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Models for decision support in allocation of social resources

by

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Models of individual and collective managerial activities are proposed. The model of an individual deals with productive and consumptive activities, as well as savings, which are constrained by financial balance and disposable time resources. The individual is assigning time to activities in such a way that his utility is maximum. Under assumptions formulated unique optimum individual and collective strategies, which take into account competition from other individuals or organizations, can be explicitly derived. They are Pareto optimal, efficient and stable (called effective).

Possible applications of proposed methods for negotiations and decision support systems are discussed.

Keywords: *Social choice theory, Decision Support, Negotiations.*

1. Introduction

From the descriptive point of view a large part of human decisions is concerned with allocation of time and capital resources to different actions or activities. The decisions of participation in these activities are taken under certainty or –

on the basis of imperfect information regarding the future state of the world, as well as – the incomplete knowledge of individual or collective preferences or utilities.

The normative approach, which is the essence of modern decision and utility theory rests on the belief that there are rational people who act or want to act accordingly to the given goals or utilities. This belief is expressed, e.g. in the concept of "economic man" originated by J. Bentham and J. Mill.

It is believed also that rational people prefer to rely on theory when inconsistency of theory and intuition is detected in their activities. Such inconsistencies may happen e.g. in gambling, insurance etc. The famous Allais paradox is an example. Mac Crimmon [9] reported e.g. that when he has presented problems of the kind devised by Allais to upper–middle–level executives his subjects regarded most of the deviations from theory as mistakes and were ready to correct them if given the opportunity.

The belief that theory may correct intuition and improve the process of decision making is encouraging for people who are engaged in modelling and constructing decision support systems. However, there is a lack of formalized models in many branches of social sciences, which could be applied and implemented by computerized support systems. The vast area of management of organizations, in particular, is dominated by descriptive models mostly.

For that reason in the present paper an attempt has been made to construct formalized models of individual and collective behaviour suitable for decision making and computerized decision support implementation.

The models employ some concepts developed by G.S. Becker [1] and M.D. Intriligator [2, 3] and are based on a number of assumptions concerning existence of utilities. It is shown that for each model a unique strategy, which is cooperative (Pareto optimal) stable and efficient (called effective), exists. The effectiveness is understood here as an ultimate standard against which the management can be evaluated.

An interesting feature of the theory presented is that effective strategies can be explicitly derived in many cases and they do not depend on the analytic form of individual utilities.

The concrete examples of application of proposed models, for negotiation and decision support systems, are given.

2. Models of Individual Activities

The present section is concerned with the behaviour of an individual who is a producer, acting also on the market, where he can sell the outcome Y of his productive activity, buy the (necessary for production) capital K and buy consumption goods and services (the behaviour of an individual with income obtained by employment or by investing capital will be studied later). Much research in that direction (see [1]) was done by the so called "household economists". Since in the present paper one is interested in decision making a rather relatively simple model of household has been used.

The individual's net income becomes

$$I = Y - \omega_K K, \quad Y = p\Phi(T_p, K), \tag{1}$$

where $\Phi(T_p, K)$ – production function, T_p – working time, p – product price, ω_K – capital price. This income should be balanced with consumption cost πT_c (assumed proportional to consumption time T_c) and savings S:

$$I = S + \pi T_c = S + \pi(T - T_p), \tag{2}$$

where T – given individual's time resource.

It is natural to assume the production function to be "constant return to scale" (otherwise one could increase outcome by changing scale of units, e.g replacing 1$ by 100 cents and 1h. by 60 min). Then production function can be rewritten

$$\Phi(T_p, K) = T_p\phi(u), \quad u = \frac{K}{T_p} \tag{3}$$

where $\phi(\cdot)$, called time productivity, is assumed to be strictly concave, increasing, $\phi(0) = 0$.

The individual's first objective is to find such $u = \hat{u}$, which maximizes his consumption + savings function:

$$\bar{\phi}(u) = pT_p\phi(u) - \omega_K T_p u.$$

Due to strict concavity of $\bar{\phi}(u)$ a unique optimum value $\hat{u}$ exists and can be derived by solving e.g. $\bar{\phi}'(u) = \frac{\omega_k}{p}$.

For example, when Φ is Cobb–Douglas, i.e. $\phi(u) = \beta u^\alpha$, $0 < \alpha < 1$, $\beta > 0$; one gets

$$\hat{u} = \left(\frac{\alpha\beta p}{\omega_k}\right)^{\frac{1}{1-\alpha}}. \tag{4}$$

With (p, ω_k, π, s) given exogeneously and production function $\phi(u)$, the individual's strategy of allocation of time T between production and consumption can be predetermined. Indeed, by (1) - (3) one gets

$$T_p = \frac{\pi T + S}{\pi + P}, \quad T_c = T - T_p = \frac{PT - S}{\pi + P}, \tag{5}$$

where

$$P = p\phi(\hat{u}) - \omega_k \hat{u}.$$

The necessary capital endowment for production becomes

$$K = \hat{u} T_p.$$

According to (5) when the individual observes that his consumption is too low (high) he can reduce (increase) the value of S.

In the more general model one gets a number (m) of alternative activities A_j, $j = 1, ..., m$, to choose from, which can be classified as productive or consumptive. In order to explain the choice mechanism an assumption is needed.

ASSUMPTION 1 (EVALUATION OF OUTCOMES) *To each activity taking place within the time T and yielding Y_j the individual attaches the value of $B_j = \frac{Y_j}{T}$, $\forall j$.*

Since by (3) the outcomes $Y_j = p_j T_j \phi(\hat{u}_j)$, the B_j coefficients can be regarded as product of : market price p_j, time productivity $b_j = \phi(\hat{u}_j)$, and the individual's choice coefficient $a_j = T_j/T$, $\sum_j a_j = 1$, i.e. $B_j = a_j b_j p_j$, $\forall j$.

The a_j coefficients can be regarded as the individual's willingness to spend the part $a_j T$ of total time resource T on the alternative A_j (when p_j, b_j are ignored). They can be also regarded as attractiveness measure of A_j.

