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Elementary submatrix operations on polynomial matrices and their use in the analysis of k–th order systems.

by

Piotr Ostalczyk

Institute of Control
Technical University of Łódź.

Elementary submatrix column (row) operations – ESCO's (ESRO's) are defined. They generalize the well known elementary column (row) operations. The list of ESO's (ESCO's and ESRO's) and their properties are given. The proposed ESO's can be useful in k–th order systems analysis because of their compact expression.

1. Introduction

The elementary column (row) operations on polynomial matrices are a useful tool in polynomial matrix theory [1], [2], [4], [5], [6], [7], [11]. There are three fundamental elementary column (row) operations:

(a) multiplication of any column (row) by a non–zero element $\alpha \in F$,

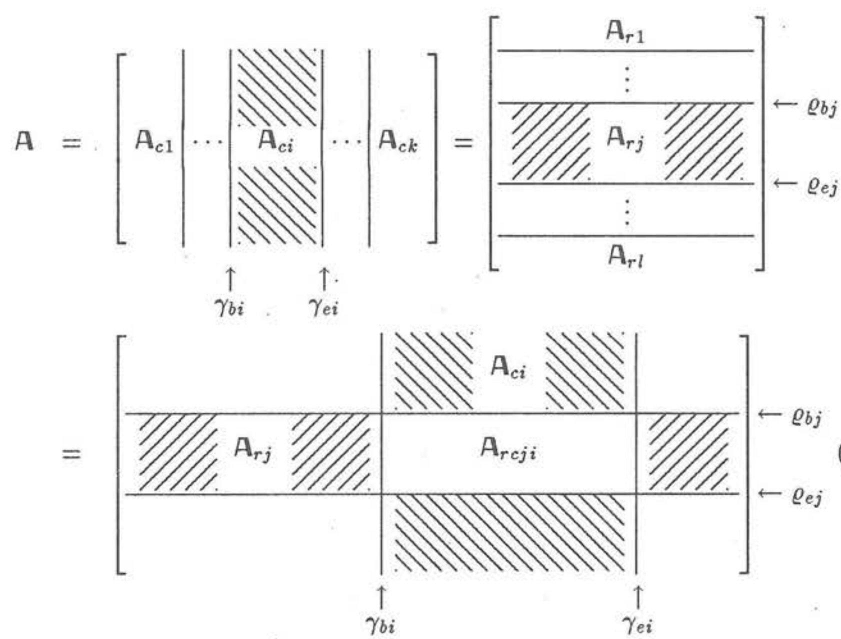
(b) addition to any column (row) of a multiple by any $\beta \in F$ of any other column (row),

(c) interchange of any two columns (rows), where F is the field of coefficients of the polynomials.

These elementary operations can be used in investigations concerning system equivalence [1], [3], [9], [11], standard forms of a matrix [5], [7] where the desirable resulting form is in general obtained by a combination of the elementary row and column operations. It seems convenient to group the elementary operations mentioned above in the manner proposed in this paper.

2. Preliminaries

Let a polynomial matrix $\mathsf{A} \in R[s]^{m\times n}$ be given, where $R[s]^{m\times n}$ is a set of $m \times n$ polynomial matrices of one variable s with real coefficients. Let A be partitioned as follows



(1)

where submatrix A_{ci}, $i \in \underline{k}$ ($\underline{k} = \{1, 2, \ldots, k\}$) is formed out of consecutive columns $\gamma_{bi}, \gamma_{bi}+1, \ldots, \gamma_{ei}$ of A, submatrix A_{rj}, $j \in \underline{l}$ of rows $\varrho_{bj}, \varrho_{bj}+1, \ldots, \varrho_{ej}$ of A and submatrix A_{rcji} of elements of A being simultaneously elements of A_{ci} and A_{rj}. For clarity, these submatrices will be denoted as

$$\mathsf{A}_{ci} \;=\; \mathsf{A}_{ci}(\gamma_{bi}, \gamma_{ei}), \qquad \gamma_{ei} \geq \gamma_{bi} \quad i \in \underline{k} \tag{2}$$

$$\mathsf{A}_{rj} = \mathsf{A}_{rj}(\varrho_{bj}, \varrho_{ej}) \qquad \varrho_{ej} \geq \varrho_{bj} \quad j \in \underline{l} \tag{3}$$

$$\mathsf{A}_{rcji} = \mathsf{A}_{rcji}(\varrho_{bj}, \varrho_{ej}, \gamma_{bi}, \gamma_{ej}) \qquad i \in \underline{k}, j \in \underline{l} \tag{4}$$

where subscripts c and r inform that the pairs of numbers in brackets refer to numbers of extreme columns and rows respectively. The subscript b designates the first row/column, while e designates the last row/column.

