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Sensitivity analysis for minimum weight base of a matroid

by

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Many different combinatorial optimization problems can be formulated as a problem of finding a minimum weight base of an appropriate matroid and solved by the greedy algorithm.
This paper addresses some postoptimality analysis questions for this problem. Given the minimum weight base we want to calculate the maximum increase and decrease of the weight of each matroid element which preserves the optimality of this base. A method of computing such a data perturbations, called tolerances of matroid elements, is described. Some other postoptimality analysis questions, which can be solved when the tolerances of elements are known, are discussed.

1. Introduction

Many different combinatorial optimization problems can be formulated as a problem of calculating a minimum weight base of an appropriate matroid (see e.g. [3]). It is well known that such a problem can be efficiently solved by the greedy algorithm. This paper addresses some problems belonging to the *postoptimality analysis* (see [1]). It is assumed that the minimum weight base of the matroid is known and we are looking for maximum individual changes of the weights of matroid elements which do not forfeit the optimality of the base. Such a maximum perturbations of elements weights are called the *tolerances* of matroid elements. The tolerances represent the sensitivity of the optimal solution with respect to individual changes of problem data. They can be also useful in answering some other postoptimality analysis questions consisting in introducing additional restrictions on the optimal solution of the considered problem.

The question of calculating tolerances of problem data for other combinatorial optimization problems such as the shortest path problem, the minimum spanning tree problem, the minimum cost network flow problem, the binary knapsack problem, the traveling salesman problem, was addressed in several papers [2,4,5,6,7].

This paper is organized as follows. In Section 2 some necessary facts from the matroid theory are recollected. In Section 3 the problem of calculating the tolerances of matroid elements is stated and a method of solving it is described. Section 4 contains a discussion of two other postoptimality analysis problems. These problems can be regarded as the minimum weight base problems with side constraints and the notion of the tolerances of elements allows a very fast calculation of lower bounds for its optimal solution values.

2. Notation and necessary elements of matroid theory

Let S be a finite set, $|S| = m$, and $\mathcal{I}$ be a collection of subsets of S.
A pair $M = (S, \mathcal{I})$ is called a *matroid* on S if the following conditions are satisfied:

(I1) $\emptyset \in \mathcal{I}$

(I2) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$

(I3) If U, V belong to $\mathcal{I}$ with $|U| = |V| + 1$ then there exists $x \in U \backslash V$ such that $V \cup \{x\} \in \mathcal{I}$.

The members of $\mathcal{I}$ are called *independent sets*; all other subsets of S are called *dependent*.

A *base* of M is a maximal independent subset of S; the collection of bases is denoted by $\mathcal{B}(M)$ or $\mathcal{B}$.

A *circuit* of M is a minimal dependent subset of S; the collection of circuits is denoted by $\mathcal{C}(M)$ or $\mathcal{C}$.

It is well known that a matroid can be defined in various equivalent ways. Two frequently used definitions consist in a description of properties of $\mathcal{B}(M)$ and $\mathcal{C}(M)$. These properties are stated in the following theorems (see e.g. [8]).

THEOREM 1 *(Base axioms)*

A non-empty collection $\mathcal{B}$ of subsets of S is the set of bases of a matroid on S if and only if for any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$ it follows that there exists $y \in B_2 \backslash B_1$ such that $(B_1 \cup \{y\}) \backslash \{x\} \in \mathcal{B}$.

THEOREM 2 *(Circuit axioms)*

A collection $\mathcal{C}$ of subsets of S is the set of circuits of a matroid on S if and only if the following conditions are satisfied:

(C1) If $X \neq Y \in \mathcal{C}$ then $X \not\subseteq Y$

(C2) If C_1, C_2 are distinct members of $\mathcal{C}$ and $z \in C_1 \cap C_2$ then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \backslash \{z\}$.

We will also need some property stronger than $(C2)$ in Theorem 2, which is sometimes called a strong circuit axiom (see e.g. [8]):

THEOREM 3 *If C_1, C_2 are distinct circuits of a matroid M and $x \in C_1 \cap C_2$ then for any element y of $C_1 \backslash C_2$ there exists a circuit C such that $y \in C \subseteq (C_1 \cup C_2) \backslash \{x\}$.*

Examples of matroids

1^o Cycle matroids of graphs.