Generally, the modelling of choice coefficients is not easy. A possible approach (see [3]) is to evaluate separately each alternative A_j from the point of view of a given criterion. When there is a set of n criteria given one gets a table A of numbers a_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, m$:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}, \quad \sum_j a_{ij} = 1, \; a_{ij} \geq 0, \; \forall i, j$$

For evaluation of the relative importance of each criterion the weight vector $w \triangleq (w_1, \ldots, w_i, \ldots, w_n)$, $w_i > 0$, $\forall i$, $\sum_i \omega_i = 1$ is introduced. Then, the preference vector $a \triangleq (a_1, \ldots, a_j, \ldots, a_m)$ becomes:

$$a = wA, \tag{6}$$

where

$$a_j = \sum_{i=1}^{n} w_i a_{ij}, \quad \forall j. \tag{7}$$

The relation (6) (7) can be also used in probabilistic choice model where the numbers a_j should be regarded as probabilities (see [3]), while $Y_j = a_j p_j b_j T$, $\forall j$ – the expected outcomes.

It should be noted that in (7) all the criteria contribute to a_j in the additive form. As a result the model does not explain the rejection mechanism. When e.g one regards prices p_j or b_j as criteria the activity should be rejected when $p_j = 0$, or $b_j = 0$ (e.g. due to individual's illness). For that reason one assumes here $B_j = a_j b_j p_j$ rather than $B_j = a_j$, with extended number of criteria (incorporating prices and time productivities).

EXAMPLE 1 Consider the problem of finding attractiveness of three locations for vacations using three criteria: sightseeing, social & cultural life and entertainment. Assume the weight vector $w = (0.4, 0.3, 0.3)$ and

$$A = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.3 & 0.1 & 0.6 \end{pmatrix}.$$

By (6) (7) one gets $a = (0.32, 0.40, 0.28)$.

It should be noted that in practical situations it may be difficult to derive B_j coefficients by formal analysis only. It is possible that people know (from experience mostly) how large outcome one can expect using T_j units of time for a concrete action or activity.

After determining B_j the next important problem is to find time–allocation strategy $x \triangleq (x_1, \ldots, x_m) \in E^m$, which maximizes individual's utility function $U(x)$. It is natural to assume that for a single activity with outcome Y_j the utility is a function $F(x_j, Y_j)$. For a program composed of several activities the additive, with respect to $F(x_j, Y_j)$, form of utility seems to be most appropriate. Since utility should not depend on change of units of measurements the "constant return to scale" form of F, i.e. $Y_j f(\frac{x_j}{Y_j})$ can be also used.

Assumption 2 (evaluation of utility) *The individual evaluates the utility of a program of activities by the formula*

$$U(x) = \sum_{j=1}^{m} Y_j f(\frac{x_j}{Y_j}), \quad Y_j = B_j T, \ \forall j, \tag{8}$$

where $f(z)$ is strictly concave, differentiable and $f'(z) > 0$ (i.e. f is increasing along with z).

It should be observed that strict concavity of f can be also interpreted as "risk averse" attitude of the individual (a person is called *risk averse* if consequence $(x_1 + x_2)/2$ is preferred to a lottery yielding either x_1 or x_2 each with probability of 0.5).

Definition 1 *The time allocation strategy $x = \hat{x}$, is called optimum when*

$$U(\hat{x}) = \max_{x \in \Omega} U(x), \tag{9}$$

where

$$\Omega = \{x_j \mid \sum_{j=1}^{m} x_j \leq T, \ x_j \geq 0, \ \forall j\}.$$

Assertion 1 (existence of optimum strategy) *Under Assumptions 1, 2 the optimum, unique strategy*

$$\hat{x}_j = \frac{B_j}{B} T, \quad \forall j, \quad B = \sum_{j=1}^{m} B_j \tag{10}$$

exists, and

$$U(\hat{x}) = BTf[1/B], \tag{11}$$

Proof: Since f is strictly concave in Ω and $f'(\cdot) > 0$, $\forall j$ there is no stationary point in Ω. The constraints $x_j \geq 0$, $\forall j$ are not active, i.e. the necessary (Kuhn–Tucker) conditions of optimality: $Y_j f'(\cdot)_{\hat{x}} \leq 0$, $\hat{x}_j Y_j f'(\cdot)_{\hat{x}} = 0$, $\hat{x}_j \geq 0$, $\forall j$, do not hold for $\hat{x}_j = 0$, $\forall j$. Then the only active constraint is

$$\sum_{j=1}^{m} x_j = T. \tag{12}$$

As a result the problem boils down to standard Lagrange technique with equality constraints (12) and the necessary conditions of optimality become

$$\frac{dU}{dx_j} = f'\left[\frac{x_j}{Y_j}\right] - \lambda = 0, \quad \forall j \tag{13}$$

where λ – Lagrange multiplier.

Obviously the strategy (10) is the unique strategy that satisfies (12) and (13). Indeed, setting (10) in (13) one gets

$$f'\left[\frac{\hat{x}_j}{Y_j}\right] = f'\left[\frac{1}{B}\right] = \text{const}, \quad \forall j.$$

Setting (10) in (8) one gets (11). ∎

It should be observed that $\hat{x}$ does not depend on the individual form of utility function f. Observe that assumption of strict concavity is essential here. When f is linear, i.e. $f(u) = \beta u$, β =const; the total time resources are assigned to the activity $j = j_0$ with the largest value of B_j: i.e. $B_{j_0} = \max_j B_j$.

It should be also noted that strategy (10) assigns time resources to the continuous (in time) activities. The discrete actions, which are indivisible in time, require the fixed time T_j as well as $\hat{u}_j T_j$ units of capital, and they do not take part in optimization.

After removing the discrete, obligatory or routine activities (from the total set of activities) the individual is left with a disposable T of resources and a number of optional activities which require making choices or decisions. The following example illustrates how such a decision is made in the case of vacations planning process.