I denotes $k \times k$ identity matrix.

3. Elementary submatrix column (row) operations – ESCO's (ESRO's)

In this section definitions of ESCO's (ESRO's) are given. These are of three kinds.

I. Interchange of two submatrices $\mathsf{A}_{ci} = \mathsf{A}_{ci}(\gamma_{bi}, \gamma_{ei})$ and $\mathsf{A}_{cj} = \mathsf{A}_{cj}(\gamma_{bj}, \gamma_{ej})$ of A where

$$\gamma_{ei} - \gamma_{bi} = \gamma_{ej} - \gamma_{bj}, \quad \gamma_{ei} < \gamma_{bj} \quad \text{or} \quad \gamma_{ej} < \gamma_{bi} \tag{5}$$

denoted by

$$P[\mathsf{A}_{ci}, \mathsf{A}_{cj}] = P[\mathsf{A}_{ci}(\gamma_{bi}, \gamma_{ei}) \leftrightarrow \mathsf{A}_{cj}(\gamma_{bj}, \gamma_{ej})] \tag{6}$$

If $\gamma_{ei} < \gamma_{bj}$ then this ESCO performed on a matrix A is equivalent to post-multiplying A by $n \times n$ block matrix $\mathsf{M}^{RI} = \mathsf{M}^{RI}[\mathsf{A}_{ci}, \mathsf{A}_{cj}]$ – a right elementary block matrix (REBM).

$$\mathsf{M}^{RI} = \begin{array}{c}\left[\begin{array}{c|c|c|c|c} \mathsf{I}_{\gamma_{bi}-1} & & & & \\ \hline & \cdot & & \mathsf{I}_{\gamma_{ej}-\gamma_{bj}+1} & \\ \hline & & \mathsf{I}_{\gamma_{bj}-\gamma_{ei}-1} & & \\ \hline & \mathsf{I}_{\gamma_{ei}-\gamma_{bi}+1} & & & \\ \hline & & & & \mathsf{I}_{n-\gamma_{ej}} \end{array}\right] \begin{array}{l} \\ \leftarrow \gamma_{bi} \\ \leftarrow \gamma_{ei} \\ \leftarrow \gamma_{bj} \\ \leftarrow \gamma_{ej} \end{array} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \gamma_{bi} & \gamma_{ei} & \gamma_{bj} & \gamma_{ej} \end{array}\end{array} \tag{7}$$

where all remaining submatrices not explicitly shown are zero. This matrix clearly is unimodular.

II. Postmultiplication of a submatrix $\mathsf{A}_{ci} = \mathsf{A}_{ci}(\gamma_{bi}, \gamma_{ei})$ by a nonzero $(\gamma_{ei} - \gamma_{bi} + 1) \times (\gamma_{ei} - \gamma_{bi} + 1)$ polynomial unimodular submatrix $\mathsf{C}(s)$. This ESCO will be denoted by

$$P[\mathbf{A}_{ci} \times \mathbb{C}] = P[\mathbf{A}_{ci}(\gamma_{bi}, \gamma_{ei}) \times \mathbb{C}] \tag{8}$$

and is equivalent to postmultiplication of $\mathbf{A}$ by $n \times n$ block matrix $\mathbf{M}^{RII}$ (REBM)

$$\mathbf{M}^{RII} = \begin{array}{c} \left[\begin{array}{c|c|c} \mathbf{I}_{\gamma_{bi}-1} & & \\ \hline & \mathbb{C}(s) & \\ \hline & & \mathbf{I}_{n-\gamma_{ej}} \end{array}\right] \begin{array}{l} \leftarrow \gamma_{bi} \\ \leftarrow \gamma_{ei} \end{array} \\ \uparrow \qquad \uparrow \\ \gamma_{bi} \qquad \gamma_{ei} \end{array} \in R[s]^{n\times n} \tag{9}$$