Let $G = (V, E)$ be a graph with a set of vertices V and a set of edges E. For $X \subseteq E$ we assume that $X \in \mathcal{I}$ if and only if the subgraph (V, X) does not contain a cycle. Then $\mathcal{I}$ is a collection of independent sets of a matroid on E called the *cycle matroid* of the graph G and denoted $M(G)$. The set of cycles of G is the set of circuits of $M(G)$ whereas the set of spanning forests of G corresponds to the set of bases of $M(G)$.

2^o Matroids of matrices.

Let A be a matrix and S denote the set of columns of A. For $X \subseteq S$ assume that $X \in \mathcal{I}$ if all columns belonging to X are linearly independent. Then $\mathcal{I}$ is a collection of independent sets of a matroid on S, called the *matroid of a matrix* A. The bases of this matroid correspond to maximal subsets of linearly independent columns of A.

3^o Uniform matroids.
Let S be a set of cardinality m and let $\mathcal{I}$ be a set of all subsets of cardinality $\leq$ k. Then $(S, \mathcal{I})$ is a matroid on S called the *uniform matroid* and denoted $U_{k,m}$.

Many other examples of matroids can be found in [8].

In the remaining part of this section we will recollect some facts from the matroid theory which will be necessary in the following. Although most of them are standard results we give also proofs for some of them for completeness.

The following facts are very well known:

PROPOSITION 1 *If B_1, B_2 are bases of the matroid then $\mid B_1 \mid = \mid B_2 \mid$.*

PROPOSITION 2 *If A is independent in M then for $x \in S, A \cup \{x\}$ contains at most one circuit.*

PROPOSITION 3 *If B is a base of M and $x \in S \backslash B$ then there exists a unique circuit $C = C(x, B)$ such that $x \in C(x, B) \subseteq B \cup \{x\}$.*

The circuit $C(x, B)$ is called a *fundamental circuit* of x in the base B.

PROPOSITION 4 *Let $C_1, \ldots, C_k$ be distinct fundamental circuits in the base B of the matroid M. Then*
$C_i \not\subset \bigcup_{j \neq i} C_j$ *for* $i = 1, \ldots, k$.

PROOF: Let $C_i = C(x_i, B), i = 1, \ldots, k$, and assume that for some $i_o, C_{i_o} \subseteq \bigcup_{j \neq i_o} C_j$. But this implies that $x_{i_o} \in B \cup \bigcup_{j \neq i_o} \{x_j\}$ which is impossible, because $x_{i_o} \notin B$ and $x_{i_o} \notin \bigcup_{j \neq i_o} \{x_j\}$ as the circuits considered are different. ∎

LEMMA 1 *Let $C_1, \ldots, C_k$ be distinct circuits of a matroid M on S with*

$$C_i \not\subset \bigcup_{j \neq i} C_j \quad \text{for } i = 1, \ldots, k \tag{1}$$

Then for any $T \subseteq S$ such that $\mid T \mid < k, (\bigcup_{i=1}^{k} C_i) \backslash T$ is a dependent subset of S.

PROOF: From Theorem 2 it follows that the lemma holds for $k = 2$. Suppose that it holds for $k = l \geq 2$ and we will prove that this implies that it is also true for $k = l + 1$.

Assume the contrary. This means that there exists a collection of $l + 1$ distinct circuits $\bar{C}_1, \ldots, \bar{C}_{l+1}$ satisfying (1) and a set $\bar{T} \subseteq S$ such that $|\bar{T}| < l + 1$ and

$$\bigcup_{j=1}^{l+1} \bar{C}_j \backslash \bar{T} \in \mathcal{I} \tag{2}$$

Observe that (1) is equivalent to the fact that for any $\bar{C}_j, j = 1, \ldots, l+1$, there exists $\bar{x}_j$ such that $\bar{x}_j \in \bar{C}_j$ and $\bar{x}_j \notin \bar{C}_i, i \neq j$. Consider an arbitrary circuit $\bar{C}_p, p \in \{1, \ldots, l+1\}$ and let $t_p \in \bar{T} \cap \bar{C}_p$. There are two possibilities:

(i) $t_p \notin \bar{C}_j, j \neq p$,

(ii) $t_p \in \bar{C}_j$ for some $j \in \{1, \ldots, l+1\}, j \neq p$.