EXAMPLE 2 Find allocation of vacation time T among three locations (with preferences derived in Example 1) and costs equal $\pi_i = \pi$ \$/day, $i = 1, 2, 3$. Assume also the time productivities of services at these locations to be:

$$b_1 = 0.4b, \quad b_2 = 0.35b, \quad b_3 = 0.25b, \quad \text{respectively,}$$

b – the average time productivity of tourist services.

By (10) one gets

$$\hat{x}_j = T_j = \frac{a_j b_j}{\sum_j a_j b_j} T, \quad \forall j, \quad \text{where} \quad a_1 = 0.32,\ a_2 = 0.40,\ a_3 = 0.28.$$

The optimum allocation of time intervals become

$$T_1 = 0.38T, \quad T_2 = 0.41T, \quad T_3 = 0.21T,$$

It should be noted that the model studied so far was constructed for an individual living out of his productive labour. It is, however, possible to construct a similar model for the individual living out of capital K (invested in different activities).

For that purpose one should replace Assumption 1 by Assumption 1' which to each activity with outcome Y_j attaches a value, $\tilde{B}_j = Y_j/K$, $\forall j$. In other words the roles of working time and capital resources are presently reversed.

Since the outcome Y_j can be written also $Y_j = p_j K_j \bar{\phi}(\hat{v}_j)$, $\hat{v}_j = T_j/K_j = \hat{u}_j^{-1}$, $\forall j$, the coefficients $\tilde{B}_j$ can be regarded as product of p_j, capital productivities $\tilde{b}_j = \bar{\phi}(\hat{v}_j)$ and individual choice coefficient $\tilde{a}_j = K_j/K$, $\sum_j \tilde{a}_j = 1$, $\forall j$.

Then, replacing (8) by

$$U(z) = \sum_{j=1}^{m} Y_j \tilde{f}(\frac{z_j}{Y_j}), \quad Y_j = \tilde{B}_j K, \quad \forall j, \tag{14}$$

where $\tilde{f}(\cdot)$ is strictly concave, increasing,

$$z \triangleq \{z_1, \ldots, z_m\} \in E^m \text{ – capital allocation strategy;}$$

and defining $\hat{z}$:

$$U(\hat{z}) = \max_{z \in \Omega} U(z), \quad \Omega = \{z_j \mid \sum_j z_j \leq K, z_j \geq 0, \forall j\}$$

one gets

$$\hat{z}_j = \frac{\tilde{B}_j K}{\tilde{B}}, \quad \tilde{B} = \sum_j \tilde{B}_j, \tag{15}$$

$$U(\hat{z}) = \tilde{B} K \tilde{f}[1/\tilde{B}].$$

The working–time endowments necessary for production become

$$T_j = \hat{z}_j \hat{v}_j, \quad j = 1, \ldots, m. \tag{16}$$

3. Risky Actions and Lotteries

So far the outcomes of the actions were regarded as deterministic. In many cases the outcomes $\tilde{Y}_j$, $\forall j$, are accomplished with certain, known probabilities $q_1, \ldots, q_i, \ldots, q_m$, $q_i \geq 0$, $\sum_i q_i = 1$. The decision maker finds out that accomplishment of an outcome is equivalent to gambling at a lottery. The lottery can be imagined as a circle with unit circumference subdivided into arcs of lengths $q_1, \ldots, q_m$ and a "fair" pointer which spins around. When it comes a stop in the arc of length q_j the price $\tilde{Y}_j$ is the outcome.

When using the lottery model to study risky actions one is using it only once and the outcome indicates a single alternative chosen. Contrary to the approach used in Section 2, where Assumption 2 postulates the existence of utility, in the case of lottery the utility is not postulated but it is proven, under a number of assumptions, that utility exists.

Following Von Neumann's and Morgenstern's axiomatic theory of utility many researchers introduce a number of axioms or assumptions. For example, Luce and Raiffa [8] introduce 5 assumptions concerned with: 1. ordering of alternatives, 2. reduction of compound lotteries, 3. continuity of alternatives, 4. substitutability of lotteries, 5. transitivity among lotteries. Under these assumptions they have proved that a real valued utility function U exists.

Specifically, for an ordered set of outcomes

$$\tilde{Y}_1 \succsim \tilde{Y}_2 \succsim \ldots \succsim \tilde{Y}_m,$$

they find number u_j, such that $U(\tilde{Y}_i) = 1$, $U(\tilde{Y}_j) = u_j$, $1 < j < m$, $U(\tilde{Y}_m) = 1$, and

$$U(q_1\tilde{Y}_1, q_2\tilde{Y}_2, \ldots, q_m\tilde{Y}_m) = \sum_{j=1}^{m} q_j u_j \tag{17}$$

The function U is invariant under positive linear transformations. Thus the theorem guarantees that whenever the assumptions hold, there exists a utility function preserving order and satisfying the expectation principle: the utility of a lottery equals expected utility of its outcomes.

An interesting problem is to derive time–allocation strategy for continuous activities, regarded as repeated discrete, risky actions, by using a lottery. One can imagine, e.g. in a "thought experiment", that the total time T is divided into small n subintervals $\Delta T = T/n$. For each subinterval the lottery is run to choose one out of m alternatives with the outcomes $\tilde{Y}_j = b_j p_j T$, and probabilities a_j

each. When the pointer comes to rest at the j–th arc (with the length a_j) n_j times out of n, the frequency $\frac{n_j}{n} = \frac{n_j \Delta T}{n \Delta T} = \frac{T_j^{(n)}}{T} = a_j^{(n)}$, and in the limit $\lim_{n\to\infty} a_j^{(n)} = a_j$, $\forall j$.

In order to compare the utility (8) (in deterministic case) with the corresponding utility $\tilde{U}$ obtained with the lottery model observe that when $U(\tilde{Y}_j) = b_j p_j T f[1/B]$, according to (17), one gets

$$\tilde{U} = \lim_{n\to\infty} \sum_j a_j^{(n)} U(\tilde{Y}_j) = \sum_j a_j b_j p_j T f[1/B] = BTf[1/B] = U.$$

It should be noted that assuming formally $a_j^{(n)} \to a_j$ where a_j are obtained by the "criteria weighting" method (6) (7) requires additional assumptions. As follows from paper [3] by Intriligator the axioms called: 1. Existence of Individual Probabilities, 2. Unanimity Preserving for a Loser, 3. Strict and Weighted Sensitivities to Probability for Each Criterion are here appropriate.