such that $\mathbb{C}^{-1}(s) \in R[s]^{(\gamma_{ei}-\gamma_{bi}+1)\times(\gamma_{ei}-\gamma_{bi}+1)}$ so that

$$(\mathbf{M}^{RII})^{-1} \in R[s]^{n\times n} \tag{10}$$

III. For $j \not\equiv i$ addition to a submatrix $\mathbf{A}_{ci} = \mathbf{A}_{ci}(\gamma_{bi}, \gamma_{ei})$ of another submatrix $\mathbf{A}_{cj} = \mathbf{A}_{cj}(\gamma_{bj}, \gamma_{ej})$ postmultiplied by a polynomial $(\gamma_{ej} - \gamma_{bj} + 1) \times (\gamma_{ei} - \gamma_{bi} + 1)$ submatrix $\mathbb{C}(s) \in R[s]^{(\gamma_{ej}-\gamma_{bj}+1)\times(\gamma_{ei}-\gamma_{bi}+1)}$. This ESCO will be denoted

$$P[\mathbf{A}_{ci} + \mathbf{A}_{cj} \times \mathbb{C}] = P[\mathbf{A}_{ci}(\gamma_{bi}, \gamma_{ei}) + \mathbf{A}_{cj}(\gamma_{bj}, \gamma_{ej}) \times \mathbb{C}] \tag{11}$$

and is equivalent to postmultiplication of $\mathbf{A}$ by $n \times n$ block matrix

$$\mathbf{M}^{RIII} = \begin{array}{c} \begin{array}{r} \\ \\ \gamma_{bj} \rightarrow \\ \\ \gamma_{ej} \rightarrow \end{array} \left[\begin{array}{c|c} \mathbf{I}_k & \\ \hline \boxed{\mathbb{C}(s)} & \mathbf{I}_{n-k} \end{array}\right] \\ \uparrow \qquad \uparrow \\ \gamma_{bi} \qquad \gamma_{ei} \end{array} \tag{12}$$

where $\gamma_{ei} \leq k \leq \gamma_{bj}$ if $\gamma_{ei} < \gamma_{bj}$
and $\gamma_{ej} \leq k \leq \gamma_{bi}$ if $\gamma_{ej} > \gamma_{bi}$. (13)

Elementary submatrix row operations – ESRO's are similarly defined. One should replace "column" by "row" and "postmultiplication" by "premultiplication". Thus the ESRO's analogous to those of (6), (8), (11) are as follows

I. $L[\mathbf{A}_{ri}, \mathbf{A}_{rj}] = L[\mathbf{A}_{ri}(\varrho_{bi}, \varrho_{ei}) \leftrightarrow \mathbf{A}_{rj}(\varrho_{bj}, \varrho_{ej})]$ (14)

where $\varrho_{ej} - \varrho_{bj} = \varrho_{ei} - \varrho_{bi}$, $\varrho_{ei} < \varrho_{bj}$ or $\varrho_{ej} < \varrho_{bi}$ (15)

II. $L[\mathbb{C} \times \mathbf{A}_{ri}] = L[\mathbb{C} \times \mathbf{A}_{ri}(\varrho_{bi}, \varrho_{ei})]$ (16)

III. $\quad L[\mathsf{A}_{ri} + \mathsf{C} \times \mathsf{A}_{rj}] = L[\mathsf{A}_{ri}(\varrho_{bi}, \varrho_{ei}) + \mathsf{C} \times \mathsf{A}_{rj}(\varrho_{bj}, \varrho_{ej})]$
where $\varrho_{ei} < \varrho_{bj}$ or $\varrho_{ej} < \varrho_{bi}$. (17)

The above ESRO's are equivalent to premultiplication of A by left elementary block matrices – LEBM's M^{LI}, M^{LII}, M^{LIII} created analogously to REBM's.

Some obvious properties of ESCO's (ESRO's) will be stated.

4. Main properties of the elementary submatrix operations

1° *Any ESCO (ESRO) is a finite product of elementary column (row) operations.*

This follows immediately from the fact that the REBM (LEBM) related to the ESCO (ESRO) is unimodular and hence a finite product of right (left) elementary matrices [1] which is equivalent to ESCO (ESRO) being a finite product of elementary column (row) operations.