In case (i) we obtain a contradiction with the assumption that the lemma holds for $k = l$, because from (2) it follows immediately that

$$(\bigcup_{j=1}^{l+1} \bar{C}_j \backslash \bar{C}_p) \backslash (\bar{T} \backslash \{t_p\}) \in \mathcal{I}.$$

In case (ii) we have also this contradiction. Observe that then $\bigcup_{j=1}^{l+1} \bar{C}_j \backslash \{t_p\}$ must contain l distinct circuits satisfying (1), because for any $j \in \{1, \ldots, l + 1\} \backslash \{p\}, (\bar{C}_j \cup \bar{C}_p) \backslash \{t_p\}$ contains (according to Theorem 3) a circuit C^j with $\bar{x}_j \in C^j$. These contradictions establish the lemma. ∎

Let $B \in \mathcal{B}$ and $x \in B$. Define

$$W(x, B) = \{y \in S \backslash B : x \in C(y, B)\}$$

The set $W(x, B)$ is called a *fundamental cutset* of x for the base B.
The following fact holds:

LEMMA 2 *Let B be a base of a matroid $M = (S, \mathcal{I})$ and consider $x \in B$, $y \in S \backslash B$. The following statements are equivalent:*

(i) $(B \backslash \{x\}) \cup \{y\} \in \mathcal{B}$

(ii) $x \in C(y, B)$

(iii) $y \in W(x, B)$

In Section 3 we will need also Hall's theorem on transversals (see e.g. [8]). Let I be a finite index set and let $\mathcal{A} = (A_i : i \in I)$ be a family of subsets of a finite set S. A set T is called a *transversal* of $\mathcal{A}$ if there is a bijection $\pi : T \to I$ such that $x \in A_{\pi(x)}$ for all $x \in T$.

THEOREM 4 *The finite family of sets $(A_i : i \in I)$ has a transversal if and only if for all $J \subseteq I$*

$$| A(J) | \geq | J |$$

where $A(J) = \bigcup(A_j : j \in J)$.

3. Sensitivity analysis for the minimum weight base

Many different combinatorial optimization problems can be formulated as a problem of finding a matroid base of minimum weight (see [3] for various examples).

Assume that we are given a *weight function* $w : S \to \mathbb{R}$. For $x \in S$ we will call $w(x)$ a *weight* of the element x.

The weight of subset $Q \subseteq S$ is defined as

$$v(Q) = \sum_{x \in Q} w(x) \tag{3}$$

The minimum weight base problem is stated as follows

$$\text{(MBP)} \qquad \min_{B \in \mathcal{B}} v(B)$$

The problem of computing the minimum weight base of a matroid is closely connected to the well known *greedy algorithm*. We will describe it using a so called *independence oracle* which for $Q \subseteq S$ gives the value of Boolean function *IND*, where

$$IND(Q) = \begin{cases} \textit{true} & \text{if and only if} \quad Q \in \mathcal{I} \\ \textit{false} & \text{otherwise} \end{cases}$$

THE GREEDY ALGORITHM

1^o $Q := \emptyset; v := 0;$

Order the elements of S in such a way that

$$w(x_1) \leq w(x_2) \leq \ldots \leq w(x_{|S|});$$

2^o **for** $i := 1$ **to** $| S |$ **do**
if $IND(Q \cup \{x_i\}) = true$ **then** $Q := Q \cup \{x_i\}, v := v + w(x_i)$;

3^o $B^o := Q; v(B^o) := v$;

It is well known (see e.g. [3]) that the GREEDY algorithm solves the MBP. Moreover,

THEOREM 5 *The following statements are equivalent :*

(i) $(E, \mathcal{F})$, *where* E *is a finite set,* $\mathcal{F} \subseteq 2^E, \mathcal{F} \neq \emptyset$, *is a matroid on* E.

(ii) For any nonnegative $w : E \to \mathbb{R}^+$ *the GREEDY algorithm solves the problem* $\max\{v(F) : F \in \mathcal{F}\}$.

Consider now two bases B', B'' of the matroid M. Theorem 1 states that any element x of B' can be exchanged with a properly chosen element y of B'' such that $(B' \backslash \{x\}) \cup \{y\}$ is a base of M. The following theorem says that one can establish a one-to-one correspondence between such pairs of elements. This fact is mentioned in [8] without proof. We will state it here with a proof for completeness.