It should be also noted that the Assumption 2 (reduction of compound lotteries) of Luce and Raiffa requires that the separate actions which create activity, should be regarded as "statistically independent".

4. Model with Competition

In the model studied in Section 2 one did not take into account the impact of the competition (i.e. the impact of other individuals attempting to accomplish the same actions at the same time). For example, in the case of tourism n individuals may try to get access to the common system of m services S_j, $j = 1, \ldots, m$, such as hotels, restaurants, theaters etc. having limited capacities C_j (e.g. number of rooms or seats $\times$ days or hours).

When the capacity C_j is less then demand $\sum_\nu x_{\nu j}$, $\forall j$, where x_{ij} – demand claimed by the i–th individual for service at S_j, a part of demand is not satisfied. As a result queues, conflicts and discomfort among the demanders follow. In other words due to competition the utility of each individual decreases. The main problem, therefore, is that of defining utility with competition and finding access strategies $x_{ij} = \hat{x}_{ij}$, $\forall i, j$, which maximize utility in the admissible sets:

$$\Omega_i = \{x_{ij} \mid \sum_j x_{ij} \le T_i,\ x_{ij} \ge 0,\ \forall i, j\}, \quad \forall i \tag{18}$$

where T_i – given individual time resources.

When dealing with competition it is necessary to define a measure of access M_i, say $\Psi_i[C_j, (\sum_\nu x_{\nu j})^{-1}]$, which is an increasing function of C_j and decreasing along with demand $\sum_\nu x_{\nu j}$.

Assuming that measure to be "constant return to scale" one can write

$$M_i = C_j \Psi_i \left(\frac{C_j}{\sum_\nu x_{\nu j}} \right), \tag{19}$$

where $C_j = c_j C$, C – total capacity of the system, $\sum_j c_j = 1$.

It is natural to assume that f_i, Ψ_i functions should enter in the resulting formula for utility, in the multiplicative form (when the accomplishment of an activity due to competition is impossible the resulting utility must be zero). In the same way as in Assumption 2 one can assume an additive form of evaluation of a program of activities.

ASSUMPTION 3 (EVALUATION OF UTILITY WITH COMPETITION) *In the presence of competition the individual evaluates a program of activities by the formula:*

$$U_i(x_i) = CT_i \sum_{j=1}^{m} B_j f_i \left(\frac{x_{ij}}{B_j T_i} \right) \Psi_i \left(\frac{CB_j}{\sum_\nu x_{\nu j}} \right), \quad \forall i, \tag{20}$$

where $B_j = a_j b_j p_j c_j$, $f_i(\cdot), \Psi_i(\cdot)$ *are strictly concave, differentiable and* $\|\text{grad } U_i(x_i)\| > 0$, $\forall i$.

A simple example of $U_i(x_i)$ is the expected number of successful demanders (i.e. people who have accomplished access when supply is less than demand):

$$U_i(x_i) = \sum_{j=1}^{m} \frac{C_j x_{ij}}{\sum_\nu x_{\nu j}}, \quad \forall i. \tag{21}$$

When $C_j < \sum_\nu x_{\nu j}$, the $C_j / \sum_\nu x_{\nu j}$ can be regarded as probability of access to S_j.

In a way similar to that used in Assertion 1 one can prove the following Assertion 2.

ASSERTION 2 (EXISTENCE OF COMPETITION STRATEGIES) *Under Assumption 3 the unique set of optimum strategies, maximizing (20) in* Ω_i*:*

$$\hat{x}_{ij} = \frac{B_j}{B} T_i, \quad B = \sum_j B_j, \quad \forall i, j \tag{22}$$

exists, and

$$U_i(\hat{x}_i) = BCT_i f_i(1/B)\Psi_i\left(\frac{BC}{\sum_\nu T_\nu}\right), \quad \forall i. \tag{23}$$

PROOF: One has to maximize n functionals $U_i(x_i)$, defined on the Ω_i sets of m-dimensional vector space E^m.

Since the partial derivatives

$$\frac{\partial U_i}{\partial x_{ij}} = -\frac{B_j^2 C^2 T_i}{(\sum_\nu x_{\nu j})^2}\Psi_i'(\cdot)f_j(\cdot) + C\Psi(\cdot)f_i'(\cdot), \quad \forall i,j \tag{24}$$

exist and the $\|\text{grad } U_i(x_i)\| > 0$, $\forall i$, there is no stationary point in Ω_i. Since the constraints $x_{ij} \geq 0$, $\forall i,j$ are not active (i.e. the Kuhn–Tucker conditions do not hold for $\hat{x}_{ij} = 0$, $\forall i,j$) the only active constraints remain

$$\sum_j x_{ij} = T_i, \quad \forall i. \tag{25}$$

As a result the problem can be solved by Lagrange multipliers technique. In other words one has to prove that for each index j and $x_{ij} = \hat{x}_{ij}$, $\forall i$

$$\left.\frac{\partial U_i}{\partial x_{ij}}\right|_{x_{ij}=\hat{x}_{ij}} = \lambda_i, \quad \forall j \quad . \tag{26}$$

where λ_i a number (the same for all j).

It can be easily checked that for strategies (22) conditions (25) (26) hold. Due to strict concavity of $U_i(x_i)$ in Ω_i, $\forall i$, these strategies are unique. ■

It should be noted that $\hat{x}_{ij}$ $\forall i,j$ do not depend on the particular forms of individual utilities $f_i(\cdot)$, $\Psi_i(\cdot)$ $\forall i$. These strategies are cooperative (Pareto optimal) what means that nobody, from all the competing individuals, can increase his utility by departing from his optimal strategy. It is believed that cooperativeness contributes much to reaching a consensus in case the access induced conflicts arise.