2° *Any elementary operation is a special case of ESO.*

Any column (row) $\boldsymbol{a}_{ci}$ $(\boldsymbol{a}_{ri})$ of A can be treated as an $m \times 1$ $(1 \times n)$ submatrix of A and in proposed notation $\boldsymbol{a}_{ci}$ $(\boldsymbol{a}_{ri})$ can be expressed as follows $\boldsymbol{a}_{ci} = \mathsf{A}_{ci}(\gamma_{bi}, \gamma_{bi})$, $(\boldsymbol{a}_{ri} = \mathsf{A}_{ri}(\varrho_{i1}, \varrho_{i1}))$. Thus any elementary column (row) operation is an ESCO (ESRO).

3° *Each ESO is invertible.*

For every ESCO (ESRO) there exists an inverse ESCO (ESRO) which neutralizes its application. A product of REBM's (LEBM's) corresponding to them is equal to a unity matrix.

4° *For $m = n$, each ESO is reversible.*

If $m = n$, for every ESCO (ESRO) there exists a reverse ESRO (ESCO) such that the product of their EBM's is equal to a unity matrix.

5° *A polynomial matrix is unimodular if and only if it is a finite product of EBM's.*

6° *The ESO performed on a polynomial matrix does not change its normal rank.*

The statements 4°, 5°, 6° are simple consequence of 1°.

The ESO's will be applied to derive some properties concerning the multivariable k–th order type linear systems [9].

5. The k–th order type linear systems

By the definition, [9], an m–input/m–output strictly proper system S described by the $m \times m$ transfer function matrix $\mathbb{G}_0(s)$ is said to be a multivariable k–th order type system if and only if

$$\det \mathbb{G}_0(s) \not\equiv 0 \tag{18}$$

$$\mathbb{G}_0^{-1}(s) \;=\; s^k \mathbb{A}_{m,k} + s^{k-1}\mathbb{A}_{m,k-1} + \ldots + s\mathbb{A}_{m,1} + \mathbb{A}_{m,0} \tag{19}$$

where $\det \mathbb{A}_{m,k} \not\equiv \mathbb{O}$, $\mathbb{A}_{m,i}$ $i \in \underline{k}$ are real, constant $m \times m$ matrices.

This system can be described by the state–space equations of the form [6], [9], [11]

$$\dot{\boldsymbol{x}}(t) = \mathbb{A}_0 \boldsymbol{x}(t) + \mathbb{B}_0 \boldsymbol{u}(t) \tag{20}$$

$$\boldsymbol{y}(t) = \mathbb{C}_0 \boldsymbol{x}(t) \tag{21}$$

where

$$\mathbb{A}_0 = \begin{bmatrix} \mathbb{O} & \mathbb{I}_m & \mathbb{O} & \ldots & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I}_m & \ldots & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \ldots & \mathbb{O} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \ldots & \mathbb{I}_m \\ -\mathbb{A}_{m,k}^{-1}\mathbb{A}_{m,0} & -\mathbb{A}_{m,k}^{-1}\mathbb{A}_{m,1} & -\mathbb{A}_{m,k}^{-1}\mathbb{A}_{m,2} & \ldots & -\mathbb{A}_{m,k}^{-1}\mathbb{A}_{m,k-1} \end{bmatrix}$$

$$\mathbb{B}_0 = \begin{bmatrix} \mathbb{O} \\ \mathbb{O} \\ \mathbb{O} \\ \vdots \\ \mathbb{O} \\ \mathbb{A}_{m,k}^{-1} \end{bmatrix} \tag{22}$$

$$\mathbb{C}_0 = \begin{bmatrix} \mathbb{I}_m & \mathbb{O} & \mathbb{O} & \ldots & \mathbb{O} \end{bmatrix}$$

It is assumed that:

1) $\det\{\mathbb{G}_0^{-1}(s)\}$ has only distinct eigenvalues

2) There are two $m \times m$ unimodular matrices $\mathbf{L}$, $\mathbf{P}$ such that $\mathbf{L}\mathbf{G}_0^{-1}(s)\mathbf{P} = \mathbf{S}_0(s)$, where $\mathbf{S}_0(s)$ has the Smith canonical form [1], [2], [5].

