf{A}_2, \mathbf{B}_2$ and $\mathbf{C}_2$ are the matrices of the appropiate dimensions. $Z_2(.,.)$ is the corresponding dimension state vector and $q \leq \rho < \rho_N$ where ρ_N is order of the original system.

Then the ρth order RM is similarly expressed as

$$Z_1(n+1) = \mathbb{A}_1 Z_1(n) + \mathbb{B}_1 \mathbf{X}_1(n) \tag{15}$$

$$\mathbf{Y}_1(n) = \mathbb{C}_1 Z_1(n) \tag{16}$$

where $x_1(.,.) \in R^{p\times 1}, \mathbf{Y}_1(.,.) \in R^{q\times 1}$ are the input and output vectors of the RM. $\mathbb{A}_1$, $\mathbb{B}_1$ and $\mathbb{C}_1$ are the matrices of the appropiate dimensions. $Z_1(.,.)$ is the corresponding dimension state vector of the RM.

The augmented system is then obtained

$$\begin{bmatrix} Z_1(n+1) \\ Z_2(n+1) \end{bmatrix} = \begin{bmatrix} \mathbb{A}_1 & \mathbb{O} \\ \mathbb{O} & \mathbb{A}_2 \end{bmatrix} \begin{bmatrix} Z_1(n) \\ Z_2(n) \end{bmatrix} + \begin{bmatrix} \mathbb{B}_1 & \mathbb{O} \\ \mathbb{O} & \mathbb{B}_2 \end{bmatrix} \begin{bmatrix} \mathbf{X}_1(n) \\ \mathbf{X}_2(n) \end{bmatrix} \tag{17}$$

$$[\mathbf{Y}_1(n) - \mathbf{Y}_2(n)] = [\mathbb{C}_1 \ - \ \mathbb{C}_2] \begin{bmatrix} Z_1(n) \\ Z_2(n) \end{bmatrix} \tag{18}$$

From (17,18), it is clear that for a chosen asymptotic stability model ($\mathbb{A}_2$ is a stable matrix), the augmented system which is incorporated with the state matrix $\begin{bmatrix} \mathbb{A}_1 & \mathbb{O} \\ \mathbb{O} & \mathbb{A}_2 \end{bmatrix}$ is asymptotically stable if and only if the RM is asymptotically stable ($\mathbb{A}_1$ is a stable matrix). The minimization of the L.H.S. of (12) when the number of samples N approaches to infinity is equivalent to the minimization a cost function J_h defined as

$$J_h = \text{trace}(\mathcal{Q}\ \mathcal{R}) \tag{19}$$

where

$$\mathcal{R} = \begin{bmatrix} \mathbb{C}_1^T R_2 \mathbb{C}_1 & -\mathbb{C}_1^T R_2 \mathbb{C}_2 \\ -\mathbb{C}_2^T R_2 \mathbb{C}_1 & \mathbb{C}_1 R_2 \mathbb{C}_1 \end{bmatrix} \in R^{2\rho\times 2\rho}$$

and

$$\mathcal{Q} = \lim_{N\to\infty} \begin{bmatrix} Z_1(n) \\ Z_2(n) \end{bmatrix} \begin{bmatrix} Z_1^T(n) & Z_2^T(n) \end{bmatrix} \in R^{2\rho\times 2\rho}$$

is a positive definite matrix which satisfies the Lyapunov equation

$$\mathcal{A}\ \mathcal{Q} + \mathcal{Q}\ \mathcal{A}^T + \mathcal{V} = 0 \tag{20}$$

where

$$\mathcal{A} = \begin{bmatrix} \mathsf{A}_1 & \mathbf{O} \\ \mathbf{O} & \mathsf{A}_2 \end{bmatrix} \in R^{2\rho \times 2\rho}$$

and

$$\mathcal{V} = \begin{bmatrix} \mathbf{B}_1 \vee \mathbf{B}_1^T & \mathbf{B}_1 \vee \mathbf{B}_2^T \\ \mathbf{B}_2 \vee \mathbf{B}_1^T & \mathbf{B}_2 \vee \mathbf{B}_2^T \end{bmatrix} \in R^{2\rho \times 2\rho}$$

is a positive definite matrix with

$$\vee = \lim_{N \to \infty} [\mathbf{X}_2^T(n)\mathbf{X}_2(n)].$$

The parameters of the RM can then be obtained from the first–order necessary condition for a Lagrangian function formed from (19) and (20) to be an extremum. It is seen that a lot of computations is involved in solving two modified Lyapunov equations which are coupled by a projector. It is also mentioned that due to the lack of sufficient conditions, an absolute minimum value of the formed Lagrangian function may not be obtained and hence an unique solution may not be expected.