It should be also noted that "increasing of $f_i(\cdot)\Psi_i(\cdot)$ condition" is essential. As shown in [5, 6], when one drops it unstable strategies may follow. Such a situation happens when $\Psi_i(\cdot)$ is decreasing fast (so U_i loses concavity) and the equality constraints (25) are enforced.

EXAMPLE 3 Consider "n–individuals × two services" model with equality constraint for i–th individual, i.e.

$$x_{i1} + x_{i2} = T_i, \quad x_{i1}, x_{i2} \geq 0. \tag{27}$$

Introduce simpler notation:

$$x_i \triangleq \frac{x_{i1}}{T_i}, \quad x_{i2} = (1 - x_i)T_i$$

and assume

$$U_i(x_i) = a_{i1}x_i\Psi_i(x_i) + a_{i2}(1 - x_i)\Psi_i(1 - x_i), \tag{28}$$

where

$$\Psi_i(x_i) = \left(x_i + \sum_{\nu \neq i}^{n} x_\nu \frac{T_\nu}{T_i}\right)^{-r}.$$

Observe that for $r > 1$ the function $\phi_i = x_i\Psi_i(x_i)$ attains a single maximum for

$$\bar{x}_i = (r - 1)^{-1} \sum_{\nu \neq i}^{n} x_\nu \frac{T_\nu}{T_i}.$$

In Fig 1. the function (28) for $r = 3$ $\sum_{\nu \neq i}^{n} x_\nu \frac{T_\nu}{T_i} = 0.2$ is shown. For $a_{i1} = a_{i2}$ the $U_i(x_i)$ has two equal peak values, for x_i^1 and x_i^2 respectively.

In the case of $r = 1$ and $a_{i1} = a_{i2}$ the optimum strategy is $\hat{x}_i = 1/2$. For $r = 3$ that strategy becomes unstable. Indeed, when a_{i1}/a_{i2} increases slightly the optimum strategy becomes $\hat{x}_i = x_i^1$ while for slightly decreased a_{i1}/a_{i2} the strategy jumps to $\hat{x}_i = x_i^2$. In other words, when a_{i1}/a_{i2} approaches 1 a bifurcation of optimal strategies occurs. Obviously the bifurcations make it impossible to reach a stable program of activities what can be disastrous for an individual (an old story of a donkey, who has starved to death because he could not make his mind as to eat barley or oats, is an example).

Obviously the problem can be stabilized at an expense of resigning from part of disposable resources. (i.e. an efficiency) by decreasing T_i in (27) to the value $\bar{T}_i \leq x_i^1 + x_i^2$.

It should be also noted that unstable processes may follow, as well, in models without competition but with $f(\cdot)$ having a maximum point within Ω set. Such a situation may happen when the consumption time is taking too much out of individual time resources (compare [7]).

$$U_i(x_i)/a_{i_1} = x_i(x_i+0.2)^{-3} + (1-x_i)(1.2-x_i)^{-3} \cdot a_{i_2}/a_{i_1}$$

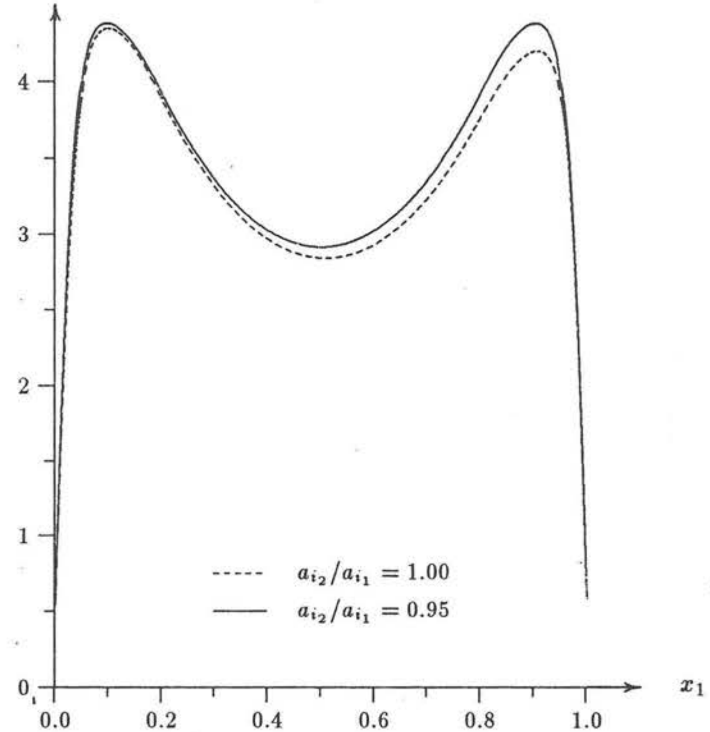

Fig. 1.

DEFINITION 2 *The individuals are called efficient when they employ all the disposable resources T_i, i.e.*

$$\sum_{j=1}^{m} x_{ij} = T_i, \quad \forall j.$$

They are called effective when in addition x_{ij} strategies are cooperative and stable.

The Assertion 2 shows that under Assumption 3 a unique set of effective strategies exists.

5. Model of Organizations

By an organization one understands here a voluntary collective of individuals with a chosen leader who is making decisions in the name of collective. The leader is also organizing (i.e. directing implementation of decisions) and awarding individuals (out of organization income). In the model studied there is n organizations given with N_i, $i = 1, \dots, n$, individuals each.

The utilities of the individuals are of the form (20), i.e.

$$U_{il}(x_{il}) = CT_{il} \sum_{j=1}^{m} B_{jl} f_{il} \left(\frac{x_{ilj}}{B_{jl} T_{il}} \right) \Psi_{il} \left(\frac{CB_{jl}}{\sum_{\nu} x_{\nu lj}} \right), \tag{29}$$

where

$$B_{jl} = a_{jl} b_j c_j p_j, \quad \forall l, j, \quad B_l = \sum_j B_{jl}.$$

Each individual has an admissible set of activities

$$\Omega_{il} = \{ x_{ilj} \mid \sum_j x_{ilj} \le T_{il},\ x_{ilj} \ge 0,\ \forall i, l, j \}$$

It is assumed that the leader (as an individual) possesses the utility of the form (20) but instead of personal he uses the collective, aggregated resources and is motivated by the aggregated preferences. Constructing the model of organization, in such a way, one avoids a difficult problem of assigning utilities to organizations.