Let $\mathbf{A}^{(0)} = s\mathbf{I}_m - \mathbf{A}_0 =$

$$
= \begin{bmatrix}
s\mathbf{I}_m & -\mathbf{I}_m & \mathbf{O} & \dots & \mathbf{O} \\
\mathbf{O} & s\mathbf{I}_m & -\mathbf{I}_m & \dots & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & s\mathbf{I}_m & \dots & \mathbf{O} \\
\vdots & \vdots & \vdots & & \vdots \\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & -\mathbf{I}_m \\
\mathbf{A}_{m,k}^{-1}\mathbf{A}_{m,0} & \mathbf{A}_{m,k}^{-1}\mathbf{A}_{m,1} & \mathbf{A}_{m,k}^{-1}\mathbf{A}_{m,2} & \dots & s\mathbf{I}_m + \mathbf{A}_{m,k}^{-1}\mathbf{A}_{m,k-1}
\end{bmatrix} \tag{23}
$$

be the $mk \times mk$ characteristic matrix of $\mathbf{A}_0$ partitioned as follows

$$
\mathbf{A}^{(0)} = \left[\mathbf{A}_{c,1}^{(0)} \mid \mathbf{A}_{c,2}^{(0)} \mid \dots \mid \mathbf{A}_{c,k}^{(0)}\right] = \begin{bmatrix} \mathbf{A}_{r,1}^{(0)} \\ \hline \vdots \\ \hline \mathbf{A}_{r,k}^{(0)} \end{bmatrix} \tag{24}
$$

where $\mathbf{A}_{c,i}^{(0)}$, $\mathbf{A}_{r,j}^{(0)}$, $i, j \in \underline{k}$ are $mk \times m$ and $m \times mk$ matrices, respectively. For convenience it is assumed that

$$
\mathbf{A}_{c,k+1}^{(0)} = \mathbf{B}_0, \qquad \mathbf{A}_{r,k+1}^{(0)} = \mathbf{C}_0 \tag{25}
$$

and the following notation is introduced. For any matrix $\mathbf{E}^{(l-1)}$, $l = 1, 2, \dots$

$$
\mathbf{E}^{(l-1)} = \left[\mathbf{E}_{c,1}^{(l-1)} \mid \mathbf{E}_{c,2}^{(l-1)} \mid \dots \mid \mathbf{E}_{c,k}^{(l-1)}\right] = \begin{bmatrix} \mathbf{E}_{r,1}^{(l-1)} \\ \hline \vdots \\ \hline \mathbf{E}_{r,k}^{(l-1)} \end{bmatrix}
$$

a block matrix $\mathbf{E}^{(l)}$

$$
\mathbf{E}^{(l)} = \left[\mathbf{E}_{c,1}^{(l)} \mid \mathbf{E}_{c,2}^{(l)} \mid \dots \mid \mathbf{E}_{c,k}^{(l)}\right] = \begin{bmatrix} \mathbf{E}_{r,1}^{(l)} \\ \hline \vdots \\ \hline \mathbf{E}_{r,k}^{(l)} \end{bmatrix} \tag{26}
$$

denotes $\mathbf{E}^{(l-1)}$ after l–th ESO.

Now the ESO's will provide a nice way of deriving some basic properties of the system S.

a) The system S is controllable.

The system S is controllable if and only if the controllability matrix $\mathbf{W}^{(0)}$ of a pair $\{\mathbf{A}_0, \mathbf{B}_0\}$

$$\mathbf{W}^{(0)} = \left[\mathbf{A}_{c,1}^{(0)} \mid \mathbf{A}_{c,2}^{(0)} \mid \ldots \mid \mathbf{A}_{c,k}^{(0)} \mid \mathbf{A}_{c,k+1}^{(0)}\right] \tag{27}$$

is of full rank for every s [7].

The following ESCO's

$$P_l \left[\mathbf{A}_{c,l}^{(l-1)} + \mathbf{A}_{c,k+1}^{(l-1)} \times (-\mathbf{A}_{m,l-1})\right] \qquad l \in \underline{k} \tag{28}$$

$$P_{k+1} \left[\mathbf{A}_{c,k+1}^{(k)} \times (-\mathbf{A}_{m,k})\right] \tag{29}$$

$$P_l \left[\mathbf{A}_{c,2k+2-l}^{(l-1)} + \mathbf{A}_{c,2k+3-l}^{(l-1)} \times (s\mathbf{I}_m)\right] \qquad l \in k+2, k+3, \ldots, 2k+1 \tag{30}$$

bring the matrix $\mathbf{W}^{(0)}$ to the form

$$\mathbf{W}^{(2k+1)} = [\mathbf{O} \mid -\mathbf{I}_{mk}] \tag{31}$$

which is of full rank for every s, thus the system S is controllable.