THEOREM 6 *Let* $B', B'' \in \mathcal{B}$. *Then there exists a bijection* $\psi : B' \backslash B'' \to B'' \backslash B'$ *such that*

$$(B'' \backslash \{\psi(x)\}) \cup \{x\} \in \mathcal{B} \quad \textit{for any } x \in B' \backslash B'' \tag{4}$$

PROOF: Denote $R = B' \backslash B'', Q = B'' \backslash B'$. Consider $r \in R$. The set $B'' \cup \{r\}$ contains a fundamental circuit $C(r, B'')$. Let

$$Q_r = Q \cap C(r, B'') \tag{5}$$

A bijection satisfying (4) exists if and only if the family $(Q_r, r \in R)$ has a transversal, i.e., by Hall's theorem, if and only if $| Q(I) | = | \bigcup(Q_i : i \in I) | \geq | I |$ for every $I \subseteq R$.
Suppose that $(Q_r, r \in R)$ has no transversal. Then there must exist some $I_o \subseteq R$ such that $| Q(I_o) | < | I_o |$. Consider a set $S_o = \bigcup_{r \in I_o} C(r, B'') \backslash Q(I_o)$. It is easy to see that $S_o \subseteq B'$, but according to Proposition 4 and Lemma 1, $S_o \notin \mathcal{I}$, which contradicts the assumption that $(Q_r, r \in R)$ has no transversal. This contradiction establishes the theorem. ∎

For $B', B'' \in \mathcal{B}$ denote

$$\Psi(B', B'') = \{\psi : \psi \text{ is a bijection } B'\backslash B'' \to B''\backslash B' \text{ satisfying (4)}\} \qquad (6)$$

From the optimality of B^o we have the following obvious fact:

PROPOSITION 5 *Let B^o be the minimum weight base and $B \in \mathcal{B}$. For any $\psi \in \Psi(B, B^o)$ and arbitrary $x \in B\backslash B^o$*

$$w(x) \geq w(\psi(x)) \text{ and } (B^o \cup \{x\})\backslash\{\psi(x)\} \in \mathcal{B}.$$

For any $y \in B^o\backslash B$

$$w(y) \leq w(\psi^{-1}(y)) \text{ and } (B^o\backslash\{y\}) \cup \{\psi^{-1}(y)\} \in \mathcal{B}.$$

Example

Consider an undirected graph $G = (V, E)$ given in Fig. 1.

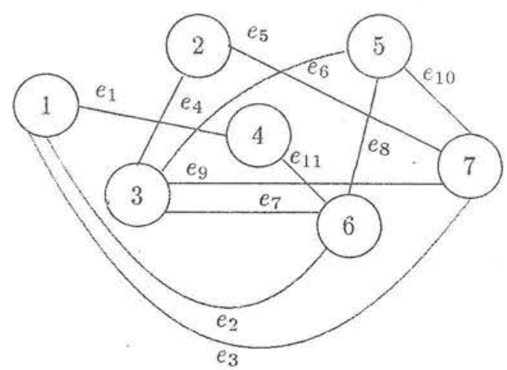

Fig. 1.

The weights of edges are given in the following table:

e	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}
$w(e)$	7	4	11	6	3	2	9	4	10	5	12

Let $M = (E, \mathcal{F})$ be a cycle matroid of the graph G, i.e., $\mathcal{F}$ is a set of forests in G and $\mathcal{B}(M)$ is a set of spanning trees. The minimum weight base of this matroid contains all edges of the minimum spanning tree. In this case

$$B^o = \{e_1, e_2, e_5, e_6, e_8, e_{10}\}$$

and $v(B^o) = 25$.

Let us consider another base of M, for instance

$$B = \{e_1, e_3, e_4, e_7, e_9, e_{10}\}$$

Theorem 6 states that the sets of bijections $\Psi(B^o, B)$ and $\Psi(B, B^o)$ are non-empty. In this case each set contains exactly one element: $\Psi(B^o, B) = \{\psi'\}$, $\Psi(B, B^o) = \{\psi''\}$, where ψ', ψ'' are defined as follows:

x	e_2	e_5	e_6	e_8
$\psi'(x)$	e_7	e_4	e_9	e_3

x	e_3	e_4	e_7	e_9
$\psi''(x)$	e_2	e_5	e_8	e_6

Observe that $\psi' \neq (\psi'')^{-1}$, so in this case one cannot establish a one-to-one correspondence ϕ between the bases B^o and B such that after an exchange of any corresponding pair of elements x and $\phi(x)$ both of the sets $(B^o \cup \{\phi(x)\}) \backslash \{x\})$ and $(B \cup \{x\}) \backslash \{\phi(x)\}$ belong to $\mathcal{B}$. ■

Assume now that for a matroid $M = (S, \mathcal{I})$ with a weight function w an optimal solution B^o of the MBP is known.
A fundamental *postoptimality analysis* problem consists in calculating ranges within which weights of elements may be varied individually without forfeiting the optimality of B^o. Such a maximum increase (decrease) of $w(x)$ which preserves the optimality of B^o will be called *upper* (*lower*) *tolerance* of x with respect to B^o and denoted $t^+(x, B^o)$ $(t^-(x, B^o))$.