Assumption 4 (aggregation of individual resources and preferences) *The leaders of organizations, evaluating the programs of activities, use the utility (20) with the aggregated resources*

$$\tilde{T}_i = \sum_{l=1}^{N_i} T_{il}, \quad \forall i \tag{30}$$

and averaged preferences

$$\frac{\tilde{B}_j}{\tilde{B}} = \frac{1}{\sum_l T_{il}} \sum_{l=1}^{N_i} \frac{B_{jl}}{B_l} T_{il} \tag{31}$$

When the individual time–resources are equal $T_{il} = \tilde{T}_i$, $\forall li$

$$\frac{\tilde{B}_j}{\tilde{B}} = \frac{1}{N_i} \sum_{l=1}^{N_i} \frac{B_{jl}}{B_l} \tag{32}$$

It should be observed that assumptions (31) or (32) require a democratic form of management. The leader should be equally sensitive to individual preferences $\frac{B_{jl}}{B_l}$, $\forall ij$ within the organization.

In the case of probabilistic model Assumption 4 should be supported by additional set of axioms. When e.g. $b_j p_j c_j$ =const $\forall j$ the formula (32) can be written as

$$a_j = \frac{1}{N_i} \sum_{l=1}^{N_i} a_{jl}, \quad \forall j \tag{33}$$

and the leader's problem boils down to the social choice problem. The last is that of obtaining a rule which uses the individual probabilities a_{jl} to determine the social probabilities a_j. As shown by M.D. Intriligator [3] such a rule, satisfying the axioms: 1. existence of social probabilities, 2. unanimity preserving for a loser, 3. strict and equal sensitivity to individual probabilities; exists and is given by (33).

When studying organizations it is also necessary to assume that individuals have an interest in joining the organization.

ASSUMPTION 5 (JOINING ORGANIZATION) *The individual will join an organization when it ensures him accomplishment of a program of activities with bigger savings (than he could earn acting individually).*

In particular, the individual's wage ω_j for 1h of work at j–th activity should be not less than the net income the individual could earn as a producer, i.e. $\omega_j \geq p_j b_j(\hat{u}_j) c_j - \omega_k \hat{u}_j$. For that reason the B_{jl} coefficients in (29) can be written in the form including wages, i.e.

$$B_{jl} = a_{jl}(\omega_j + \omega_k \hat{u}_j), \quad \forall j, l \tag{34}$$

DEFINITION 3 *The leaders are called efficient when they employ all the aggregated resources i.e.*

$$x_{ij} = \sum_{l=1}^{N_i} x_{ilj}, \quad \forall i, j \tag{35}$$

They are called effective when they are efficient and their strategies are cooperative and stable.

ASSERTION 3 (EXISTENCE OF EFFECTIVE STRATEGIES) *Under Assumption 4 the unique set of optimum, effective strategies*

$$\hat{x}_{ij} = \frac{\tilde{B}_j}{\tilde{B}}\tilde{T}_i, \quad \forall i,j, \quad \tilde{B} = \sum_j \tilde{B}_j, \tag{36}$$

$$\hat{x}_{ilj} = \frac{B_{jl}}{B_l}T_{il}, \quad B_l = \sum_j B_{jl}, \tag{37}$$

where T_i, B_j *are defined by (30) (31);*
exists, and

$$U_{il}(\hat{x}_{il}) = B_l C T_{il} f_i(1/B_l)\Psi_i\left(\frac{B_l C}{\sum_{\nu,l} T_{\nu l}}\right), \quad \forall i,l, \tag{38}$$

$$U_i(\hat{x}_i) = \tilde{B}C\tilde{T}_i f_i(1/\tilde{B})\Psi_i\left(\frac{\tilde{B}C}{\sum_\nu \tilde{T}_\nu}\right), \quad \forall i. \tag{39}$$

PROOF: By (27) and Assertion 2 there exists a unique set of individual effective strategies $\hat{x}_{ilj}$, $\forall ilj$, as well as the leaders optimum strategy $\hat{x}_{ij}$, $\forall ij$ expressed by (37) (36) respectively.

It remains to prove that these strategies are efficient. Indeed, by (37) and (31) one gets

$$\sum_l \hat{x}_{ilj} = \sum_l \frac{B_{jl}}{B_l}T_{il} = \frac{\tilde{B}_j}{\tilde{B}}\sum_l \tilde{T}_{il}$$

Since $\sum_l \tilde{T}_{il} = \tilde{T}_i$, by (36) one obtains

$$\sum_{l=1}^{N_i} \hat{x}_{ilj} = \hat{x}_{ij}, \quad \forall i,j$$

and the strategies are efficient. They are also stable and cooperative, i.e. effective. ■

EXAMPLE 4 Consider again two tourists (or group of tourists labeled $l = 1,2$) and three locations, without competition; problem studied in Example 1, 2.

In order too minimize vacation costs (e.g. to get a discount) the tourists have agreed to act collectively and assumed the total vacation time to be $T = 14$ days.

After revealing preferences they would find

$$B_{11} = 0.43T, \quad B_{12} = 0.36T, \quad B_{13} = 0.21T$$
$$B_{21} = 0.21T, \quad B_{22} = 0.36T, \quad B_{23} = 0.43T$$

According to the aggregation of preferences principle (31) they get

$$B_1 = 0.32T, \quad B_2 = 0.36T, \quad B_3 = 0.32T$$

The optimum individual strategies (38) become

$$\hat{x}_{11} = 6 \text{ days}, \quad \hat{x}_{12} = 5 \text{ days}, \quad \hat{x}_{13} = 3 \text{ days}$$
$$\hat{x}_{21} = 3 \text{ days}, \quad \hat{x}_{22} = 5 \text{ days}, \quad \hat{x}_{23} = 6 \text{ days}.$$

The efficient strategy of collective reservation (35) becomes

$$\hat{x}_1 = 9 \text{ days}, \quad \hat{x}_2 = 10 \text{ days}, \quad \hat{x}_3 = 9 \text{ days}.$$

6. Applications – Negotiations & Decision Support

An important sphere of applications of the methodology proposed deals with allocation of labour and capital resources. In many fields, such as Government expenditures for education, health, R & D systems etc, the market mechanisms do not work satisfactorily and decisions regarding allocation of budget among different activities (conducted by universities, hospitals, research institutes etc.) are left to the bureaucracy. Such a practice is generally opposed by the scientific community and often, before the final decision regarding allocation of budget is taken, the programs of activities are formulated in the form of proposals, followed by reviewing and negotiation processes. To facilitate the negotiation and decision processes the computerized support systems can be used.