b) The Smith canonical form of $\mathbf{A}^{(0)}$ equals to

$$\operatorname{diag}\{\mathbf{I}_{m(k-1)}, \mathbf{S}_0(s)\} \tag{32}$$

The following ESO's

$$\begin{aligned}
&P_l \left[\mathbf{A}_{c,k-l}^{(l-1)} + \mathbf{A}_{c,k-l+1}^{(l-1)} \times (s\mathbf{I}_m)\right] \qquad l \in \underline{k-1} \\
&L_l \left[\mathbf{A}_{r,k}^{(l-1)} + \mathbf{A}_{r,2k-l-1}^{(l-1)} \times (s^{l-k+1}\mathbf{I}_m + s^{l-k}\mathbf{A}_{m,k}^{-1}\mathbf{A}_{m,k-1} + \ldots \right. \\
&\qquad \left. \ldots + \mathbf{A}_{m,k}^{-1}\mathbf{A}_{m,2k-l-1})\right] \quad l = k, \ldots, 2k-2 \\
&L_{2k-1} \left[\mathbf{A}_{r,k}^{(2k-2)} \times (\mathbf{A}_{m,k})\right] \\
&P_l \left[\mathbf{A}_{c,l-2k+1}^{(l-1)} \leftrightarrow \mathbf{A}_{c,l-2k+2}^{(l-1)}\right] \qquad l = 2k, \ldots, 3k-2 \\
&P_l \left[\mathbf{A}_{c,l-3k+2}^{(l-1)} \times (-\mathbf{I}_m)\right] \qquad l = 3k-1, \ldots, 4k-3 \\
&P_{4k-2} \left[\mathbf{A}_{c,k}^{(4k-3)} \times (\mathbf{P})\right] \\
&L_{4k-1} \left[\mathbf{A}_{c,k}^{(4k-2)} \times (\mathbf{L})\right]
\end{aligned} \tag{33}$$

performed on the matrix $\mathbf{A}^{(0)}$ lead to the form (32).

c) For a given matrix polynomial equation $\mathbf{G}_0^{-1}(s) = 0$ there exist k solution matrices $\mathbf{T}_i$ such that

$$\mathbf{A}_{m,k}\mathbf{T}_i^k + \mathbf{A}_{m,k-1}\mathbf{T}_i^{k-1} + \ldots + \mathbf{A}_{m,1}\mathbf{T}_i + \mathbf{A}_{m,0} = \mathbf{O} \qquad i \in \underline{k} \tag{34}$$

where the matrices $\mathsf{T}_{i,j}$

$$\mathsf{T}_{i,j} = \mathsf{T}_i - \mathsf{T}_j \qquad i \neq j, \quad i,j \in \underline{k} \tag{35}$$

are nonsingular [4], [6]. Then the transformation matrix $\mathsf{T}^{(0)}$

$$\mathsf{T}^{(0)} = \begin{bmatrix} \mathsf{I}_m & \mathsf{I}_m & \dots & \mathsf{I}_m \\ \mathsf{T}_1 & \mathsf{T}_2 & \dots & \mathsf{T}_k \\ \mathsf{T}_1^2 & \mathsf{T}_2^2 & \dots & \mathsf{T}_k^2 \\ \vdots & \vdots & & \vdots \\ \mathsf{T}_1^{k-1} & \mathsf{T}_2^{k-1} & \dots & \mathsf{T}_k^{k-1} \end{bmatrix} =$$

$$= \left[\mathsf{T}_{c,1}^{(0)} \mid \mathsf{T}_{c,2}^{(0)} \mid \dots \mid \mathsf{T}_{c,k}^{(0)}\right] = \begin{bmatrix} \mathsf{T}_{r,1}^{(0)} \\ \hline \mathsf{T}_{r,2}^{(0)} \\ \hline \mathsf{T}_{r,3}^{(0)} \\ \hline \vdots \\ \hline \mathsf{T}_{r,k}^{(0)} \end{bmatrix} \tag{36}$$

is nonsingular and

$$\left[\mathsf{T}^{(0)}\right]^{-1} \mathsf{A}_0 \mathsf{T}^{(0)} = \operatorname{diag}\{\mathsf{T}_1, \mathsf{T}_2, \dots, \mathsf{T}_k\} \tag{37}$$