Let

$$v_x = \min\{v(B) : B \in \mathcal{B}, x \notin B\}$$

and

$$v^x = \min\{v(B) : B \in \mathcal{B}, x \in B\}.$$

In calculations of v_x and in the following we use a standard convention that if the minimization problem is infeasible, then its optimal value is equal to ∞ .

Proposition 6 *If $x \in B^o$, then $t^-(x, B^o) = \infty$ and*

$$t^+(x, B^o) = v_x - v(B^o).$$

If $x \in S \backslash B^o$, then $t^+(x, B^o) = \infty$ and

$$t^-(x, B^o) = v^x - v(B^o).$$

Proof: Consider an element $x \in B^o$. It is obvious that the weight of x may decrease arbitrarily without forfeiting the optimality of B^o, so $t^-(x, B^o) = \infty$. If the weight of x increases individually, then the difference between $v(B^o)$ and a weight of any base containing x remains unchanged, but the difference $v(B) - v(B^o)$ for $B \in \mathcal{B}$ with $x \notin B$ decreases in the same way. So the maximum increase of the weight of x, for which $v(B^o)$ is not greater than $v(B)$ for $B \in \mathcal{B}$, is equal to $v_x - v(B^o)$. The proof of the second part of the proposition is analogous. ■

Proposition 6 allows to calculate the tolerances of single matroid element by computing values of v_x or v^x. Each such a value can be found by a simple modification of the GREEDY algorithm in $O(m)$ calls of independence oracle. So if the weights of matroid elements have been already ordered then the tolerances of all matroid elements can be computed in $O(m^2)$ calls of independence oracle. Observe, that Proposition 6 in fact does not exploit properties of the MBP. An alternate method of calculating tolerances of matroid elements follows from the necessary and sufficient optimality conditions for the base B^o stated in Lemma 3.

Lemma 3 *The following statements are equivalent:*

(i) B^o is a minimum weight base of M.

(ii) For any $x \in B^o, w(x) \leq w(y)$ for all $y \in W(x, B^o)$.

(iii) For any $y \in S \backslash B^o, w(y) \geq w(x)$ for all $x \in C(y, B^o)$.

Proof: The equivalence of (ii) and (iii) follows immediately from the definitions of sets $W(x, B^o), C(y, B^o)$. Also the implication (i) $\Rightarrow$ *(ii)* is an immediate consequence of Lemma 2.

To prove that (ii) $\Rightarrow$ *(i)* assume that B^o satisfies (ii) but there exists another base B, such that $v(B) < v(B^o)$. From Theorem 6 it follows that the set $\Psi(B, B^o)$ is nonempty. Consider any bijection $\psi \in \Psi(B, B^o)$. If $v(B) < v(B^o)$ then for at least one element $y \in B\backslash B^o, w(y) < w(\psi(y))$, but this contradicts the assumption (ii), because $(B^o\backslash\{\psi(y)\})\cup\{y\} \in \mathcal{B}$ and from Lemma 2 it follows that $y \in W(\psi(y), B^o)$. ■

An immediate consequence of Lemma 3 is the following fact:

LEMMA 4 *If $x \in B^o$, then $t^-(x, B^o) = \infty$ and*

$$t^+(x, B^o) = \min\{w(y) : y \in W(x, B^o)\} - w(x) \tag{7}$$

If $x \in S\backslash B^o$, then $t^+(x, B^o) = \infty$ and

$$t^-(x, B^o) = w(x) - \max\{w(y) : y \in C(x, B^o), y \neq x\} \tag{8}$$

To calculate tolerances of elements using Lemma 4 one needs two families of subsets of S:
the *family of fundamental cutsets*

$$\mathcal{F}_{cu}(B^o) = (W(x, B^o) : x \in B^o) \tag{9}$$

and the *family of fundamental circuits*

$$\mathcal{F}_{ci}(B^o) = (C(y, B^o) : y \in S\backslash B^o) \tag{10}$$

Let us introduce a numbering of elements of subsets B^o and $S\backslash B^o$, i.e., $B^o = \{x_1, \ldots, x_b\}, S\backslash B^o = \{y_1, \ldots, y_n\}$, where $b =\mid B^o \mid, n =\mid S\backslash B^o \mid$. The families $\mathcal{F}_{cu}(B^o), \mathcal{F}_{ci}(B^o)$ may be represented now in the *fundamental matrix* $A(B^o)$, where

$$A(B^o) = [a_{ij}] \qquad i = 1, \ldots, b, \quad j = 1, \ldots, n$$

$$a_{ij} = \begin{cases} 1 & \text{if} \qquad x_i \in C(y_j, B^o) \\ 0 & \text{otherwise} \end{cases}$$

Now the family $\mathcal{F}_{ci}(B^o)$ corresponds to columns of $A(B^o)$ and the family $\mathcal{F}_{cu}(B^o)$ corresponds to rows of $A(B^o)$.