As regards the continuous–time model, the input and output data supplied by the linear dynamical operators may be in other than time domain. The convolution integral (summation) still remains in the Hankel operator norms which maps the past input to the future output [24] The Hankel operator norms are used not only for the system analysis and design but also for the model reduction. It is seen from (7) that a cost function can be directly defined on the error in the impulse responses. The minimization of the defined cost function can then be carried out. In such a case, the impulse responses $\mathsf{H}_1(.,.)$ and $\mathsf{H}_2(.,.)$ are to be expressed in the Hankel matrix form using the Markov parameter [19] or in any form of their norms viz., the L_1 norm, The Euclidean (L_2) norm, the Frobenius (trace) norm and the Tchebysheff (L_∞) norm [20]. The error in the impulse responses and the IE can then be expressed in the corresponding form of the norm used and hence the defined cost functions can similarly be expressed. The RM parameters can then be obtained in the proper domain as functions of the model parameters and the input and output data of the system.

Although the configuration of the system to be reduced can be ignored in finding the RM parameters, the computation to be carried out in the minimization of the cost function belonging to any one of the above mentioned possibilities becomes complicated due to the nonlinear computational procedure involved.

Alternatively, since the IE accounts for the difference between the parameters of the RM and the model, a technique for determining the solution of the estimation problem involving a simpler computational procedure is described below.

1.3. Proposed technique of the estimation problem

The difference between the parameter (λ_{sv}) of the RM and that of the model (λ_{mv}) is defined as $\Delta\lambda_v = \lambda_{sv} - \lambda_{mv}$ for $v = 1, 2, \ldots\ r$ where r is the number of the parameters to be determined. Expanding the impulse response $\mathbf{H}_1(.,.)$ of the RM about the parameters of the impulse response of the model $\mathbf{H}_2(.,.)$ in the neighbourhood of $\mathbf{H}_2(.,.)$ by the use of Taylor's series expansion, eqn.(7) is written as

$$\left[\sum_{m=1}^{M} \frac{1}{m!}[\sum_{v=1}^{r} \Delta\lambda_v D_v]^m\ \mathbf{H}_1(n)\right] \mathbf{X}_1(n) = \mathbf{H}_2(n)\,[\mathbf{X}_2(n) - \mathbf{X}_1(n)] \qquad (21)$$

$$n = 0, 1, \ldots, N$$

where M is the number of terms taken in the Taylor's series expansion. D_v is the symbolic forms of the partial derivatives applied to $\mathbf{H}_1(.,.)$ w.r.t. the parameter λ_v of the impulse response of the RM. If IE and D_v are known, the $\Delta\lambda_v$ $(v = 1, 2, \ldots, r)$ can be evaluated.

It is observed that (21) expresses a set of multivariable polynomial equations. If N samples are taken together, the number of equations N is usually greater than the number of unknown r, this set of equations has generally no unique solution for $\Delta\lambda_v$ $(v = 1, 2, \ldots, r)$. An approximate solution can, however, be found by the use of the standard least squares error method but a large computer memory would be required. The equations can be solved in each sample and an arithmetic average would then be taken to avoid the overloading of the memory. If each sample is considered separately, a conjecture should be used since there are r unknowns in a single equation. The principle of equal effects [21] method which permits every variable to contribute the same amount to the error on the R.H.S. of (22) can be used as such a conjecture. It is, however, found that the steepest descent [22] method which permits all the variables to be considered together contributing simultaneously to the error gives better results in finding the parameter departures.

2. Estimation of the parameters of the reduced model

The parameters of the RM can be estimated by finding the aforesaid parameter departures $\Delta\lambda_v$ $(v = 1, 2, \ldots, r)$. As the calculation of the parameter departures requires the knowledge of the input error IE and the gradients D_v, the evaluations of the IE and D_v are considered first.