The following ESO's performed on $\mathsf{T}^{(0)}$

$$\begin{aligned} &L_l\left[\mathsf{T}_{r,k-l+1}^{(l-1)} + \mathsf{T}_{r,k-l}^{(l-1)} \times (-\mathsf{T}_1)\right] && l \in \underline{k-1} \\ &P_l\left[\mathsf{T}_{c,l-k+2}^{(l-1)} + \mathsf{T}_{c,1}^{(l-1)} \times (-\mathsf{I}_m)\right] && l = k, \dots, 2k-2 \\ &P_l\left[\mathsf{T}_{c,l-2k+2}^{(l-1)} \times (\mathsf{T}_{l-2k+2,1}^{-1})\right] && l = 2k, \dots, 3k-3 \end{aligned} \tag{38}$$

give $\mathsf{T}^{(3k-2)} = \left[\mathsf{T}_{c,1}^{(3k-2)} \mid \mathsf{T}_{c,2}^{(3k-2)} \mid \dots \mid \mathsf{T}_{c,k}^{(3k-2)}\right] = \begin{bmatrix} \mathsf{T}_{r,1}^{(3k-2)} \\ \hline \mathsf{T}_{r,2}^{(3k-2)} \\ \hline \mathsf{T}_{r,3}^{(3k-2)} \\ \hline \vdots \\ \hline \mathsf{T}_{r,k}^{(3k-2)} \end{bmatrix} =$

$$= \left[\begin{array}{c|ccc} \mathsf{I}_m & \mathsf{O} & \dots & \mathsf{O} \\ \hline \mathsf{O} & \mathsf{I}_m & \dots & \mathsf{I}_m \\ \mathsf{O} & \mathsf{T}_2' & \dots & \mathsf{T}_k' \\ \vdots & \vdots & & \vdots \\ \mathsf{O} & (\mathsf{T}_2')^{k-2} & \dots & (\mathsf{T}_k')^{k-2} \end{array}\right] \tag{39}$$

where

$$\mathsf{T}'_i = \mathsf{T}_{i,1}\,\mathsf{T}_i\;\;\mathsf{T}_{i,1}^{-1} \qquad i = 2,\ldots,k$$

and the matrices $\mathsf{T}'_{i,j}$

$$\mathsf{T}'_{i,j} = \mathsf{T}'_i - \mathsf{T}'_j \qquad i \neq j, \quad i,j \in 2,\ldots,k \tag{40}$$

are nonsingular [2]. Continuing this procedure one easily proves that $\mathsf{T}^{(0)}$ is nonsingular. Therefore state-variable transformation matrix $\mathsf{T}^{(0)}$ is obtained giving convenient parallel submatrix realization of S.

6. Conclusions

ESO's proposed in this paper as a generalization of elementary operations can be helpful in investigations concerning block matrices [3], [4], [8], [10], [12].

It should be noted that in performing any elementary column (row) operation on selected column (row) of A in general one can change desirably only one entry of this column (row) when all remaining ones alternate out of control. And so it is in the ESO case. Influencing any submatrix A_{rcji} of A_{rj} (A_{ci}) by ESCO (ESRO) one changes all remaining submatrices of A_{rj} (A_{ci}).

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Elementarne przekształcenia macierzowe na macierzach wielomianowych i ich zastosowanie do analizy systemów k–tego rzędu

Zdefiniowano elementarne przekształcenia macierzowe ESCO (ESRO) na kolumnach (rzędach) macierzy. Uogólniają one znane elementarne przekształcenia na kolumnach (rzędach). Określono przekształcenia ESO (tzn. ESCO i ESRO) oraz podano ich właściwości. Zaproponowane przekształcenia moga być użyteczne w analizie systemów k–tego rzędu.

Элементарные матричные преобразования на многочленных матрицах и их применение к анализу k–ого порядка

Определены элементарные матричные преобразования ESCO (ESRO) на столбцах (рядах) матрицы. Они обобщают известные элементарные преобразования на столбцах (рядах). Определены преобразования ESO (т.е. ESCO и ESRO), а также приведены их свойства. Предложенные преобразования могут быть полезными в анализе систем k–ого порядка.