The matrix $A(B^o)$ can be constructed in $O(b \cdot n)$ calls of independence oracle because the value of a single element a_{ij} can be determined by verifying whether $(B^o\backslash\{x_i\}) \cup \{y_j\} \in \mathcal{I}$. It can be also constructed in $O(n)$ calls of the so called *circuit oracle* which for $x \in S$ and $Q \in \mathcal{I}$ gives an empty set if $Q \cup \{x\} \in \mathcal{I}$ or the single circuit contained in $Q \cup \{x\}$ otherwise.
Given the fundamental matrix one can compute from Lemma 4 the tolerances of all elements in $O(b \cdot n)$ comparisons.

The calculation of tolerances can be performed in an efficient way if we have an auxiliary graph called a *transmuter*. We will define a transmuter in a similar way as in [7], where such an approach was proposed to sensitivity analysis for minimum spanning trees and shortest path trees:

For a given matroid base B a transmuter $T(B)$ is a directed acyclic graph which contains one vertex e_x of in-degree zero for any $x \in B$, one vertex e_y of out-degree zero for any $y \in S \backslash B^o$ and an arbitrary number of additional vertices. Moreover, there exists a path from vertex e_x to vertex e_y if and only if $y \in W(x, B)$.

A binary transmuter $BT(B)$ is a transmuter in which the in-degree and out-degree of any vertex is less than or equal to two. It is obvious that any transmuter can be transformed to a binary transmuter by enlarging the number of additional vertices.
In some cases a transmuter can be constructed in an efficient way using appropriate data structures (see for example [7]).

Given a transmuter $BT(B^o)$ with T vertices one can compute all tolerances with respect to B^o in $O(T)$ steps using the following simple procedures:

PROCEDURE TRANS–L to calculate lower tolerances for $y \in S \backslash B^o$

1^o Assign labels $l(e_x) := w(x)$ to all vertices $e_x, x \in B^o$. All other vertices remain temporarily unlabeled.

2^o For any vertex e, for which the adjacent in-vertices are labeled, assign a label $l(e)$ equal to the maximum of labels of in-vertices.

3^o Calculate $t^-(y, B^o) = w(y) - l(e_y), y \in S \backslash B^o$.

PROCEDURE TRANS–U to calculate upper tolerances for $x \in B^o$.

1^o Assign labels $l(e_y) := w(y)$ to all vertices $e_y, y \in S \backslash B^o$. All other vertices remain temporarily unlabeled.

2^o For any vertex e, for which the adjacent out-vertices are labeled, assign a label $l(e)$ equal to the minimum of labels of out-vertices.

3^o Calculate $t^+(x, B^o) = l(v_x) - w(x), x \in B^o$.

Example (continued)

Consider a cycle matroid of the graph G of Fig. 1 with the minimum weight base $B^o = \{e_1, e_2, e_5, e_6, e_8, e_{10}\}$. Then $S \backslash B^o = \{e_3, e_4, e_7, e_9, e_{11}\}$. Introduce a natural numbering of elements of these subsets. The fundamental matrix with respect to B^o is as follows

$$A(B^o) = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{vmatrix}$$

The transmuter $T(B^o)$ is shown in Fig. 2.

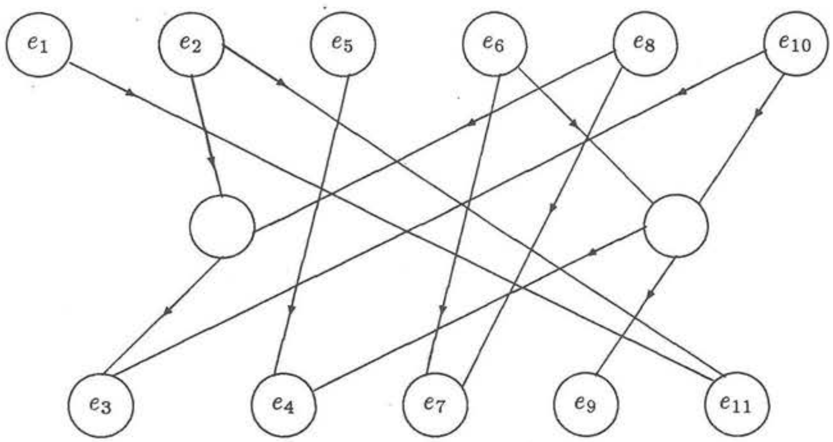

Fig. 2.