2.1. Evaluation of the IE

The transfer function matrix of the ρth order model in Z–domain consists of $q \times p$ entries $\mathcal{H}_{ji}(z)$ for $j = 1, 2, \ldots, q$ output and $j = 1, 2, \ldots, p$ input. Each entry is assumed to be a proper rational polynomials and is given by

$$\mathcal{H}_{2_{ji}}(z) = \frac{y_j(z)}{x_{2_i}(z)} = (1 - z^{-1}) \left[\frac{\sum_{v=0}^{\rho} \xi_{iv} s^v}{s(1 + \sum_{\sigma=1}^{\rho} \zeta_{j\sigma} s^\sigma)} \right]^* \tag{22}$$

$$j = 1, 2, \ldots, q; \quad i = 1, 2, \ldots, p$$

where $(1 - z^{-1})/s$ appears due to the presence of the Z.O.H. element and superscript "$*$" stands for the Z–transform of a given function in S–domain. The expression in (22) is defined on the assumption that all inputs other than the ith input are zero. After some mathematical manipulations, (22) is written as

$$\mathcal{H}_{2_{ji}}(z) = \frac{\sum_{v=0}^{\rho+1} a_{iv} z^{-v}}{1 + \sum_{\sigma=1}^{\rho+1} b_{j\sigma} z^{-\sigma}} \quad j = 1, 2, \ldots, q; \quad i = 1, 2, \ldots, p \tag{23}$$

where $\sum_{v=0}^{\rho+1} a_{iv} z^{-v}$ and $(1 + \sum_{\sigma=0}^{\rho+1} b_{j\sigma} z^{-\sigma})$ are coprime polynomials in which the real coefficients a_{iv} and $b_{j\sigma}$ are expressed as the functions of the model parameters $\zeta_{j\sigma}$ and ξ_{iv} and of the sampling period T. It is seen that (23) is an ARMA model whose inverse exists. Moreover, the transfer function of the model–inverse is a proper rational polynomials. For ensuring a desired output, the requested input is given in Z–domain by

$$\mathrm{x}_{2_i}(z) = \mathcal{H}_{2ij}(z) \mathrm{y}_j(z) \quad j = 1, 2, \ldots, q; \quad i = 1, 2, \ldots, p \tag{24}$$

where $\mathcal{H}_{2ij}(z)$ stands for the transfer function of the SISO model–inverse which can readily be obtained from $\mathcal{H}_{2ji}(z)$.

It is noted that in the case of an MIMO model, each input contributes to all of the outputs and each output is effected by all the inputs. If the model delivers the same output as given by the system, the requested input becomes

$$\mathrm{x}_{2_i}(z) = \sum_{j=0}^{q} \mathcal{H}_{2ij}(z)\mathrm{y}_j(z) \quad i = 1, 2, \ldots, p \tag{25}$$

The expression for the requested input $x_{2_i}(n)$ applicable to the model at the nth instant of time is found as

$$x_{2_i}(n) = \sum_{j=1}^{q}\sum_{\sigma=0}^{\rho+1} B_{j\sigma} y_j(n-\sigma) - \sum_{v=1}^{\rho+1} A_{iv} x_{2_i}(n-v) \quad i = 1, 2, \ldots, p \tag{26}$$

where $B_{j\sigma} = b_{j\sigma}/a_{i0}$ and $A_{iv} = a_{iv}/a_{i0}$ which are functions of the model parameters expressed via a_{iv} and $b_{j\sigma}$.

IE can be evaluated by subtracting the original input of the system $x_{1_i}(n)$ from the requested input to the model $x_{2_i}(n)$ obtained from (26).

2.2. Evaluation of the gradient

The gradients can be evaluated by taking the derivatives of the Stirling's interpolation formula [21] of the each entry $h_{1_{ij}}(.,.)$ of the impulse response matrix of the reduced system around that of the model $h_{2_{ji}}(.,.)$.

The impulse response of the RM is given by

$$\begin{aligned} h_{1_{ji}}(n,.) &= \mathbf{Z}^{-1}\left\{(1-z^{-1})\left[\frac{\sum_{v=0}^{\rho}(\xi_{jv} + \Delta\xi_{jv})s^{v}}{s[1+\sum_{\sigma=1}^{\rho}(\zeta_{i\sigma} + \Delta\zeta_{i\sigma}s^{\sigma}]}\right]^{*}\right\} = \\ &= g_{1_{ji}}(n,.) - g_{1_{ji}}(n-1,.) \end{aligned} \tag{27}$$

$$j = 1, 2, \ldots, q; \quad i = 1, 2, \ldots, p$$

where

$$g_{1_{ji}}(n,.) = \mathbf{Z}^{-1}\left\{\left[\frac{\sum_{v=0}^{\rho}(\xi_{jv} + \Delta\xi_{jv})s^{v}}{s[1+\sum_{\sigma=1}^{\rho}(\zeta_{i\sigma} + \Delta\zeta_{i\sigma})s^{\sigma}]}\right]^{*}\right\}, \qquad g_{1_{ji}}(n-1,.)$$

is a sampling period delay of $g_{1_{ji}}(n,.)$ and $\mathbf{Z}^{-1}$ stands for the inverse Z-transform.