As a concrete example consider the allocation of research funds. The model consists of l research sponsors S_ν, $\nu = 1, \ldots, l$, which offer the grants to finance activities, proposed by n research organizations O_i, $i = 1, \ldots, n$. The organizations have T_i labour resources (number of researchers $\times$ 1 year) each. The capital "capacities" of sponsors are $C_\nu = K_\nu / c$, where K_ν – capital, c – cost of one researcher/year.

Each organization proposes a strategy $x_i \triangleq (x_{i1}, \ldots, x_{in})$, $\forall i$, concerned with allocation of labour resources (T_i) among m projects $\{P_j\}_1^m$ contracted with S_ν, in such a way that the utility $U_i(x_i)$ is maximum, subject to the constraint $\sum_j x_{ij} \leq T_i$. The O_i representatives, negotiating contracts with S_ν, are concerned, first of all, with the attractiveness (a_j) and time productivities (b_j). They take into account also the future prices (p_j) of outcomes and the expected financial support of $\{P_j\}$ by sponsors, expressed by $c_j = K_j / K_\nu$, $j = 1, \ldots, m$ coefficients. As a result the utility of O_i according to (20) becomes

$$U_i(x_i) = K_\nu T_i \sum_{j=1}^{m} B_j f_i \left(\frac{x_{ij}}{B_j T_i}\right) \Psi_i \left(\frac{K_\nu B_j}{\sum_i x_{ij}}\right), \quad i = 1, \ldots, n \tag{40}$$

where $B_j = a_j p_j b_j c_j$, $j = 1, \ldots, m$.

The optimum strategy for O_i:

$$\hat{x}_{ij} = \frac{B_j}{B} T_i, \quad \forall i, j,$$

can be implemented when the sponsor supplies the necessary capital K_ν, i.e. when he signs a contract with O_i financing the program $\{P_j\}_1^m$.

However, the sponsor may have a different view than O_i regarding the preferences (in terms of a_j, c_j coefficients) of $\{P_j\}$. For example he may prefer the applied to theoretical projects, contrary to what O_i prefers. He may look at O_i as qualified labour resources and if there is a competition (for the same stock of resources the other sponsors compete) his utility $V_\nu(z_\nu)$ according to (20) becomes

$$V_\nu(z_\nu) = K_\nu T_i \sum_{j=1}^{m} A_j \bar{f}_\nu \left(\frac{z_{\nu j}}{A_j K_\nu}\right) \bar{\Psi}_\nu \left(\frac{T_i A_j}{\sum_\nu z_{\nu j}}\right), \quad \nu = 1, \ldots, l, \tag{41}$$

where $A_j = \bar{c}_j p_j \bar{b}_j \bar{a}_j$, $\bar{b}_j = b_j / \hat{u}_j$, $\forall j$, while $\bar{c}_j$, $\bar{a}_j$ are generally different from a_j, c_j.

The optimum sponsor strategy, according to (22),

$$\hat{z}_{\nu j} = \frac{A_j}{A} K_\nu, \quad A = \sum_j A_j, \quad \forall j \tag{42}$$

can be regarded as the capital offered (supplied) by S_ν to $\{P_j\}$. When it is not less the demand for capital $\hat{u}_j \hat{x}_{ij}$, $\forall j$, claimed by O_i, i.e. $\hat{z}_{\nu j} \geq \hat{u}_j \hat{x}_{ij}$, $\forall j$, where

$$\hat{x}_{ij} = \frac{B_j}{B} T_i, \quad \forall j \tag{43}$$

the sponsor is ready to sign the contract.

In the opposite case the projects can be rejected or negotiated. The negotiations can, in particular, result in "rapprochement of points of view" between O_i and S_ν, i.e. bringing $\bar{a}_j$, $\bar{c}_j$ closer to a_j, c_j, $\forall j$. It is also possible to formulate and include within the program $\{P_j\}$ new projects, which have greater values of $A_j / B_j = \frac{\bar{a}_j \bar{c}_j}{a_j c_j} \hat{u}_j$.

The relations

$$\hat{z}_{\nu j} = \hat{u}_j \hat{x}_{ij}, \quad \forall j \tag{44}$$

can be called the symmetric arbitration scheme.

One can easily prove the following assertion.

ASSERTION 4 *When* $K_\nu/T_i = A/B$ *and* $A_j/B_j = \hat{u}_j$, $\forall j$ *the arbitration scheme (44) exists and*

$$U_i(\hat{x}_i) = \max_{x_i \in \Omega_i} U_i(x_i) = BK_\nu T_i f_i(1/B)\Psi_i\left(\frac{BK_\nu}{\sum_i T_i}\right),$$

$$V_\nu(\hat{z}_\nu) = \max_{z_\nu \in \Omega_\nu} V_\nu(z_\nu) = AK_\nu T_i \bar{f}_\nu(1/A)\bar{\Psi}_\nu\left(\frac{AT_i}{\sum_\nu K_\nu}\right).$$

Observe that under arbitration scheme the product $U_i(\hat{x}_i)V_\nu(\hat{z}_\nu)$ is maximum and $\hat{x}_i$, $\hat{z}_\nu$ represents, as well, the solution of Nash bargaining problem.