This is a binary transmuter containing two additional vertices. The labels produced by the procedure TRANS–L are indicated in Fig. 3.

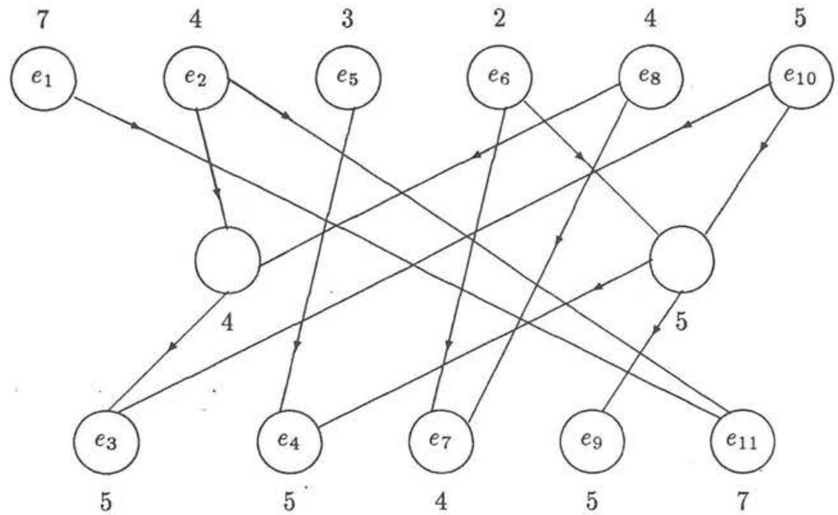

Fig. 3.

Now it is easy to calculate $t^-(e, B^o)$ for $e \notin B^o$ according to Step 3^o in the procedure TRANS-L.

e	e_3	e_4	e_7	e_9	e_{11}
$t^-(e, B^o)$	6	1	5	5	5

In a similar way we can use $T(B^o)$ in the procedure TRANS-U and we obtain the following values for $t^+(e, B^o), e \in B^o$.

e	e_1	e_2	e_5	e_6	e_8	e_{10}
$t^+(e, B^o)$	5	7	3	4	5	1

4. Some other sensitivity analysis problems

In Section 3 the fundamental sensitivity analysis problem consisting in the calculation of individual data perturbations (tolerances) preserving the optimality of solution was considered. In this section we will show how the notion of the tolerances can be used in some other sensitivity analysis problems. The approach is based on the following fact:

THEOREM 7 *Let B^o be the minimum weight base of the matroid M and B be an arbitrary base. Then for any bijection $\psi \in \Psi(B, B^o)$ the following relations hold:*

(i) For any $y \in B \backslash B^o$

$$w(y) - w(\psi(y)) \geq t^-(y, B^o) \tag{11}$$

(ii) For any $x \in B^o \backslash B$

$$w(\psi^{-1}(x)) - w(x) \geq t^+(x, B^o) \tag{12}$$

PROOF: Consider an arbitrary bijection $\psi \in \Psi(B, B^o)$. From Lemma 2 it follows that for any $y \in B \backslash B^o, \psi(y) \in C(y, B^o)$. But by Lemma 4, $t^-(y, B^o) = w(y) - \max\{w(x) : x \in C(y, B^o), x \neq y\}$ for all $y \in S \backslash B^o$, which implies (11).
Similarly, for any $x \in B^o \backslash B, \psi^{-1}(x) \in W(x, B^o)$ and because $t^+(x, B^o) = \min\{w(y) : y \in W(x, B^o)\} - w(x)$ we have (12). ■

COROLLARY 1 *If B^o is the minimum weight base of M and $B \in \mathcal{B}$ then*

$$v(B) - v(B^o) \geq \sum_{y \in B \backslash B^o} t^-(y, B^o) \tag{13}$$

and

$$v(B) - v(B^o) \geq \sum_{x \in B^o \backslash B} t^+(x, B^o) \tag{14}$$

PROOF: Theorem 6 states that the set of bijections $\Psi(B, B^o)$ is nonempty. Taking any element ψ of $\Psi(B, B^o)$ and using (11), (12) we obtain (13), (14). ■

Assume now that we are given the minimum weight base B^o and we want to calculate the increase of the base weight caused by introduction of additional restrictions that some subset D of elements belonging to B^o must be removed from the minimum weight base or that some subset of elements $A \in S \backslash B^o$ has to be included in this base. More formally, we are interested in solving of the following problems:

$$v_D = \min\{v(B) : B \in \mathcal{B}, D \cap B = \emptyset\} \tag{15}$$

where $D \subseteq B^o$,

$$v^A = \min\{v(B) : B \in \mathcal{B}, A \subseteq B\} \tag{16}$$

where $A \subseteq S \backslash B^o, A \in \mathcal{I}$.