The impulse response of the model is similarly obtained as

$$h_{2_{ji}}(n,.) = g_{2_{ji}}(n,.) - g_{2_{ji}}(n-1,.) \tag{28}$$

where $g_{2_{ji}}(n,.) = \mathbf{Z}^{-1}\left\{\left[\frac{\sum_{v=0}^{\rho}\xi_{jv}s^{v}}{s(1+\sum_{\sigma=1}^{\rho}\zeta_{i\sigma}s^{\sigma})}\right]^{*}\right\}$, $g_{2_{ji}}(n-1,.)$ is a sampling period delay of $g_{2_{ji}}(n,.)$.

Using (27) and (28), the interpolation of the impulse response of the RM about the impulse response of the model can be performed by the use of the Stirling's formula and hence the gradients can be evaluated.

2.3. Calculation of the parameter departure

Solving of (21) is equivalent to the determination of the minimum of a function which is defined as

$$\phi(\Delta\lambda_v) = F^2(\Delta\lambda_v) \qquad n = 0, 1, \ldots, N \tag{29}$$

where

$F(\Delta\lambda_v) = \left[\sum_{m=1}^{M} \frac{1}{m!}[\sum_{v=1}^{r} \Delta\lambda_v D_v]^m \mathbf{H}_1(n)\right] \mathbf{X}_1(n) - \mathbf{H}_2(n)\,[\mathbf{X}_2(n) - \mathbf{X}_1(n)]$

The values of the parameter departures are the values of $\Delta\lambda_v (v = 1, 2, \ldots, r)$ corresponding to the minimum point of $\phi(.,.)$. During the course of finding the minimum point of $\phi(.,.)$, an iteration process can be followed and the new values of the parameter departures can be found which are given by [22]

$$\Delta\lambda_{vN} = \Delta\lambda_{v0} - \left[\frac{\phi(\Delta\lambda_v)\frac{\partial\phi(\Delta\lambda_v)}{\partial(\Delta\lambda_v)}}{\sum_{v=1}^{r}\left[\frac{\partial\phi(\Delta\lambda_v)}{\partial(\Delta\lambda_v)}\right]^2}\right]_{\Delta\lambda_v = \Delta\lambda_{v0}} \tag{30}$$

where $\Delta\lambda_{v0}$ is the previous value of $\Delta\lambda_v$. The iteration process is stopped whenever $\mid \Delta\lambda_{vN} - \Delta\lambda_{v0} \mid$ attains a preassigned error limit or the function $\phi(.,.)$ is near to a stationary point.

After determining the parameter departures, the RM parameters can be found by adding the parameter departures to the parameters of the model.

3. Example

Consider a system with zero conditions [7] given by

$$G(s) = \frac{8s^2 + 6s + 2}{s^3 + 4s^2 + 5s + 2} \tag{31}$$

The parameters of a RM are determined by adopting the present approach with the use of the data obtained from system input and output measurements but not using the transfer function of the given system. Let the input of the system be a unit step signal. The system response $y_1(n) = y(n)$ to that input signal is computed and shown in Fig.2 with the sampling period $T = 0.2$ Sec.

The chosen model is a second order one having the damping ratio $\xi_0 = 1$, the undamped resonant frequency $\omega_0 = 2$, $a_0 = \frac{dy(t)}{dt} \mid_{t=0} = 2$ and $b_0 = y(t) \mid_{t=0} = 1$.