Problems of Nash type are described e.g. in [8]. The problem discussed presently can be also regarded as composed of several bilateral bargaining games, taking place between O_i, S_ν; $\forall i, \nu$. Each game is cooperative in the sense that preplay messages (concerning $\{P_j\}$ preferences) and discussions, as well as binding agreements between players are possible. The games are nonstrictly competitive so there is a "room" for negotiations. It is worthwhile to mention that Nash solution is unique under the assumptions of: 1. Invariance of players' utilities to linear transformations, 2. Pareto optimality, 3. Independence of irrelevant alternatives (i.e. elimination of points, other then solution, from the bargaining set, which do not change the solution), 4. Symmetry.

It should be also mentioned that the symmetric arbitration scheme (44) is not the only arbitration scheme possible. However, since it comprises the symmetric chances for both players (O_i, S_ν) and it utilizes efficiently all capital and labour resources it can be regarded as just and reasonable. The advantage of the methodology proposed, based on (42) (43) (44) and Assertion 4, is the explicit form of arbitration, negotiations and decisions formulae.

As mentioned already, in order to implement the negotiation processes a negotiation support system can be used.

The main objectives of the negotiations & decision support system could be to:

a) collect and exchange information $(a_j,\ c_j,\ \bar{a}_j,\ \bar{c}_j)$ between O_i and S_ν.

b) mediate, i.e. propose arbitration scheme.

c) recalculate strategies when the number of activities is changed.

It should be noted that the explicit form (36) of effective strategies is very useful but it can be applied for the case when all O_i, S_ν organizations have a similar preference structure, i.e. when $a_{ij} = a_i$, $c_{ij} = c_i$, $\forall i, j$.

In the opposite case the problem becomes more complicated, but can be, in some cases, solved numerically. In particular, for the utility functions of the form (21):

$$U_i(x_i) = \sum_j \frac{c_{ij}}{\sum_\nu x_{ij}}, \quad c_{ij} \text{ given numbers, } \forall i, j,$$

the optimum strategies $\hat{x}_i$, as shown in [6] can be derived numerically.

It should be also noted that a simple version of the proposed decision support system, has been already checked experimentally and applied to allocate research funds among different research teams at Systems Research Institute of Polish Academy of Sciences. It was described in [4].

There is also a number of other applications possible. They were described in details elsewhere (see [5, 6]) and are mentioned here only.

1. **Employment systems.** The model deals with m employers, with given capacities, and n groups of employees (e.g. trade unions) who want to get access to jobs.

2. **Access to public services.** The model deals with n regions and m public services (such as health, education, recreation, water supply, waste disposal etc.) financed by local regional budgets in proportion to populations. The problem is to find the best allocation of budgets among services, taking into account social preferences.

3. **Political systems.** The model deals with n parties which compete in elections at m regions. The number of candidates x_{ij} nominated by i–th party at j–th region is limited. The objective of each party is to maximize the expected number of candidates elected, taking into account that social preferences of each region are given. Using the proposed methodology the stability of the cabinet, depending on parliament support, can be also studied.

It should be noted that in all the applications described the negotiation & decision support systems are not supposed to replace or direct the managers or decision makers, but rather – to help them to reach a consensus, by facilitating an exchange or transformation of information, if necessary.

References

[1] BECKER G.S., A Theory of the Allocation of Time, *The Economic Journal* Sept. (1965), **LXXV**, 413–517.

[2] INTRILIGATOR M.D., A probabilistic model of social choice, *Review of Economic Studies* **40** (1973), 553–560.

[3] INTRILIGATOR M.D., Probabilistic models of choice, *Math. Social Sciences* **2** (1982), 157–166.

[4] KULIKOWSKI R., JAKUBOWSKI A., WAGNER D., Interactive system for collective decision making, *System Analysis, Modelling and Simulation* **3** (1986), 13

[5] KULIKOWSKI R., Access competition and disequilibrium in economic and political systems, Submitted for XVII Intern. Conference on Macromodels '90.

[6] KULIKOWSKI R., Equilibrium and Disequilibrium in Negotiation of Public Joint Venture, in: Systems Analysis a. Computer Science. Proc. of II Polish–Spanish Conf., Rozalin, Sept. 1990. R. Kulikowski, J.Rudnicki, eds. OMNITECH Press, Warszawa, 1990.

[7] KULIKOWSKI R., Optimization and negotiation of complex service program, *Proc. of IFAC Congress,* Tallinn August (1990).

[8] LUCE R.D., RAIFFA H., Games and Decisions, New York, Wiley (1959).

[9] MAC CRIMMON K.R., Descriptive and normative implications of the decision theory postulates, In: K. Borch (ed.) Risk and uncertainty, New York, MacMillan (1967).

Modele dla wspomagania decyzji rozdziału zasobów społecznych

Zaproponowano modele indywidualnych i zbiorowych działań zarządzających. Model dla jednostek obejmuje działania produkcyjne i konsumpcyjne oraz oszczędności, powiązane bilansami finansowymi i rozporządzalnymi zasobami czasowymi. Jednostka przydziela czas na działania w taki sposób, aby jej użyteczność była największa. Przy sformułowanych założeniach można określić jednoznaczne strategie optymalne indywidualne i zbiorowe biorące pod uwagę konkurencję innych jednostek lub organizacji. Są one optymalne w sensie Pareto, sprawne i stabilne (czyli efektywne).

Przedyskutowano możliwe zastosowania zaproponowanych metod w systemach negocjacji i wspomagania decyzji.

Модели для автоматизирования принятия решений по распределению общественных ресурсов

Предложены модели индивидуальных и коллективных управленческих действий. Единичная модель охватывает производственную и потребительскую деятельность, связанную финансовым и распорягаемым ресурсным балансами. Единица определяет время на действие таким образом, чтобы её эффективность была наибольшей. При сформулированных предпосылках можно определить однозначные оптимальные индивидуальные и коллективные стратегии, берущие во внимание конкуренцию других единиц, либо организаций. Они являются оптимальными в смысле Парето, действенными и устойчивыми (т.е. эффективными)

Рассмотрена возможность применения предлагаемых методов в системах переговоров и автоматизированного принятия решений.