The above problems may be regarded as the MBP with side constraints and can be solved by greedy-type algorithms. But if we know the tolerances $t^+(x,B), t^-(x,B), x \in S$, then the lower bounds of the values v_D, v^A can be easily calculated due to the following facts:

COROLLARY 2 *Let $D \subseteq B^o$. Then for any base B_D satisfying the condition $D \cap B_D = \emptyset$*

$$v(B_D) \geq v(B^o) + \sum_{x \in D} t^+(x, B^o) \tag{17}$$

COROLLARY 3 *Let $A \subseteq S \backslash B^o$ and $A \in \mathcal{I}$. Then for any base B^A satisfying the condition $A \subseteq B^A$*

$$v(B^A) \geq v(B^o) + \sum_{x \in A} t^-(x, B^o) \tag{18}$$

Proofs of above facts are immediate from (13), (14).

Example (continued)

Consider two bases of the cycle matroid of the graph shown on Fig. 1. : the minimum weight base $B^o = \{e_1, e_2, e_5, e_6, e_8, e_{10}\}$ and $B = \{e_1, e_3, e_4, e_7, e_9, e_{10}\}$ with $v(B^o) = 25, v(B) = 48$,

$$B^o \backslash B = \{e_2, e_5, e_6, e_8\},$$

$$B \backslash B^o = \{e_3, e_4, e_7, e_9\},$$

$$v(B) - v(B^o) = 23.$$

Using the values of tolerances previously obtained we can now calculate right--hand sides of inequalities (13), (14).

$$\sum_{e \in B^o \backslash B} t^+(e, B^o) = 19$$

$$\sum_{e \in B \backslash B^o} t^-(e, B^o) = 17$$

Assume now that we are introducing a restriction that the edges e_2, e_6, e_8 may not appear in the minimum weight base. From Corollary 2 it follows that an increase of the solution value (with respect to $v(B^o)$) caused by this restriction is equal at least to

$$t^+(e_2, B) + t^+(e_6, B) + t^+(e_8, B) = 16.$$

So the minimum weight base which does not contain e_2, e_6, e_8 has the weight of at least 41. In fact, the weight of such a base is exactly equal to 41 for $B_D = \{e_1, e_3, e_4, e_5, e_7, e_{10}\}$.

In a similar way we can calculate the effect of additional constraint requiring that some edges, say e_4, e_7, must appear in the solution. According to Corollary 3, the weight of such a base must be equal at least to

$$v(B^o) + t^-(e_4, B^o) + t^-(e_7, B^o) = 31.$$

Also in this case the bound is tight and $B^A = \{e_1, e_2, e_4, e_5, e_6, e_7\}$.

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Analiza wrażliwości dla bazy o minimalnej wadze w matroidzie

Liczne zadania optymalizacji kombinatorycznej występujące w badaniach operacyjnych można sformułować jako problem wyznaczania bazy o minimalnej wadze w odpowiednim matroidzie. Praca dotyczy analizy pooptymalizacyjnej dla tego problemu. Zagadnienie polega na znalezieniu dla danej optymalnej bazy matroidu takich dopuszczalnych przyrostów i zmniejszeń wag pojedyńczych elementów matroidu, które nie naruszają optymalności tej bazy. Podana jest metoda wyznaczania tych zmian wag nazwanych tolerancjami elementów. Dyskutowane jest wykorzystanie tolerancji elementów w innych zadaniach analizy pooptymalizacyjnej.

Анализ чувствитиельности для базиса матроида с минимальным весом

Многие задачи дискретной оптимизации могут быть поставлены как задачи нахождения базиса матроида с минимальным весом. В работе рассмотрены некоторые вопросы постоптимального анализа этой проблемы.

Описан метод нахождения индивидуальных возмущений весов элементов, для которых данный базис остается минимальным. Рассматриваются другие проблемы постоптимального анализа, которые могут быть решены, когда такие возмущения известны.