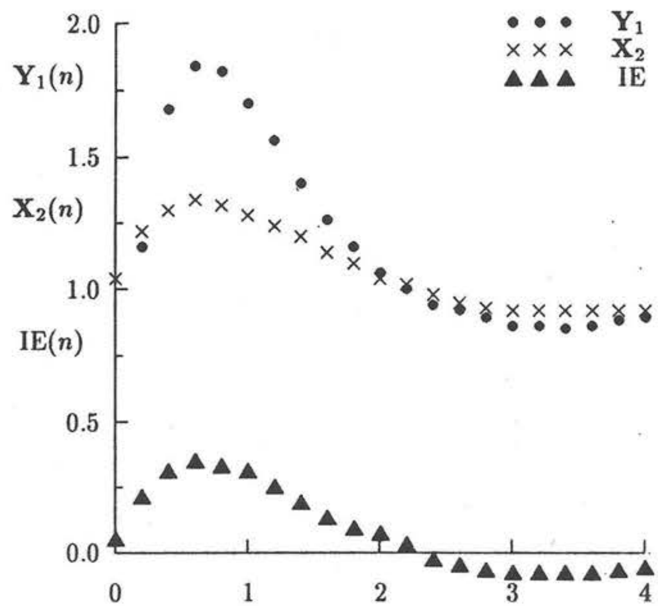

Figure 2. Characteristics of $Y_1(n)$, $X_2(n)$, and $IE(n)$

3.1. Evaluation of the IE

The transfer function of the model in Z–domain is

$$H_2(z) = \frac{Y(z)}{X_2(z)} = (1 - z^{-1})[\frac{\omega_0^2(a_0 s + b_0)}{s(s^2 + 2\xi_0\omega_0 s + \omega_0^2)}]^* \tag{32}$$

For $\xi_0 = 1$, (32) is written after some mathematical manipulations as

$$\frac{Y(z)}{X_2(z)} = \frac{A_1 z^{-1} + A_2 z^{-2} + A_3 z^{-3}}{1 + B_1 z^{-1} + B_2 z^{-2} + B_3 z^{-3}} \tag{33}$$

where

$$\begin{aligned}
A_1 &= b_0(1 - \exp(-\omega_0 T)) + \omega_0(a_0\omega_0 - b_0)T\exp(-\omega_0 T) \\
A_2 &= b_0[\exp(-2\omega_0 T) - 1] - 2\omega_0(a_0\omega_0 - b_0)T\exp(-\omega_0 T) \\
A_3 &= [b_0[1 - \exp(-\omega_0 T)] + \omega_0(a_0\omega_0 - b_0)T]\exp(-\omega_0 T) \\
B_1 &= -[2\exp(-\omega_0 T) + 1] \\
B_2 &= [\exp(-\omega_0 T) + 2]\exp(-\omega_0 T) \\
B_3 &= -\exp(-2\omega_0 T)
\end{aligned}$$

From (33), the requested input of the model is obtained as

$$\begin{aligned} x_2(n) &= \frac{1}{A_1}[y(n+1) + B_1 y(n) + B_2 y(n-1) + B_3 y(n-2) \\ &\quad - A_2 x_2(n-1) - A_3 x_2(n-2)] \end{aligned} \tag{34}$$

where A_1, A_2, A_3, B_1, B_2 and B_3 are given in (33).

The IE(n) can be evaluated by subtracting the unit step input signal of the system from the requested input of the model $x_2(n)$ calculated from (34) by using the computed values of $y(n)$ (shown in Fig.2). The computed results of the requested input $x_2(n)$ and $IE(n)$ are also shown in Fig.2.

3.2. Evaluation of the gradient

The impulse response of the reduced system is

$$\begin{aligned} h_1(n,.) &= \mathbf{Z}^{-1}\left\{(1-z^{-1})[\frac{\omega^2(as+b)}{s(s^2+2\xi\omega s+\omega^2)}]^*\right\} = \\ &= g_1(n,.) - g_1(n-1,.) \end{aligned} \tag{35}$$

where

$$g_1(n,.) = \mathbf{Z}^{-1}\left\{[\frac{\omega^2(as+b)}{s(s^2+2\xi\omega\ s+\omega^2)}]^*\right\}$$

The expressions for $g_1(.,.)$ for different ranges of ξ are

$$\begin{aligned} g_1(n,.) &= b\cdot 1(n) - \frac{\exp(-\xi\omega nT)}{\sqrt{1-\zeta^2}}[(b\xi - a\omega)sin(\omega nT\sqrt{1-\xi^2})] \\ &\quad + b\sqrt{1-\xi^2}\ cos(\omega nT\sqrt{1-\xi^2}) \\ &\quad \text{for } 0 < \xi < 1 \end{aligned} \tag{36}$$

$$\begin{aligned} g_1(n,.) &= b[1(n) - \exp(-\omega nT)] + \omega(a\omega - b)nT\exp(-\omega nT) \\ &\quad \text{for } \xi = 1 \end{aligned} \tag{37}$$

$$\begin{aligned} g_1(n,.) &= b\cdot 1(n) - \frac{\exp(-\xi\omega nT)}{\sqrt{\zeta^2-1}}[(b\xi - a\omega)\ sh(\omega nT\sqrt{\xi^2-1} \\ &\quad + b\sqrt{\xi^2-1}ch(\omega nT\sqrt{\xi^2-1}) \\ &\quad \text{for } \xi > 1 \end{aligned} \tag{38}$$

The impulse response of the model is similarly obtained

$$h_2(n,.) = g_2(n,.) - g_2(n-1,.) \tag{39}$$

where $g_2(n,.) = b_0[1(n) - \exp(-\omega_0 nT)] + \omega_0(a_0\omega_0 - b_0)nT\exp(-\omega_0 nT)$.

Using(35)-(39), the Stirling's interpolation of the impulse response of the RM about the model parameters is carried out considering $\Delta\xi = \Delta\omega = \Delta a = \Delta b = 0.25$. The impulse response of the model and all the partial and mixed partial derivatives of the impulse response of the RM up to second order around the model parameters are calculated.

3.3. Calculation of the parameter departure

Considering only three terms in the Taylor's series expansion, (21) is written as

$$\sum_{k=0}^{n} F(\alpha,\beta,\gamma,\delta) = 0 \quad n = 0, 1, \ldots, 20 \tag{40}$$

where the parameter departures are defined as $\alpha = \omega - \omega_0$, $\beta = \xi - \xi_0$, $\gamma = a - a_0$ and $\delta = b - b_0$ and

$$\begin{aligned} F(.) &= x_1(n-k)[\sum_{m=1}^{2} \frac{1}{m!}(\sum_{j=1}^{4} P_j Dj)_0^m h_1(k,.)] \\ &\quad - h_2(k,.)[x_2(n-k) - x_1(n-k)] \end{aligned} \tag{41}$$

where
$P_j = \alpha, D_j = D_\xi$ for $j = 1$, $P_j = \beta, D_j = D_\omega$ for $j = 2$,
$P_j = \gamma, D_j = D_a$ for $j = 3$, $P_j = \delta, D_j = D_b$ for $j = 4$,
and the index "0" of $(.,.D_j)$ indicates that the partial and mixed derivatives are evaluated at the point $(\xi_0, \omega_0, a_0, b_0)$.

Using (40) and following the iteration process mentioned in (30) with reference to (29), the new values of the parameter departures are calculated with the starting values of the departures as $(0,0,0,0)$ and with the derivatives of the function $\phi(.,.)$ obtained as

$$\frac{\partial\phi(\alpha,\beta,\gamma,\delta)}{\partial P_j} = \tag{42}$$

$$2F(\alpha,\beta,\gamma,\delta)\left\{x_1(n-k)[D_j + P_j D_j^2 + \sum_{i=1 i\neq j}^{3} P_j D_{ij}^2]h_1(k,.)\right\}$$

The steepest descent method is applied at each value of $n (n = 0, 1, \ldots, 20)$ and for each value of n, one set of values comprised of α, β, γ and δ is calculated. The arithmetic average value of each of the parameter departures is found as $\alpha = -0.0816$, $\beta = 0.0141$, $\gamma = 0.0084$ and $\delta = -0.2847$. The transfer function of the RM is then

$$R(s) = \frac{(2.0141)^2 \times (2.0084 \times s + 0.7153)}{s^2 + 2 \times (0.9184) \times (2.0141) \times s + (2.0141)^2} \tag{43}$$

The impulse response of the system as well as of the RM obtained by using the proposed approach and of the RM obtained by Lucas [7] are computed at different instants of time and are given in the Table 1.

Table 1.

Time	Impulse response of	Impulse response of the reduced models obtained by	
(Sec.)	the original system	proposed approach	Lucas method
0.0	8.000	8.1460	7.6000
0.2	3.940	3.8820	4.0600
0.4	1.574	1.4080	1.7600
0.6	0.260	0.0680	0.4000
0.8	-0.411	-0.5786	-0.3190
1.0	-0.701	-0.8212	-0.6250
1.2	-0.775	-0.8450	-0.6860
1.4	-0.734	-0.7605	-0.6200
1.6	-0.638	-0.6340	-0.5000
1.8	-0.523	-0.5020	-0.3700
2.0	-0.409	-0.3820	-0.2580
2.2	-0.306	-0.2820	-0.1655
2.4	-0.218	-0.2020	-0.0980
2.6	-0.146	-0.1410	-0.0510
2.8	-0.089	-0.0960	-0.0200
3.0	-0.045	-0.0640	-0.0035
3.2	-0.012	-0.0410	+0.0055
3.4	+0.011	-0.0260	+0.0090
3.6	+0.027	-0.0153	+0.0096
3.8	+0.038	-0.0087	+0.0085
4.0	+0.044	-0.0045	+0.0067

It is observed from the table that the calculated impulse response of the reduced models obtained by adopting the two approaches shows different amounts of error at different instants of time. Determination of an average error in each case would be convenient for comparison of the accuracy obtained in each approach. With $h_r(k)$ and $h(k)$ as the calculated impulse responses of the RM and actual system respectively, a simple average error is defined as $\eta = \frac{1}{N+1}\sum_{k=0}^{N} \mid h_r(k) - h(k) \mid$, where N is the number of sampling instants. The values of the two average errors calculated from the results obtained by adopting the proposed approach and the Lucas method are respectively $\eta_1 = 0.0516$, $\eta_2 = 0.1056$.

It is seen that the result obtained by adopting the present approach is more closer to the actual values than those obtained by following the Lucas method. Such closeness has been obtained in the present approach by considering only three terms of the Taylor's series expansion of the impulse response of the RM. The results obtained are found encouraging and acceptable.

4. Conclusion

A new approach to obtain a RM from a high-order system is described. In this approach, an RM is assumed to have the same input and output characteristics as those of the high-order system and then a modified MRT is used to determine the RM parameters. Modification in the MRT lies in the fact that an input error is defined instead of the output error as used in the conventional MRT. This consideration allows one to use the real convolution formula which has been found convenient in the mathematical manipulation of the problem and it gives rise to an IE to be made effective at the assumed model input to ensure the same output of the high-order system and the model. This IE which is found to arise due to (i) the departures in the parameters of the RM from those of the model and (ii) the difference between the orders of the high-order system and the RM, has been considered to find the departures in the parameters of the RM and the model. The departures have been calculated by the use of the Taylor's series expansion of the impulse response of the RM around the model parameters and the IE. Only three terms in the expansion series have been considered in the present analysis and the results obtained have been found satisfactory. Better accuracy can, of course, be obtained by considering the higher terms in the series expansion at the cost of complexity in the computation. This possibility of trimming the errors in the results, however, is not observed in the first group of the existing reduction techniques. Different forms of the RM may, however, be obtained using the present approach depending upon the conjecture to be used for solving the multivariable polynomial equation obtained from the series expansion. It may also be noted that the proposed approach does not require the knowledge of the configuration of the system to be reduced but utilizes its input and output data for estimation of the parameters of the RM with a simple computational process.

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Metoda redukcji modelu liniowego

W pracy opisano metodę redukcji modelu o wysokim rzędzie do modelu o niższym rzędzie. Przy założeniu, że zredukowany model daje tę samą odpowiedź, co model wysokiego rzędu, poszukuje się jego struktury za pomocą zmodyfikowanej metody identyfikacji z odniesieniem do modelu. Przy przekształceniach użyto efektywnych wzorów sumowania do obliczania błędów na wejściu modelu. Parametry zredukowanego modelu wyznaczono minimalizując błąd wejściowy i rozwijając dyskretną odpowiedź impulsową w szereg Taylora. Otrzymany zredukowany model zadowalająco dobrze przybliża odpowiedź impulsową modelu wysokiego rzędu.

Метод редуцирования модели

В работе описан метод редуцирования модели высокого порядка к модели низшего порядка. При предпосылке, что редуцированная модель дает такой же результат, что и модель высокого порядка, производится поиск её структуры с помощью модифицированного метода идентификации по отношению к модели. При преобразованиях используются эффективные формулы суммирования для вычисления ошибок на входе модели. Параметры редуцированной модели определены посредствам минимизации входной ошибки и преобразования выходного импульсного дискретного ответа в ряд Тейлора. Полученная редуцированная модель весьма хорошо приближает выходной импульсный ответ модели высокого порядка.